Adaptive output feedback controller design for high-order stochastic nonlinear systems with uncertain output function and unknown growth rates

Ce Liu (749947738@qq.com)
Southeast University https://orcid.org/0000-0002-5212-2101
Junyong Zhai
Southeast University https://orcid.org/0000-0001-5122-3819

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Ce Liu · **Junyong Zhai**

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Abstract This paper concentrates on the adaptive output feedback controller design for a class of high-order stochastic nonlinear systems(SNSs) with uncertain output function. Firstly, a homogeneous output feedback controller for the nominal system is designed through the technique of adding a power integrator. Secondly, a well-designed dynamic gain is introduced into the controller to ensure the original SNSs globally asymptotically stable(GAS) in probability. Besides, the proposed control strategy can be also extended to upper-triangular SNSs. Finally, two numerical examples illustrate the effectiveness of the proposed method.

Keywords High-order stochastic nonlinear systems · dynamic gain · uncertain output function · homogeneous output feedback controller

1 Introduction

The research on linear systems theory has a history of decades and has been successfully applied in various defense and industrial control. However, the dynamic characteristics of linear systems are no longer sufficient to explain many common nonlinear phenomena in actual engineering, such as chemical reactions, vehicle suspension or robotic systems and so on. Thus the stability of nonlinear systems has been a research focus in the control engineering field and many results have been accomplished. As we all know, it is not realistic to access all the system states, the shortcomings of state feedback have become obvious, so how to stabilize the systems or achieve the expected system performance via output feedback has attracted more and more attention [1–3]. Based on a non-separation principle method, the work [1] considered how to globally stabilize the nonlinear systems via output feedback control. The work [2] constructed two higher-dimensional homogeneous observers to estimate both the low-order and high-order terms, which was further extended to the output polynomial form in the work [3].

The study on output feedback controller design of deterministic systems has achieved fruitful results. Nevertheless, stochastic noises or disturbances are widespread in economic systems and many other practical ones, which are often modeled by stochastic differential equations. The stochastic Lyapunov stability theory proposed in [4] is widely used for the research on SNSs under various structures or growth conditions. Since the result on global output-feedback stabilization of SNSs had been proposed in [5], an output feedback controller was put forward in [7] to make SNSs suffering from linearly bounded unmeasurable states GAS in probability. Futhermore, efforts have been made in [6] towards how to design continuous controller via output feedback for SNSs with unmeasurable states-dependent growth condition and uncertain control coefficients. With the help of adaptive control methods, the literature [7] has achieved the adaptive output-feedback stabilization of SNSs whose nonlinear functions satisfy linear unmeasured states-dependent growth. The works [8] paid attention to the global stability of SNSs whose nonlinearities satisfied linear growth conditions with nonnegative smooth functions. After the homogeneity theory had been used to solve stabilization problem of deterministic systems in [9–11], scholars in [12–15] relaxed the power order restriction and generalized the concepts and theorems of global stabilization to a class of high-order SNSs. To mention a few, [14] investigated the problem of global finite-time controller design in the stochastic case where drift and diffusion terms satisfy the homogeneous growth conditions, and [15] designed a stabilizing

C. Liu · J. Zhai
Key Laboratory of Measurement and Control of CSE, Ministry of Education, School of Automation, Southeast University,
Nanjing, Jiangsu 210096, China
E-mail: jyzhai@seu.edu.cn

controller and further relaxed the nonlinear growth condition.

Additionally, the output signals are usually measured by sensors in actual engineering applications as mentioned in [16], which leads to the uncertainty of output function. Taking this into consideration, problem of controller design by output feedback was handled in [16–21], when the output function is no longer precise. But except for the study on deterministic systems in [22], there is few related research with respect to the output feedback stabilization for high-order systems with uncertain output function, and the aforementioned results cannot be used for the cases when system parameters are uncertain in high-order SNSs. Therefore in this paper, we consider the following SNSs

$$\begin{aligned} dz_i &= \left(z_{i+1}^{p_i} + f_i(z,\nu)\right)dt + g_i^T(z,\nu)dw, \\ i &= 1,\dots,n-1, \\ dz_n &= \left(\nu^{p_n} + f_n(z,\nu)\right)dt + g_n^T(z,\nu)dw, \\ y &= h(z_1), \end{aligned} \tag{1}$$

where $z=(z_1,\dots,z_n)^T \in \mathbb{R}^n$ is the system state, $\nu \in \mathbb{R}$ represents control input, and $y \in \mathbb{R}$ is a C^1 uncertain output function satisfying $h(0)=0$, $p_i \in \mathbb{R}^{\geq 1}_{odd}$. w is an r-dimensional standard Wiener process defined on a probability space (Ω, $\mathscr{F}$, $\mathscr{F}_t$, $\mathscr{P}$). The drift function $f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and the diffusion function $g_i^T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{1\times r}$ satisfy unknown homogeneous growth condition with $f_i(0)=0$, $g_i(0)=0$. The highlights of this article are fourfold:

1. Compared to research on the systems whose nonlinearities satisfy linear or homogeneous growth condition with known rates in [8,23], the unknown homogeneous growth condition on the drift and diffusion terms brings difficulty to solve the problem. Inspired by [19], we introduce a novel dynamic gain into the reduced-order observer to handle more general SNSs;
2. Due to the uncertain output function, the system state z_1 cannot be accurately obtained, the controller proposed in [10] is no longer in effect. Besides, the proposed method in [14] cannot be used because of the appearance of $\frac{\partial^2 y}{\partial z_1^2}$ in the Hessian terms. Thus the stabilization of high-order SNSs considered in this paper are more general than the results in the existing literature;
3. Because of the complex structure of the nonlinear terms, it is challenging to show the existence and uniqueness of the solution. Inspired by the stochastic Barbalat's lemma in [25], the signals of the closed-loop system are demonstrated to be bounded and all the system states are proved to converge to the origin in probability;
4. Compared with the study on lower-triangular SNSs, we have extended the existed results to the controller design of upper-triangular ones so that we can handle more practical examples, such as electric circuit systems in [26] and cart-pendulum ones in [27].

Notations $\mathbb{R}^n$ is denoted as the real n-dimensional space; $\mathbb{R}_+$ is the set of all nonnegative real numbers; $\mathbb{R}^{\geq 1}_{odd}$ is a real number ≥ 1 and is ratio of two odd integers. $C^{2,1}(\mathbb{R}^n \times [0,\infty);\mathbb{R}_+)$ represents the family of all non-negative functions $V(x,t)$ on $\mathbb{R}^n \times [0,\infty)$, where $C^{2,1}$ means the functions are C^2 in x and C^1 in t. "a.s." is equivalent to "... holds almost surely on a certain probability space". $\phi \in \Xi(\mathbb{R}_+ \times \Omega)$ is established when the stochastic process $\phi(t) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ fulfills $E(\int_0^\infty \|\phi(s)\|)ds < \infty$. For real numbers, $a \wedge b = \min\{a,b\}$.

2 Preliminaries

This section contains some definitions and several lemmas utilized later.

Definition 1 [23] Consider

$$dx = f(x)dt + g^T(x)dw \tag{2}$$

where $x \in \mathbb{R}^n$ is the system sate, the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g^T : \mathbb{R}^n \to \mathbb{R}$ satisfy locally Lipschitz condition with $f(0)=0$, $g(0)=0$. The infinitesimal generator $\mathscr{L}$ of any identified $V(x) \in C^2$ with respect to system (2) is denoted as

$$\mathscr{L}V(x) = \frac{\partial V}{\partial x} f(x) + \frac{1}{2}\mathrm{Tr}\{g(x)\frac{\partial^2 V}{\partial x^2} g^T(x)\}. \tag{3}$$

Definition 2 [24] For constants $r_1 > 0,\dots,r_n > 0$ and coordinates $(x_1,\dots,x_n) \in \mathbb{R}^n$, $\forall \varepsilon > 0$:
(i) define the dilation $\Delta_\varepsilon(x) = (\varepsilon^{r_1}x_1,\dots,\varepsilon^{r_n}x_n)$ and abbreviate it as $\Delta = (r_1,\dots,r_n)$, where r_i is named as the weight of x_i.
(ii) a function $W \in C(\mathbb{R}^n,\mathbb{R})$ is homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^n \setminus \{0\}, W(\Delta_\varepsilon(x)) = \varepsilon^\tau W(x_1,\dots,x_n)$.
(iii) $\forall x \in \mathbb{R}^n$, $\|x\|_{\Delta,p} = \left(\sum_{i=1}^n |x_i|^{p/r_i}\right)^{1/p}$ denotes a homogeneous p-norm with $p \geq 1$. For convenience, $\|x\|_{\Delta,p}$ is written as $\|x\|_\Delta$ when $p = 2$.

Lemma 1 [25] For system (2), if there is a function $V(x,t) \in C^{2,1}(\mathbb{R}^n \times [0,\infty);\mathbb{R}_+)$ and a positive constant D, such that for any $t \geq 0$, the following relation holds

$$EV(x(t \wedge \bar{\sigma}_k), t \wedge \bar{\sigma}_k) \leqslant D, \lim_{k\to\infty} \inf_{\|x\|\geq k} V(x,t) = \infty,$$

where $\bar{\sigma}_k = \inf\{t \geq 0 \big| \|x(t)\| \geq k, \forall k > 0\}$, then the solution of SNSs exists and is unique in $[0,\infty)$.

Lemma 2 [25] If $x(t)$ is strongly bounded in probability, the function $\beta(x(t)) \in \Xi([0,\infty) \times \Omega)$ and is continuous, then the following holds

$$\lim_{t\to\infty} \beta(x(t)) = 0, \ a.s.$$

Lemma 3 [24] Assume that $W_1(x):\mathbb{R}^n\to\mathbb{R}$ is a homogeneous function of degree m_1 with respect to the dilation weight Δ, under which one has
(i) the homogeneous degree of $\frac{\partial W_1}{\partial x_i}$ is m_1-r_i with r_i being the homogeneous weight of x_i.
(ii) there is a constant $\bar{d}$ such that $W_1(x)\le \bar{d}\|x\|_\Delta^{m_1}$. Moreover, if $W_1(x)$ is positive definite, $W_1(x)\ge \underline{d}\|x\|_\Delta^{m_1}$, $\underline{d}>0$.
(iii) For another homogeneous function $W_2(x)$ of degree m_2 with respect to the same Δ, $W_1(x)W_2(x)$ is a homogeneous function of degree m_1+m_2 with regard to Δ.

Lemma 4 [10] For $x\in\mathbb{R}$ and $y\in\mathbb{R}$, if $r\ge 1$, one has

$$|x+y|^r\le 2^{r-1}|x^r+y^r|,\ (|x|+|y|)^{\frac{1}{r}}\le |x|^{\frac{1}{r}}+|y|^{\frac{1}{r}};$$

if $r\in\mathbb{R}_{odd}^{\ge 1}$, one gets

$$|x-y|^r\le 2^{r-1}|x^r-y^r|,\ |x^{\frac{1}{r}}-y^{\frac{1}{r}}|\le 2^{1-\frac{1}{r}}|x-y|^{\frac{1}{r}};$$

$$-(x-y)(x^r-y^r)\le -\frac{1}{2^{r-1}}(x-y)^{r+1};$$

$$\begin{aligned}|x^r-y^r|&\le r|x-y|(x^{r-1}+y^{r-1})\\&\le d|x-y|((x-y)^{r-1}+y^{r-1})\end{aligned}$$

for a constant $d>0$.

Lemma 5 [10] Let $x\in\mathbb{R}$ and $y\in\mathbb{R}$, one has

$$|x|^m|y|^n\le \frac{m}{m+n}\gamma|x|^{m+n}+\frac{n}{m+n}\gamma^{-\frac{m}{n}}|y|^{m+n} \tag{4}$$

where m, n and γ are positive constants.

3 Main Results

In this section, firstly we design an output feedback controller for the nominal nonlinear system. Then, a dynamic gain is introduced into the observer and controller to render SNSs (1) GAS in probability by virtue of the homogeneous domination method. Before moving on, the following assumptions should be imposed on the drift and diffusion terms.

Assumption 1 There are two known positive constants $\underline{o}$ and $\bar{o}$ such that

$$0<\underline{o}\le\frac{\partial h(s)}{\partial s}\le\bar{o},\ \forall s\in\mathbb{R}. \tag{5}$$

Remark 1 Assumption 1 means that the output function is differentiable and its first derivative is bounded. Compared to the Assumption in [21], the condition is further extended to situation like $h(z_1)=3z_1+\arctan(z_1)$. Under condition (5), one has

$$\frac{|y|}{\bar{o}}\le|z_1|\le\frac{|y|}{\underline{o}} \tag{6}$$

whose proof can be found in [16]. Obviously, when y tends to zero, z_1 will also tend to zero.

Assumption 2 For $i=1,\ldots,n$, there are two unknown positive constants θ_1 and θ_2, such that

$$|f_i(\cdot)|\le\theta_1\sum_{j=1}^{i}|z_j|^{\frac{r_i+\tau}{r_j}},\ \ \|g_i(\cdot)\|\le\theta_2\sum_{j=1}^{i}|z_j|^{\frac{2r_i+\tau}{2r_j}} \tag{7}$$

where $\tau\ge 0$ and $r_1=1, r_{i+1}p_i=r_i+\tau$.

Remark 2 Assumption 2 shows that system (1) has inherent nonlinearities which depend on unmeasurable states and grow with unknown homogeneous rates, and this condition is weaker than those in the existing results. For example, [15] investigated the global output feedback controller design for SNSs with output function $y=z_1$. [17] dealt with the SNSs for $p_i=1$ where the nonlinear terms are merely relevant to system output. [22] addressed the adaptive output feedback stabilization issue of deterministic systems with $\tau\ge 0$.

3.1 Output feedback controller design for nominal system

Consider the following nominal nonlinear system

$$\begin{aligned}&dx_1=x_2^{p_1}dt,\ldots,dx_{n-1}=x_n^{p_{n-1}}dt,\ dx_n=u^{p_n}dt,\\&y=h(x_1)\end{aligned} \tag{8}$$

with $x=(x_1,\ldots,x_n)^T\in\mathbb{R}^n$, $u\in\mathbb{R}$, $y\in\mathbb{R}$ being system state, control input and output, respectively. In this subsection, an output feedback controller for the nominal system (8) will be designed as follows.

Step 1: Choose the Lyapunov function V_1 as

$$V_1=\int_{y^*}^{y}(s^{\sigma/r_1}-y^{*\sigma/r_1})^{\frac{4\sigma-r_1-\tau}{\sigma}}ds \tag{9}$$

with $y^*=0, \sigma\ge 2\max_{2\le i\le n+1}\{r_ip_{i-1}\}$ and $\sigma\in\mathbb{R}_{odd}^{\ge 1}$.

With a simple calculation, one has

$$\mathscr{L}V_1=y^{4\sigma-r_1-\tau}\frac{\partial h}{\partial x_1}(x_2^{p_1}-x_2^{*p_1})+y^{4\sigma-r_1-\tau}\frac{\partial h}{\partial x_1}x_2^{*p_1}. \tag{10}$$

Letting $\xi_1=y^{\sigma/r_1}$, one chooses the virtual controller as

$$x_2^*=-\beta_1\xi_1^{r_2/\sigma},\ \beta_1\ge(\frac{n+g_0}{\underline{o}})^{1/p_1} \tag{11}$$

with a positive constant g_0 to render that

$$\mathscr{L}V_1\le -n\xi_1^4+\xi_1^{\frac{4\sigma-r_1-\tau}{\sigma}}\frac{\partial h}{\partial x_1}(x_2^{p_1}-x_2^{*p_1})-g_0\xi_1^4. \tag{12}$$

Step 2: Choose

$$V_2=V_1+T_2=V_1+\int_{x_2^*}^{x_2}(s^{\sigma/r_2}-x_2^{*\sigma/r_2})^{\frac{4\sigma-r_2-\tau}{\sigma}}ds. \tag{13}$$

Then one has

$$\begin{aligned}\mathscr{L}V_2 =&\mathscr{L}V_1 + \xi_2^{\frac{4\sigma-r_2-\tau}{\sigma}} x_3^{p_2} - \frac{4\sigma-r_2-\tau}{\sigma}\int_{x_2^*}^{x_2}(s^{\sigma/r_2} \\ &- x_2^{*\sigma/r_2})^{\frac{4\sigma-r_2-\tau}{\sigma}-1} ds \frac{\partial x_2^{*\sigma/r_2}}{\partial y}\frac{\partial h}{\partial x_1}x_2^{p_1}\end{aligned} \tag{14}$$

with $\xi_2 = x_2^{\sigma/r_2} - x_2^{*\sigma/r_2}$.

As $\frac{r_2 p_1}{\sigma} \le 1$, combined with Lemmas 4, 5 and Assumption 1, it can be obtained that

$$\begin{aligned}&\xi_1^{\frac{4\sigma-r_1-\tau}{\sigma}}\frac{\partial h}{\partial x_1}(x_2^{p_1} - x_2^{*p_1}) - \frac{4\sigma-r_2-\tau}{\sigma}\int_{x_2^*}^{x_2}(s^{\sigma/r_2} \\ &- x_2^{*\sigma/r_2})^{\frac{4\sigma-r_2-\tau}{\sigma}-1} ds \frac{\partial x_2^{*\sigma/r_2}}{\partial y}\frac{\partial h}{\partial x_1}x_2^{p_1} \\ \le& 2^{1-\frac{r_2p_1}{\sigma}}\bar{o}|\xi_1|^{\frac{2\sigma-r_1-\tau}{\sigma}}|\xi_2|^{\frac{r_2p_1}{\sigma}} + 2^{1-\frac{r_2}{\sigma}}\frac{4\sigma-r_2-\tau}{\sigma} \\ &|\xi_2|^{\frac{r_2}{\sigma}}|\xi_2|^{\frac{3\sigma-r_2-\tau}{\sigma}}\beta_1^{\frac{\sigma}{r_2}}\frac{\sigma}{r_1}|\xi_1|^{\frac{\sigma-r_1}{\sigma}}\bar{o}|\xi_2 - \beta_1^{\frac{\sigma}{r_2}}\xi_1|^{\frac{r_2p_1}{\sigma}} \\ \le& \xi_1^4 + \hat{d}_1\xi_2^4\end{aligned} \tag{15}$$

with a positive constant $\hat{d}_1$. Then one has

$$\begin{aligned}\mathscr{L}V_2 \le& -(n-1+g_0)\xi_1^4 + \xi_2^{\frac{4\sigma-r_2-\tau}{\sigma}}(x_3^{p_2} - x_3^{*p_2}) \\ &+ \xi_2^{\frac{4\sigma-r_2-\tau}{\sigma}}x_3^{*p_2} + \hat{d}_1\xi_2^4.\end{aligned} \tag{16}$$

The virtual controller is chosen as

$$x_3^* = -\beta_2\xi_2^{r_3/\sigma}, \beta_2 \ge (n-1+g_0+\hat{d}_1)^{1/p_2} \tag{17}$$

which yields

$$\begin{aligned}\mathscr{L}V_2 \le& -(n-1)(\xi_1^4+\xi_2^4) - g_0(\xi_1^4+\xi_2^4) \\ &+ \xi_2^{\frac{4\sigma-r_2-\tau}{\sigma}}(x_3^{p_2} - x_3^{*p_2}).\end{aligned} \tag{18}$$

Inductive Step: Assume that there exists a Lyapunov function V_i and a group of virtual controllers $y^*, x_2^*, \ldots, x_{i+1}^*$ defined as

$$\begin{aligned}&y^* = 0,\ \xi_1 = y^{\sigma/r_1} - y^{*\sigma/r_1} \\ &x_2^* = -\beta_1\xi_1^{r_2/\sigma},\ \xi_2 = x_2^{\sigma/r_2} - x_2^{*\sigma/r_2} \\ &\vdots \\ &x_{i+1}^* = -\beta_i\xi_i^{r_{i+1}/\sigma},\ \xi_{i+1} = x_{i+1}^{\sigma/r_{i+1}} - x_{i+1}^{*\sigma/r_{i+1}}\end{aligned} \tag{19}$$

with constants $\beta_1, \ldots, \beta_i > 0$, such that

$$\begin{aligned}\mathscr{L}V_i \le& -(n-i+1)\sum_{j=1}^{i}\xi_j^4 - g_0(\xi_1^4+\xi_2^4) \\ &+ \xi_i^{\frac{4\sigma-r_i-\tau}{\sigma}}(x_{i+1}^{p_i} - x_{i+1}^{*p_i}).\end{aligned} \tag{20}$$

By constructing Lyapunov function at $(i+1)$th step

$$\begin{aligned}V_{i+1} &= V_i + T_{i+1} \\ &= V_i + \int_{x_{i+1}^*}^{x_{i+1}}(s^{\sigma/r_{i+1}} - x_{i+1}^{*\sigma/r_{i+1}})^{\frac{4\sigma-r_{i+1}-\tau}{\sigma}}ds\end{aligned} \tag{21}$$

one has

$$\begin{aligned}\mathscr{L}T_{i+1} =& \frac{\partial T_{i+1}}{\partial x_{i+1}}x_{i+2}^{p_{i+1}} + \sum_{j=2}^{i}\frac{\partial T_{i+1}}{\partial x_j}x_{j+1}^{p_j} \\ &+ \frac{\partial T_{i+1}}{\partial y}\frac{\partial h}{\partial x_1}x_2^{p_1}.\end{aligned} \tag{22}$$

Similarly as proved in (15), one has

$$\begin{aligned}&\xi_i^{\frac{4\sigma-r_i-\tau}{\sigma}}(x_{i+1}^{p_i} - x_{i+1}^{*p_i}) + \sum_{j=2}^{i}\frac{\partial T_{i+1}}{\partial x_j}x_{j+1}^{p_j} + \frac{\partial T_{i+1}}{\partial y}\frac{\partial h}{\partial x_1}x_2^{p_1} \\ &\le \sum_{j=1}^{i}\xi_j^4 + \hat{d}_2\xi_{i+1}^4\end{aligned} \tag{23}$$

with a positive constant $\hat{d}_2$.

Substituting (23) into (21), one gets

$$\begin{aligned}\mathscr{L}V_{i+1} \le& -(n-i)\sum_{j=1}^{i}\xi_j^4 + \xi_{i+1}^{\frac{4\sigma-r_{i+1}-\tau}{\sigma}}(x_{i+2}^{p_{i+1}} - x_{i+2}^{*p_{i+1}}) \\ &+ \xi_{i+1}^{\frac{4\sigma-r_{i+1}-\tau}{\sigma}}x_{i+2}^{*p_{i+1}} + \hat{d}_2\xi_{i+1}^4 - g_0(\xi_1^4+\xi_2^4).\end{aligned} \tag{24}$$

Similarly, combining with the form of virtual controller

$$x_{i+2}^* = -\beta_{i+1}\xi_{i+1}^{r_{i+2}/\sigma},\ \beta_{i+1} \ge (n-i+\hat{d}_2)^{1/p_{i+1}}$$

(24) turns into

$$\begin{aligned}\mathscr{L}V_{i+1} \le& -(n-i)\sum_{j=1}^{i+1}\xi_j^4 - g_0(\xi_1^4+\xi_2^4) \\ &+ \xi_{i+1}^{\frac{4\sigma-r_{i+1}-\tau}{\sigma}}(x_{i+2}^{p_{i+1}} - x_{i+2}^{*p_{i+1}}).\end{aligned} \tag{25}$$

Remark 3 Different from the existed study on deterministic system, $V_i(y, x_2, \ldots, x_i)$ should be designed as a C^2 Lyapunov function to eliminate some nonsense and ill-defined terms caused by the Hessian term, which is shown in [14]. From the definitions of σ and x_{i+1}^*, the zero-division problem of both $\frac{\partial^2 x_{i+1}^{*\frac{\sigma}{r_{i+1}}}}{\partial x_j^2}(2 \le j \le i)$ and $\frac{\partial^2 x_{i+1}^{*\frac{\sigma}{r_{i+1}}}}{\partial y^2}$ will not occur, which ensures that V_i is C^2.

As a result, at step n, a Lyapunov function V_n and a positive constant $\hat{d}_3$ lead to

$$\begin{aligned}\mathscr{L}V_n \le& -\sum_{i=1}^{n-1}\xi_i^4 + \xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}(u^{p_n} - x_{n+1}^{*p_n}) \\ &+ \xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}x_{n+1}^{*p_n} + \hat{d}_3\xi_n^4 - g_0(\xi_1^4+\xi_2^4)\end{aligned} \tag{26}$$

where the virtual controller needs to satisfy

$$
\begin{aligned}
x_{n+1}^{*\frac{\sigma}{r_{n+1}}} &= -\beta_n^{\frac{\sigma}{r_{n+1}}} \xi_n \\
&= -\sum_{i=2}^{n} \beta_n^{\frac{\sigma}{r_{n+1}}} \cdots \beta_i^{\frac{\sigma}{r_{i+1}}} x_i^{\frac{\sigma}{r_i}} - \beta_n^{\frac{\sigma}{r_{n+1}}} \cdots \beta_1^{\frac{\sigma}{r_2}} y^{\frac{\sigma}{r_1}}
\end{aligned} \tag{27}
$$

with $\beta_n \geq (\hat{d}_3 + 1)^{1/p_n}$, rendering that

$$
\begin{aligned}
\mathscr{L}V_n \leq &-\sum_{i=1}^{n} \xi_i^4 - g_0(\xi_1^4 + \xi_2^4) \\
&+ \xi_n^{\frac{4\sigma - r_n - \tau}{\sigma}} (u^{p_n} - x_{n+1}^{*p_n}).
\end{aligned} \tag{28}
$$

For system (8), the following homogeneous observer is introduced to estimate the unmeasurable states

$$
\begin{aligned}
&d\eta_2 = -l_1 \hat{x}_2^{p_1} dt,\ \hat{x}_2 = (\eta_2 + l_1 y)^{\frac{r_2}{r_1}}, \\
&d\eta_i = -l_{i-1} \hat{x}_i^{p_{i-1}} dt,\ \hat{x}_i = (\eta_i + l_{i-1}\hat{x}_{i-1})^{\frac{r_i}{r_{i-1}}}, \\
&i = 3, \ldots, n
\end{aligned} \tag{29}
$$

where $l_1, \ldots, l_{n-1}$ are positive gains to be determined later, and OFC satisfies the following form

$$
u^{\frac{\sigma}{r_{n+1}}} = -\sum_{i=2}^{n} \beta_n^{\frac{\sigma}{r_{n+1}}} \cdots \beta_i^{\frac{\sigma}{r_{i+1}}} \hat{x}_i^{\frac{\sigma}{r_i}} - \beta_n^{\frac{\sigma}{r_{n+1}}} \cdots \beta_1^{\frac{\sigma}{r_2}} y^{\frac{\sigma}{r_1}}. \tag{30}
$$

Denote $\lambda_i = \eta_i + l_{i-1}x_{i-1}$ for $i = 3, \ldots, n$, $\lambda_2 = \eta_2 + l_1 y$, and the Lyapunov function U_i is adopted

$$
U_i = \int_{\lambda_i^{(4\sigma - r_{i-1} - \tau)/r_{i-1}}}^{x_i^{(4\sigma - r_{i-1} - \tau)/r_i}} (s^{\frac{r_{i-1}}{4\sigma - r_{i-1} - \tau}} - \lambda_i) ds,\ \ i = 2, \ldots, n. \tag{31}
$$

It can be verified that U_i is positive definite. Define $e_i = (x_i^{p_{i-1}} - \hat{x}_i^{p_{i-1}})^{\frac{\sigma}{r_i p_{i-1}}}$ for $i = 2, \ldots, n$.

For $i = 2$, from (31), one has

$$
U_2 = \frac{4\sigma - r_1 - \tau}{4\sigma - \tau} x_2^{\frac{4\sigma - \tau}{r_2}} - \lambda_2 x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} + \frac{r_1}{4\sigma - \tau} \lambda_2^{\frac{4\sigma - \tau}{r_1}} \tag{32}
$$

then it can be calculated that

$$
\begin{aligned}
\frac{\partial U_2}{\partial x_2} &= \frac{4\sigma - r_1 - \tau}{r_2} x_2^{\frac{4\sigma - r_1 - \tau - r_2}{r_2}} (x_2^{\frac{r_1}{r_2}} - \lambda_2), \\
\frac{\partial U_2}{\partial \eta_2} &= -(x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \lambda_2^{\frac{4\sigma - r_1 - \tau}{r_1}}), \\
\frac{\partial U_2}{\partial y} &= -l_1(x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \lambda_2^{\frac{4\sigma - r_1 - \tau}{r_1}}).
\end{aligned} \tag{33}
$$

Therefore, along (8)-(29), one can obtain that

$$
\begin{aligned}
\mathscr{L}U_2 = &\frac{4\sigma - r_1 - \tau}{r_2} x_2^{\frac{4\sigma - r_1 - \tau - r_2}{r_2}} (x_2^{\frac{r_1}{r_2}} - \lambda_2) x_3^{p_2} \\
&- l_1 \frac{\partial h}{\partial x_1} x_2^{p_1} (x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \lambda_2^{\frac{4\sigma - r_1 - \tau}{r_1}}) \\
&+ l_1 \hat{x}_2^{p_1} (x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \lambda_2^{\frac{4\sigma - r_1 - \tau}{r_1}}) \\
= &\frac{4\sigma - r_1 - \tau}{r_2} x_2^{\frac{4\sigma - r_1 - \tau - r_2}{r_2}} (x_2^{\frac{r_1}{r_2}} - \hat{x}_2^{\frac{r_1}{r_2}}) x_3^{p_2} \\
&- l_1 \frac{\partial h}{\partial x_1} e_2^{\frac{r_2 p_1}{\sigma}} (x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \hat{x}_2^{\frac{4\sigma - r_1 - \tau}{r_2}}) \\
&+ l_1 (1 - \frac{\partial h}{\partial x_1}) \hat{x}_2^{p_1} (x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \hat{x}_2^{\frac{4\sigma - r_1 - \tau}{r_2}}).
\end{aligned} \tag{34}
$$

Due to $\frac{4\sigma - r_{i-1} - \tau}{r_i p_{i-1}} \geq 3$, by Lemma 4 and Assumption 1, one has

$$
\begin{aligned}
&- l_1 \frac{\partial h}{\partial x_1} e_2^{\frac{r_2 p_1}{\sigma}} (x_2^{\frac{4\sigma - r_1 - \tau}{r_2}} - \hat{x}_2^{\frac{4\sigma - r_1 - \tau}{r_2}}) \\
= &- l_1 \frac{\partial h}{\partial x_1} |e_2|^{\frac{r_2 p_1}{\sigma}} \left| (x_2^{p_1})^{\frac{4\sigma - r_1 - \tau}{r_2 p_1}} - (\hat{x}_2^{p_1})^{\frac{4\sigma - r_1 - \tau}{r_2 p_1}} \right| \\
\leq &- l_1 m_2 e_2^4
\end{aligned} \tag{35}
$$

with a constant $m_2 = 2^{1 - \frac{4\sigma - r_1 - \tau}{r_2 p_1}} \underline{o}$.

Besides, for $i = 3, \ldots, n$, from (31), we have

$$
\begin{aligned}
U_i = &\frac{4\sigma - r_{i-1} - \tau}{4\sigma - \tau} x_i^{\frac{4\sigma - \tau}{r_i}} - \lambda_i x_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}} \\
&+ \frac{r_{i-1}}{4\sigma - \tau} \lambda_i^{\frac{4\sigma - \tau}{r_{i-1}}}
\end{aligned} \tag{36}
$$

then one has

$$
\begin{aligned}
\frac{\partial U_i}{\partial x_i} &= \frac{4\sigma - r_{i-1} - \tau}{r_i} x_i^{\frac{4\sigma - r_{i-1} - \tau - r_i}{r_i}} (x_i^{\frac{r_{i-1}}{r_i}} - \lambda_i), \\
\frac{\partial U_i}{\partial \eta_i} &= -(x_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}} - \lambda_i^{\frac{4\sigma - r_{i-1} - \tau}{r_{i-1}}}), \\
\frac{\partial U_i}{\partial x_{i-1}} &= -l_{i-1}(x_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}} - \lambda_i^{\frac{4\sigma - r_{i-1} - \tau}{r_{i-1}}}).
\end{aligned} \tag{37}
$$

Therefore, along (8)-(29), one can obtain that

$$
\begin{aligned}
\mathscr{L}U_i = &\frac{4\sigma - r_{i-1} - \tau}{r_i} x_i^{\frac{4\sigma - r_{i-1} - \tau - r_i}{r_i}} (x_i^{\frac{r_{i-1}}{r_i}} - \lambda_i) x_{i+1}^{p_i} \\
&- l_{i-1} e_i^{\frac{r_i p_{i-1}}{\sigma}} (x_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}} - \hat{x}_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}}) \\
&- l_{i-1} e_i^{\frac{r_i p_{i-1}}{\sigma}} (\hat{x}_i^{\frac{4\sigma - r_{i-1} - \tau}{r_i}} - \lambda_i^{\frac{4\sigma - r_{i-1} - \tau}{r_{i-1}}})
\end{aligned} \tag{38}
$$

where $x_{n+1} = u$.

By Lemma 4, one has

$$
\begin{aligned}
&-l_{i-1}e_i^{\frac{r_ip_{i-1}}{\sigma}}(x_i^{\frac{4\sigma-r_{i-1}-\tau}{r_i}}-\hat{x}_i^{\frac{4\sigma-r_{i-1}-\tau}{r_i}})\\
=&-l_{i-1}(x_i^{p_{i-1}}-\hat{x}_i^{p_{i-1}})\big((x_i^{p_{i-1}})^{\frac{4\sigma-r_{i-1}-\tau}{r_ip_{i-1}}}-(\hat{x}_i^{p_{i-1}})^{\frac{4\sigma-r_{i-1}-\tau}{r_ip_{i-1}}}\big)\\
\leq&-l_{i-1}2^{1-\frac{4\sigma-r_{i-1}-\tau}{r_ip_{i-1}}}(x_i^{p_{i-1}}-\hat{x}_i^{p_{i-1}})^{\frac{4\sigma}{r_ip_{i-1}}}\\
:=&-l_{i-1}m_ie_i^4 \qquad (39)
\end{aligned}
$$

where $m_i=2^{1-(4\sigma-r_{i-1}-\tau)/(r_ip_{i-1})}$.

Meanwhile, we introduce the following propositions to estimate (38), whose proof can be referred to [14] and [22] for details.

Proposition 1 For constants $\alpha_i>0, i=2,\ldots,n-1$, one has

$$
\begin{aligned}
&\frac{4\sigma-r_{i-1}-\tau}{r_i}x_i^{\frac{4\sigma-r_{i-1}-\tau-r_i}{r_i}}(x_i^{\frac{r_{i-1}}{r_i}}-\lambda_i)x_{i+1}^{p_i}\\
\leq&\frac{1}{12}\sum_{j=i-1}^{i+1}\xi_j^4+\alpha_ie_i^4+g_i(l_{i-1})e_{i-1}^4 \qquad (40)
\end{aligned}
$$

with continuous functions $g_i(l_{i-1})$ for $i=2,\ldots,n-1$ and $g_2(l_1)=0$.

Proposition 2 For the controller u, one has

$$
\begin{aligned}
&\frac{4\sigma-r_{n-1}-\tau}{r_n}x_n^{\frac{4\sigma-r_{n-1}-\tau-r_n}{r_n}}(x_n^{\frac{r_{n-1}}{r_n}}-\lambda_n)u^{p_n}\\
\leq&\frac{1}{8}\sum_{i=1}^{n}\xi_i^4+\bar{\alpha}\sum_{i=2}^{n}e_i^4+g_n(l_{n-1})e_{n-1}^4 \qquad (41)
\end{aligned}
$$

with a constant $\bar{\alpha}>0$ and a continuous function $g_n(l_{n-1})$.

Proposition 3 For positive constants $\hat{d}_4=\underline{o}2^{-\frac{4\sigma-r_1-\tau}{r_2p_1}}$ and $\hat{d}_5$, one has

$$
\begin{aligned}
&l_1(1-\frac{\partial h}{\partial x_1})\hat{x}_2^{p_1}(x_2^{\frac{4\sigma-r_1-\tau}{r_2}}-\lambda_2^{\frac{4\sigma-r_1-\tau}{r_1}})\\
\leq&l_1\hat{d}_4e_2^4+l_1\hat{d}_5(\xi_1^4+\xi_2^4). \qquad (42)
\end{aligned}
$$

Proposition 4 For $i=3,\ldots,n$, one has

$$
\begin{aligned}
&-l_{i-1}e_i^{\frac{r_ip_{i-1}}{\sigma}}(\hat{x}_i^{\frac{4\sigma-r_{i-1}-\tau}{r_i}}-\lambda_i^{\frac{4\sigma-r_{i-1}-\tau}{r_{i-1}}})\\
\leq&e_i^4+\frac{1}{16}(\xi_{i-1}^4+\xi_i^4)+h_i(l_{i-1})e_{i-1}^4 \qquad (43)
\end{aligned}
$$

with continuous functions $h_i(l_{i-1})$.

These propositions will lead to the following inequality

$$
\begin{aligned}
\mathscr{L}U\leq&\frac{1}{2}\sum_{i=1}^{n}\xi_i^4-(l_{n-1}m_n-\bar{\alpha}-1)e_n^4-\big(l_1\hat{d}_4-\alpha_2-\bar{\alpha}\\
&-g_3(l_2)-h_3(l_2)\big)e_2^4-\sum_{i=3}^{n-1}\big(l_{i-1}m_i-\alpha_i-\bar{\alpha}-1\\
&-g_{i+1}(l_i)-h_{i+1}(l_i)\big)e_i^4+l_1\hat{d}_5(\xi_1^4+\xi_2^4) \qquad (44)
\end{aligned}
$$

where $U=\sum_{i=2}^{n}U_i$.

Besides, it can be proved by Lemmas 4 and 5 as follows

$$
\begin{aligned}
&\xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}(u^{p_n}-x_{n+1}^{*p_n})\\
\leq&2^{1-\frac{r_{n+1}p_n}{\sigma}}\xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}|u^{\frac{\sigma}{r_{n+1}}}-x_{n+1}^{*\frac{\sigma}{r_{n+1}}}|^{\frac{r_{n+1}p_n}{\sigma}}\\
\leq&\tilde{c}\xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}(\sum_{i=2}^{n}|x_i^{\frac{\sigma}{r_i}}-\hat{x}_i^{\frac{\sigma}{r_i}}|)^{\frac{r_{n+1}p_n}{\sigma}}\\
\leq&\tilde{c}\xi_n^{\frac{4\sigma-r_n-\tau}{\sigma}}\big(\sum_{i=2}^{n}\tilde{c}_i(|e_i|+|\xi_i|+|\xi_{i-1}|)\big)^{\frac{r_{n+1}p_n}{\sigma}}\\
\leq&\frac{1}{3}\sum_{i=1}^{n}\xi_i^4+\tilde{\alpha}\sum_{i=2}^{n}e_i^4 \qquad (45)
\end{aligned}
$$

where $\tilde{\alpha},\tilde{c},\tilde{c}_i$ are positive constants .

Combining (28), (44) with (45) and defining $W=V_n+U$, one obtains that

$$
\begin{aligned}
\mathscr{L}W\leq&-\frac{1}{6}\sum_{i=1}^{n}\xi_i^4-(l_{n-1}m_n-\bar{\alpha}-\tilde{\alpha}-1)e_n^4\\
&-\big(l_1\hat{d}_4-\bar{\alpha}-\alpha_2-\tilde{\alpha}-g_3(l_2)-h_3(l_2)\big)e_2^4\\
&-\sum_{i=3}^{n-1}\big(l_{i-1}m_i-\bar{\alpha}-\alpha_i-\tilde{\alpha}-1-g_{i+1}(l_i)\\
&-h_{i+1}(l_i)\big)e_i^4-(g_0-l_1\hat{d}_5)(\xi_1^4+\xi_2^4). \qquad (46)
\end{aligned}
$$

By selecting

$$
\begin{aligned}
&l_{n-1}=(\frac{7}{6}+\tilde{\alpha}+\bar{\alpha})/m_n\\
&l_{i-1}=\big(\frac{7}{6}+\tilde{\alpha}+\alpha_i+\bar{\alpha}+g_{i+1}(l_i)+h_{i+1}(l_i)\big)/m_i,\\
&i=3,\ldots,n-1\\
&l_1=\big(\frac{1}{6}+\alpha_2+\bar{\alpha}+\tilde{\alpha}+g_3(l_2)+h_3(l_2)\big)/\hat{d}_4\\
&g_0=l_1\hat{d}_5 \qquad (47)
\end{aligned}
$$

(46) becomes

$$
\mathscr{L}W\leq-\frac{1}{6}\sum_{i=1}^{n}\xi_i^4-\frac{1}{6}\sum_{j=2}^{n}e_j^4. \qquad (48)
$$

Evidently (48) reveals that $\mathscr{L}W$ is negative definite. Therefore, ξ_i and e_j converge to zeros. From the definitions of ξ_i and e_j, $(y,x_2,\ldots,x_n,\eta_2,\ldots,\eta_n)^T$ converges to zero. According to Remark 1, x_1 also converges to zero, then it can be deduced that the closed-loop system (8)-(29)-(30) is GAS in probability. Besides, (8)-(29)-(30) is rewritten to be a compact form as

$$
\begin{aligned}
dX=&F(X)dt\\
=&(\frac{\partial h}{\partial x_1}x_2^{p_1},\ldots,x_n^{p_{n-1}},u^{p_n},\dot{\eta}_2,\ldots,\dot{\eta}_n)^Tdt \qquad (49)
\end{aligned}
$$

where $X:=[y,x_2,\ldots,x_n,\eta_2,\ldots,\eta_n]^T$.

Meanwhile, the dilation weight is chosen as

$$\Delta = (\underbrace{r_1, r_2, \dots, r_n}_{for\ y, x_2, \dots, x_n}, \underbrace{r_1, \dots, r_{n-1}}_{for\ \eta_2, \dots, \eta_n}). \quad (50)$$

From (27) and Definition 2, it can be certified that $x_i^*(r_1 y, r_2 x_2, \dots, r_{i-1} x_{i-1}) = \varepsilon^{r_i} x_i^*(y, x_2, \dots, x_{i-1})$, then one has

$$\begin{aligned} &V_n(r_1 y, r_2 x_2, \dots, r_n x_n) \\ =&\frac{1}{4\sigma-\tau}(\varepsilon^{r_1} y)^{4\sigma-\tau} + \sum_{i=2}^{n} \int_{\varepsilon^{r_i} x_i^*}^{\varepsilon^{r_i} x_i} (s^{\sigma/r_i} - \varepsilon^{\sigma} x_i^{*\sigma/r_i})^{\frac{4\sigma-r_i-\tau}{\sigma}} ds \\ =&\varepsilon^{4\sigma-\tau} V_n(y, x_2, \dots, x_n). \end{aligned} \quad (51)$$

Similarly from (32) and (36), one can prove that $U(\Delta_\varepsilon(X)) = \varepsilon^{4\sigma-\tau} U(X)$, under which $W = V_n + U$ is homogeneous of degree $4\sigma - \tau$ with regard to Δ. Then one can deduce from Definition 2 and Lemma 3 that the following inequalities are established

$$W \le \bar{d} \|X\|_\Delta^{4\sigma-\tau}, \ \ \mathscr{L}W \le -d_1 \|X\|_\Delta^{4\sigma} \quad (52)$$

with positive constants $\bar{d}$ and d_1.

3.2 Output feedback controller for SNSs (1)

Combining with the previous controller established for the nominal system (8), a dynamic gain is employed to stabilize SNSs (1). Firstly, introduce the following coordinates change

$$x_1 = \frac{z_1}{L^{q_1}}, \dots, x_n = \frac{z_n}{L^{q_n}} \text{ and } u = \frac{v}{L^{q_{n+1}}} \quad (53)$$

where $q_1 = 0, \ q_{i+1} p_i = q_i + 1, \ i = 1, \dots, n$ and the dynamic gain L is designed as

$$dL = L \min\{y^{4\sigma}, y^{\tau}\} dt, \ L(0) = 1. \quad (54)$$

Together with (53), system (1) turns into

$$\begin{aligned} dx_i &= \big(L x_{i+1}^{p_i} - q_i \frac{\dot{L}}{L} x_i + \frac{f_i(\cdot)}{L^{q_i}}\big) dt + \frac{g_i^T}{L^{q_i}} dw, \\ &\quad i = 1, \dots, n-1, \\ dx_n &= \big(L u^{p_n} - q_n \frac{\dot{L}}{L} x_n + \frac{f_n(\cdot)}{L^{q_n}}\big) dt + \frac{g_n^T}{L^{q_n}} dw, \\ &\quad y = h(x_1). \end{aligned} \quad (55)$$

Besides, one can calculate that

$$dy = (L \frac{\partial h}{\partial x_1} x_2^{p_1} + \frac{\partial h}{\partial x_1} f_1) dt + \frac{\partial h}{\partial x_1} g_1^T dw. \quad (56)$$

Similar to the form of (29), we construct the homogeneous observer with L as follows

$$\begin{aligned} &d\eta_2 = -L l_1 \hat{x}_2^{p_1} dt, \ \hat{x}_2 = (\eta_2 + l_1 y)^{\frac{r_2}{r_1}}, \\ &d\eta_i = -L l_{i-1} \hat{x}_i^{p_{i-1}} dt, \ \hat{x}_i = (\eta_i + l_{i-1} \hat{x}_{i-1})^{\frac{r_i}{r_{i-1}}}, \\ &\quad i = 3, \dots, n \end{aligned} \quad (57)$$

and the new controller v is designed similarly to (30)

$$\begin{aligned} v = &- L^{q_{n+1}} \beta_n \big(\hat{x}_n^{\frac{\sigma}{r_n}} + \beta_{n-1}^{\frac{\sigma}{r_n}} (\hat{x}_{n-1}^{\frac{\sigma}{r_{n-1}}} + \cdots \\ &+ \beta_2^{\frac{\sigma}{r_3}} (\hat{x}_2^{\frac{\sigma}{r_2}} + \beta_1^{\frac{\sigma}{r_2}} y^{\frac{\sigma}{r_1}}))\big)^{\frac{r_{n+1}}{\sigma}}. \end{aligned} \quad (58)$$

Define $\phi_i(\cdot) = \frac{f_i(\cdot)}{L^{q_i}}, \varphi_i(\cdot) = \frac{g_i(\cdot)}{L^{q_i}}$. Similarly, rewrite (55)-(56)-(57)-(58) as

$$\begin{aligned} dX =& \big(LF(X) - \frac{\dot{L}}{L}(0, q_2 x_2, \dots, q_n x_n, 0, \dots, 0)^T \\ &+ \big(\frac{\partial h}{\partial x_1} f_1(\cdot), \frac{f_2(\cdot)}{L^{q_2}}, \dots, \frac{f_n(\cdot)}{L^{q_n}}, 0, \dots, 0)^T\big) dt \\ &+ \big(\frac{\partial h}{\partial x_1} g_1(\cdot), \frac{g_2(\cdot)}{L^{q_2}}, \dots, \frac{g_n(\cdot)}{L^{q_n}}, 0, \dots, 0\big)^T dw \\ =& \big(LF(X) - \frac{\dot{L}}{L}(q_1 y, q_2 x_2, \dots, q_n x_n, 0, \dots, 0)^T \\ &+ \phi(X)\big) dt + \varphi^T(X) dw \end{aligned} \quad (59)$$

where $\phi(X) = \big(\frac{\partial h}{\partial x_1}\phi_1(\cdot), \phi_2(\cdot), \dots, \phi_n(\cdot), 0, \dots, 0\big)^T$, $\varphi(X) = \big(\frac{\partial h}{\partial x_1}\varphi_1(\cdot), \varphi_2(\cdot), \dots, \varphi_n(\cdot), 0, \dots, 0\big)$.

With the same Lyapunov function W adopted, one has

$$\begin{aligned} \mathscr{L}W(X) =& L \frac{\partial W}{\partial X} F(X) - \frac{\dot{L}}{L} \frac{\partial W}{\partial X} (q_1 y, q_2 x_2, \dots, q_n x_n, 0, \dots, 0)^T \\ &+ \frac{\partial W}{\partial X} \phi(X) + \frac{1}{2} \mathrm{Tr}\{\varphi(X) \frac{\partial^2 W}{\partial X^2} \varphi^T(X)\} \\ \le& - d_1 L \|X\|_\Delta^{4\sigma} + \frac{\dot{L}}{L} \sum_{i=1}^{n} \big| \frac{\partial W}{\partial X_i} q_i X_i \big| \\ &+ \sum_{i=2}^{n} \big| \frac{\partial W}{\partial X_i} \phi_i(\cdot) \big| + \big| \frac{\partial W}{\partial y} \frac{\partial h}{\partial x_1} \phi_1(\cdot) \big| \\ &+ \frac{1}{2} \mathrm{Tr}\{\varphi(X) \frac{\partial^2 W}{\partial X^2} \varphi^T(X)\}. \end{aligned} \quad (60)$$

Noting the definitions of r_i and q_i, one has

$$\begin{aligned} q_{i+1} &= \frac{1}{p_i} + \frac{1}{p_{i-1} p_i} + \cdots + \frac{1}{p_1 \cdots p_i}, \\ r_{i+1} &= \frac{\tau}{p_i} + \frac{\tau}{p_{i-1} p_i} + \cdots + \frac{1+\tau}{p_1 \cdots p_i}. \end{aligned} \quad (61)$$

As a special case mentioned in [12], one gets

$$\begin{aligned}&\frac{q_j(r_i+\tau)}{r_j}-q_i\\=&1-\frac{p_1\dots p_{j-1}+\dots+p_1\dots p_{i-1}}{\big(\tau(1+p_1+\dots+p_1\dots p_{j-2})+1\big)p_1\dots p_{i-1}},\\&\frac{q_j(2r_i+\tau)}{2r_j}-q_i\\=&\frac{1}{2}-\frac{p_1\dots p_{j-1}+\dots+p_1\dots p_{i-2}+\frac{p_1\dots p_{i-1}}{2}}{\big(\tau(1+p_1+\dots+p_1\dots p_{j-2})+1\big)p_1\dots p_{i-1}}.\end{aligned} \tag{62}$$

Letting $\iota=\min_{2\le j\le i}\{\frac{p_1\cdots p_{j-1}+\cdots+p_1\cdots p_{i-1}}{\big(\tau(1+p_1+\cdots+p_1\dots p_{j-2})+1\big)p_1\dots p_{i-1}},1\}$, it is easy to find that $\frac{q_j(2r_i+\tau)}{2r_j}-q_i\le\frac{1-\iota}{2}$, then

$$\begin{aligned}|\phi_i|=&\frac{\theta_1}{L^{q_i}}\sum_{j=1}^{i}|L^{q_j}x_j|^{\frac{r_i+\tau}{r_j}}\le\theta_1L^{1-\iota}\sum_{j=1}^{i}|x_j|^{\frac{r_i+\tau}{r_j}}\\&\le\bar\theta_1L^{1-\iota}(|y|^{\frac{r_i+\tau}{r_1}}+\sum_{j=2}^{i}|x_j|^{\frac{r_i+\tau}{r_j}}),\\|\varphi_i|=&\frac{\theta_2}{L^{q_i}}\sum_{j=1}^{i}|L^{q_j}x_j|^{\frac{2r_i+\tau}{2r_j}}\le\theta_2L^{\frac{1-\iota}{2}}\sum_{j=1}^{i}|x_j|^{\frac{2r_i+\tau}{2r_j}}\\&\le\bar\theta_2L^{\frac{1-\iota}{2}}(|y|^{\frac{2r_i+\tau}{2r_1}}+\sum_{j=2}^{i}|x_j|^{\frac{2r_i+\tau}{2r_j}}).\end{aligned} \tag{63}$$

where $\bar\theta_1=\theta_1\max\{|\frac{1}{\underline{o}}|^{\frac{r_i+\tau}{r_1}},1\},\bar\theta_2=\theta_2\max\{|\frac{1}{\underline{o}}|^{\frac{2r_i+\tau}{r_1}},1\}$.

Inspired by Lemma 3, it is reasonable to obtain from the homogeneity theory that

$$\begin{aligned}&\sum_{i=2}^{n}\big|\frac{\partial W}{\partial X_i}\phi_i(\cdot)\big|+\big|\frac{\partial W}{\partial y}\frac{\partial h}{\partial x_1}\phi_1(\cdot)\big|\le\vartheta_1L^{1-\iota}\|X\|_\Delta^{4\sigma},\\&\frac{1}{2}\mathrm{Tr}\{\varphi(X)\frac{\partial^2W}{\partial X^2}\varphi^T(X)\}\\=&\frac{1}{2}\mathrm{Tr}\{\frac{\partial h}{\partial x_1}\varphi_1\frac{\partial^2W}{\partial X_1^2}\frac{\partial h}{\partial x_1}\varphi_1^T+\sum_{i=2}^{n}\varphi_i\frac{\partial^2W}{\partial X_i\partial X_1}\frac{\partial h}{\partial x_1}\varphi_1^T\\&+\sum_{j=2}^{n}\frac{\partial h}{\partial x_1}\varphi_1\frac{\partial^2W}{\partial X_1\partial X_j}\varphi_j^T+\sum_{i=2}^{n}\sum_{j=2}^{n}\varphi_i\frac{\partial^2W}{\partial X_i\partial X_j}\varphi_j^T\}\\=&\frac{1}{2}((\frac{\partial h}{\partial x_1})^2\frac{\partial^2W}{\partial X_1^2}\varphi_1^T\varphi_1+\sum_{i=2}^{n}\frac{\partial h}{\partial x_1}\frac{\partial^2W}{\partial X_i\partial X_1}\varphi_1^T\varphi_i\\&+\sum_{j=2}^{n}\frac{\partial h}{\partial x_1}\frac{\partial^2W}{\partial X_1\partial X_j}\varphi_j^T\varphi_1+\sum_{i=2}^{n}\sum_{j=2}^{n}\frac{\partial^2W}{\partial X_i\partial X_j}\varphi_j^T\varphi_i)\\\le&\vartheta_2L^{1-\iota}\|X\|_\Delta^{4\sigma}\end{aligned} \tag{64}$$

where ϑ_1,ϑ_2 are unknown positive constants.

From Lemma 3 and (54), one can deduce that

$$\frac{\dot L}{L}\sum_{i=1}^{n}\big|\frac{\partial W}{\partial X_i}q_iX_i\big|\le d_2\|X\|_\Delta^{4\sigma}\le d_2L^{1-\iota}\|X\|_\Delta^{4\sigma} \tag{65}$$

where d_2 is a positive constant.

Substituting (64) and (65) into (60), one has

$$\begin{aligned}\mathscr{L}W(X)\le&-L(d_1-d_2L^{-\iota}-(\vartheta_1+\vartheta_2)L^{-\iota})\|X\|_\Delta^{4\sigma}\\=&-L(d_1-\vartheta L^{-\iota})\|X\|_\Delta^{4\sigma}\end{aligned} \tag{66}$$

where $\vartheta:=d_2+\vartheta_1+\vartheta_2$ is an unknown positive constant.

3.3 Stability analysis

Define $\sigma_k=\inf\{t\ge0\big|\|\gamma(t)\|\geqslant k,\forall k>0\}$ and $\gamma(t)=(L(t),X^T(t))^T$. Since (54) and (59) are locally Lipschitz, it can be inferred that the closed-loop system has a unique solution defined on $[0,\sigma_\infty)$, where $\sigma_\infty:=\lim_{k\to\infty}\sigma_k$.

Letting $t_0=\inf\{t\in[0,\sigma_\infty)\big|L(t)\ge\max\{1,(\frac{2\vartheta}{d_1})^{\frac{1}{\iota}}\}$, under which (66) becomes

$$\mathscr{L}W\le-\frac{d_1}{2}L\|X\|_\triangle^{4\sigma},\ \forall t\in[t_0,\sigma_\infty). \tag{67}$$

Then for any $(t\wedge\sigma_k)\in[t_0,\sigma_\infty)$, one has

$$\begin{aligned}E(L(t\wedge\sigma_k))-E(L(t_0))=&E(\int_{t_0}^{t\wedge\sigma_k}\dot L(s)ds)\\\le&E(\int_{t_0}^{t\wedge\sigma_k}Ly^{4\sigma}ds)\\\le&E(\int_{t_0}^{t\wedge\sigma_k}L\|X(s)\|_\triangle^{4\sigma}ds)\\\le&E(\int_{t_0}^{t\wedge\sigma_k}-\frac{2}{d_1}\mathscr{L}W(s)ds)\\\le&\frac{2}{d_1}E(W(t_0))<\infty.\end{aligned} \tag{68}$$

From (54), distinctly one deduces that L is monotonically non-decreasing, thus it is obvious that $E(L(t\wedge\sigma_k))$ is bounded in probability for any $t\in[0,\sigma_\infty)$.

From (52), $\|X(s)\|_\triangle^{4\sigma}\le(\frac{W}{\underline{d}})^{\frac{4\sigma}{4\sigma-\tau}}$, so $E(\int_0^{t_0}L\|X(s)\|_\triangle^{4\sigma}ds)\le E(\int_0^{t_0}L(\frac{W}{\underline{d}})^{\frac{4\sigma}{4\sigma-\tau}}ds)$, which means $E(\int_0^{t_0}\|X(s)\|_\triangle^{4\sigma}ds)$ is bounded a.s. It can be concluded that $E(\int_{t_0}^{t\wedge\sigma_k}\|X(s)\|_\triangle^{4\sigma}ds)$ is bounded in probability from (68), so one has proved the boundedness of $E(\int_0^{t\wedge\sigma_k}\|X(s)\|_\triangle^{4\sigma}ds)$ for any $t\in[0,\sigma_\infty)$ a.s.

Afterwards, the boundedness of W will be verified as following

$$\begin{aligned}&E(W(t\wedge\sigma_k))-E(W(0))\\=&E(\int_0^{t\wedge\sigma_k}\mathscr{L}W(s)ds)\\\le&E(\int_0^{t\wedge\sigma_k}L^{1-\iota}\vartheta||X(s)||_\triangle^{4\sigma}ds)\end{aligned} \tag{69}$$

which implies that $E(W(t\wedge\sigma_k))$ is bounded a.s.

According to the previous proof process, there exists a constant D making $EW(\gamma(t\wedge\sigma_k))\le D$ for any $t\ge0$ and $k>0$ and $\inf_{||\gamma||\ge k}V(\gamma)=\infty,\ k\to\infty$. From Lemma 1, the closed-loop system has a unique solution $\gamma(t)$ in $[0,\infty)$ and $\gamma(t)$ is strongly bounded in probability.

Suppose $t\to\infty$ and $k\to\infty$, then $E(\int_0^\infty \|X(s)\|_\Delta^{4\sigma}ds)<\infty$, which means $\|X(s)\|\in\Xi(\mathbb{R}_+\times\Omega)$, from Lemma 2 one knows that

$$\lim_{t\to\infty}(y,x_2,\ldots,x_n,\eta_2,\ldots,\eta_n)=0,\ a.s. \tag{70}$$

Since the transformation is reversible, it immediately follows that $z(t),\eta(t)$ converge to zeros when $t\to\infty$ a.s.

4 Extension

In this section, the proposed scheme is extended to deal with the upper-triangular systems, whose drift and diffusion terms satisfy the following assumption.

Assumption 3 There exist two unknown positive constants θ_1 and θ_2, such that

$$\begin{aligned}|f_i(\cdot)|&\le\theta_1(\sum_{j=i+2}^n|z_j|^{\frac{r_i+\tau}{r_j}}+|\nu|^{\frac{r_i+\tau}{r_{n+1}}}),\\ \|g_i(\cdot)\|&\le\theta_2(\sum_{j=i+2}^n|z_j|^{\frac{2r_i+\tau}{2r_j}}+|\nu|^{\frac{2r_i+\tau}{2r_{n+1}}})\end{aligned} \tag{71}$$

where r_i and τ are defined as same as in Assumption 2.

With the help of coordinates change

$$x_1=L^{q_1}z_1,\ldots,x_n=L^{q_n}z_n \text{ and } u=L^{q_{n+1}}\nu \tag{72}$$

where $q_1=0,\ q_{i+1}p_i=q_i+1,\ i=1,\ldots,n$ and L is the dynamic gain designed as

$$dL=\frac{1}{L}\min\{y^{4\sigma},y^\tau\}dt,\ L(0)=1. \tag{73}$$

Together with (72), system (1) turns into

$$\begin{aligned}dx_i&=\big(\frac{1}{L}x_{i+1}^{p_i}-q_i\frac{\dot L}{L}x_i+L^{q_i}f_i(\cdot)\big)dt+L^{q_i}g_i^Tdw,\\ &\quad i=1,\ldots,n-1,\\ dx_n&=(\frac{1}{L}u^{p_n}-q_n\frac{\dot L}{L}x_n)dt,\\ &\quad y=h(x_1)\end{aligned} \tag{74}$$

and one has

$$dy=(\frac{1}{L}\frac{\partial h}{\partial x_1}x_2^{p_1}+\frac{\partial h}{\partial x_1}f_1)dt+\frac{\partial h}{\partial x_1}g_1^Tdw. \tag{75}$$

Similarly, the new observer and controller are designed as follows

$$\begin{aligned}d\eta_2=&-\frac{1}{L}l_1\hat x_2^{p_1}dt,\ \hat x_2=(\eta_2+l_1y)^{\frac{r_2}{r_1}},\\ d\eta_i=&-\frac{1}{L}l_{i-1}\hat x_i^{p_{i-1}}dt,\ \hat x_i=(\eta_i+l_{i-1}\hat x_{i-1})^{\frac{r_i}{r_{i-1}}},\\ i=&3,\ldots,n,\\ \nu=&-\frac{\beta_n}{L^{q_{n+1}}}\big(\hat x_n^{\frac{\sigma}{r_n}}+\beta_{n-1}^{\frac{\sigma}{r_n}}(\hat x_{n-1}^{\frac{\sigma}{r_{n-1}}}+\cdots\\ &+\beta_2^{\frac{\sigma}{r_3}}(\hat x_2^{\frac{\sigma}{r_2}}+\beta_1^{\frac{\sigma}{r_2}}y^{\frac{\sigma}{r_1}}))\big)^{\frac{r_{n+1}}{\sigma}}.\end{aligned} \tag{76}$$

Following the same line as (59), one has

$$\begin{aligned}dX=&\big(\frac{1}{L}F(X)-\frac{\dot L}{L}(q_1y,q_2x_2,\ldots,q_nx_n,0,\ldots,0)^T\\ &+\bar\phi(X)\big)dt+\bar\varphi^T(X)dw\end{aligned} \tag{77}$$

where $\bar\phi(X)=\big(\frac{\partial h}{\partial x_1}\bar\phi_1(\cdot),\bar\phi_2(\cdot),\ldots,\bar\phi_{n-1}(\cdot),0,\ldots,0\big)^T$, $\bar\varphi(X)=\big(\frac{\partial h}{\partial x_1}\bar\varphi_1(\cdot),\bar\varphi_2(\cdot),\ldots,\bar\varphi_{n-1}(\cdot),0,\ldots,0\big)$, $\bar\phi_i(\cdot)=L^{q_i}f_i(\cdot)$, $\bar\varphi_i(\cdot)=L^{q_i}g_i(\cdot)$. Employing the same Lyapunov function W, from Lemma 3 one obtains that

$$\begin{aligned}\mathscr{L}W(X)\le&-d_1\frac{1}{L}\|X\|_\Delta^{4\sigma}+\frac{\dot L}{L}\sum_{i=1}^n\big|\frac{\partial W}{\partial X_i}q_iX_i\big|\\ &+\big|\frac{\partial W}{\partial y}\frac{\partial h}{\partial x_1}\phi_1(\cdot)\big|+\sum_{i=2}^n\big|\frac{\partial W}{\partial X_i}\phi_i(\cdot)\big|\\ &+\frac{1}{2}\mathrm{Tr}\{\varphi(X)\frac{\partial^2W}{\partial X^2}\varphi^T(X)\}.\end{aligned} \tag{78}$$

Recalling the derivation process of (62), similarly one has

$$\begin{aligned}&q_i-\frac{q_j(r_i+\tau)}{r_j}\\ =&-1-\frac{p_1\ldots p_i+\cdots+p_1\ldots p_{j-2}}{(\tau(1+p_1+\cdots+p_1\ldots p_{j-2})+1)p_1\ldots p_{i-1}},\\ &q_i-\frac{q_j(2r_i+\tau)}{2r_j}\\ =&-\frac{1}{2}-\frac{\frac{p_1\ldots p_{i-1}}{2}+p_1\ldots p_i+\cdots+p_1\ldots p_{j-2}}{(\tau(1+p_1+\cdots+p_1\ldots p_{j-2})+1)p_1\ldots p_{i-1}}\\ \le&-\frac{1}{2}(1+\frac{p_1\ldots p_i+\cdots+p_1\ldots p_{j-2}}{(\tau(1+\cdots+p_1\ldots p_{j-2})+1)p_1\ldots p_{i-1}}).\end{aligned} \tag{79}$$

Letting $\delta = \min_{i+2\le j\le n+1}\{\frac{p_1\cdots p_i+\cdots+p_1\cdots p_{j-2}}{(\tau(1+p_1+\cdots+p_1\cdots p_{j-2})+1)p_1\cdots p_{i-1}}, 1\}$, it can be obtained that

$$\begin{aligned}
|\phi_i| &\le \theta_1 L^{q_i}\Big(\sum_{j=i+2}^{n}\Big|\frac{x_j}{L^{q_j}}\Big|^{\frac{r_i+\tau}{r_j}}+\Big|\frac{u}{L^{q_{n+1}}}\Big|^{\frac{r_i+\tau}{r_{n+1}}}\Big)\\
&\le \theta_1 L^{-1-\delta}\Big(\sum_{j=i+2}^{n}|x_j|^{\frac{r_i+\tau}{r_j}}+|u|^{\frac{r_i+\tau}{r_{n+1}}}\Big),\\
|\varphi_i| &\le \theta_2 L^{q_i}\Big(\sum_{j=i+2}^{n}\Big|\frac{x_j}{L^{q_j}}\Big|^{\frac{2r_i+\tau}{2r_j}}+\Big|\frac{u}{L^{q_{n+1}}}\Big|^{\frac{2r_i+\tau}{2r_{n+1}}}\Big)\\
&\le \theta_2 L^{\frac{-1-\delta}{2}}\Big(\sum_{j=i+2}^{n}|x_j|^{\frac{2r_i+\tau}{2r_j}}+|u|^{\frac{2r_i+\tau}{2r_{n+1}}}\Big).
\end{aligned} \tag{80}$$

It is similarly proved as in (64) and (65) that

$$\begin{aligned}
&\frac{\dot L}{L}\sum_{i=1}^{n}\Big|\frac{\partial W}{\partial X_i}q_iX_i\Big| \le d_2L^{-2}\|X\|_\Delta^{4\sigma} \le d_2L^{-1-\delta}\|X\|_\Delta^{4\sigma},\\
&\Big|\frac{\partial W}{\partial y}\frac{\partial h}{\partial x_1}\phi_1(\cdot)\Big| + \sum_{i=2}^{n}\Big|\frac{\partial W}{\partial X_i}\phi_i(\cdot)\Big| \le \vartheta_3 L^{-1-\delta}\|X\|_\Delta^{4\sigma},\\
&\frac{1}{2}\mathrm{Tr}\{\varphi(X)\frac{\partial^2 W}{\partial X^2}\varphi^T(X)\} \le \vartheta_4 L^{-1-\delta}\|X\|_\Delta^{4\sigma}
\end{aligned} \tag{81}$$

where ϑ_3, ϑ_4 are unknown positive constants.

Substituting (81) into (78), one has

$$\mathscr{L}W(X) \le -\frac{1}{L}\big(d_1-(d_2+\vartheta_3+\vartheta_4)L^{-\delta}\big)\|X\|_\Delta^{4\sigma} \tag{82}$$

Similar to the stability analysis process in subsection 3.3, it can be proved that $\lim_{t\to\infty}\big(z(t),\eta(t),L(t)\big)=(0,0,\bar L)$ a.s.

5 Simulation

Example 1. Consider the lower-triangular SNSs as

$$\begin{aligned}
dz_1 &= \big(z_2^{\frac{7}{5}}+\frac{1}{2}z_1\sin(z_1)\big)dt+\frac{1}{2}z_1^{\frac{6}{5}}dw,\\
dz_2 &= \big(v^{\frac{5}{3}}+z_1\sin^2(z_2)\big)dt+\frac{1}{2}z_1\sin(z_2)dw,\\
y &= 5z_1+\sin(z_1).
\end{aligned} \tag{83}$$

Choosing $\tau=\frac{2}{5}$, then one has $r_2=1, r_3=\frac{21}{25}, q_3=\frac{27}{35}, \sigma=\frac{63}{25}$, and Assumptions 1 and 2 hold. The adaptive output controller and observer are designed as

$$\begin{aligned}
&d\eta_2=-Ll_1\hat x_2^{\frac{5}{3}}dt,\ \ \hat x_2=(\eta_2+l_1y)^{\frac{r_2}{r_1}},\\
&v=-\beta_2L^{q_3}(\hat x_2^{\frac{\sigma}{r_2}}+\beta_1^{\frac{\sigma}{r_2}}y^{\frac{\sigma}{r_1}})^{\frac{r_3}{\sigma}}.
\end{aligned} \tag{84}$$

The observer and controller parameters are chosen as $l_1=6,\ \beta_1=4, \beta_2=3$, and the initial value of system is chosen as $\big(z_1(0),z_2(0),\eta_2(0)\big)^T=(-0.9,1,0)^T$. The results are shown in Figures 1, 2, 3, 4.

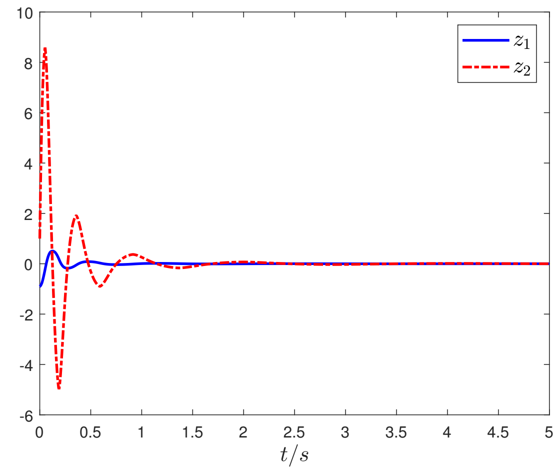

Fig. 1 The trajectories of z in lower-triangular SNSs.

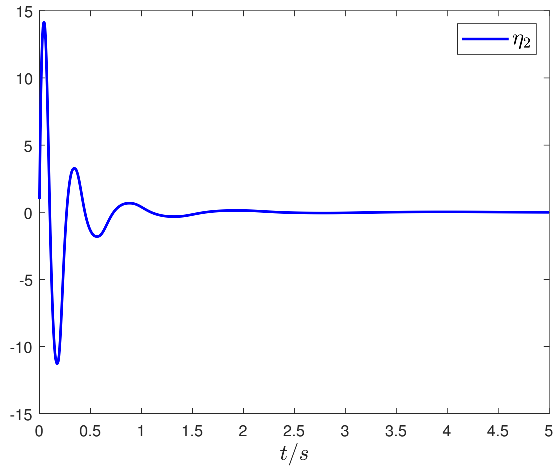

Fig. 2 The trajectory of η in lower-triangular SNSs.

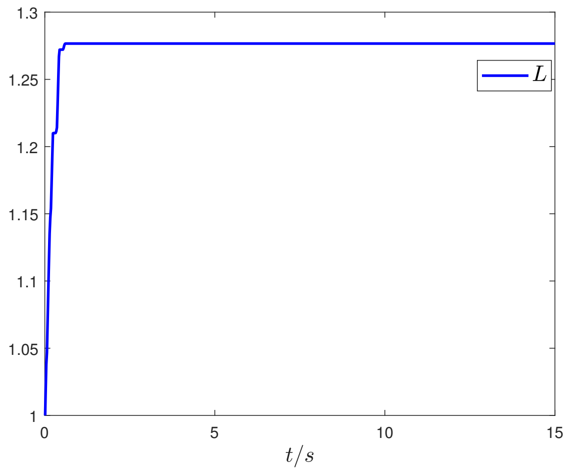

Fig. 3 The trajectory of L in lower-triangular SNSs.

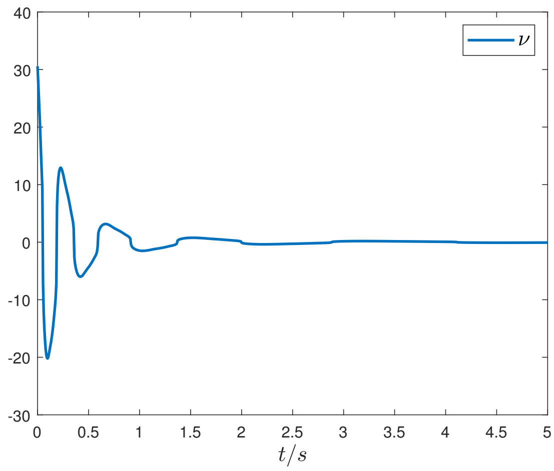

Fig. 4 The trajectory of v in lower-triangular SNSs.

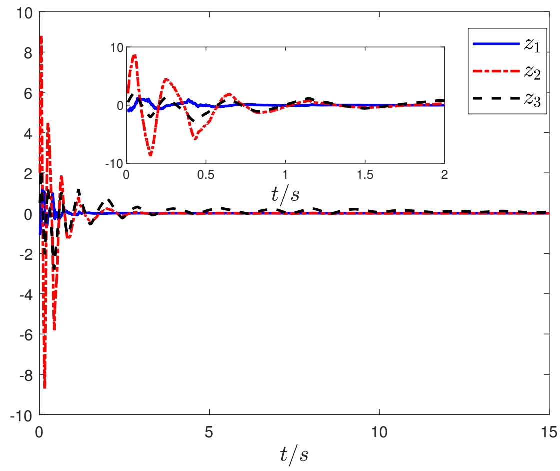

Fig. 5 The trajectories of z in upper-triangular SNSs.

Example 2. Consider the upper-triangular SNSs as

$$
\begin{aligned}
dz_1 &= \left(z_2^{\frac{11}{7}} + 0.1 z_3^2 v\right)dt + 0.1 z_3 v^{\frac{7}{5}} dw, \\
dz_2 &= \left(z_3^3 + 0.2 v^{\frac{9}{5}}\right)dt + 0.1 v^{\frac{8}{5}} dw, \\
dz_3 &= v dt, \\
y &= 1.4 z_1 + \sin(z_1).
\end{aligned} \tag{85}
$$

Choosing $\tau = \frac{2}{7}$, then one has $r_2 = \frac{7}{9}, r_3 = \frac{1}{3}, r_4 = \frac{5}{9}, q_4 = \frac{17}{11}$ and $\sigma = 3$, and Assumptions 1 and 3 hold. The adaptive output controller and observer are designed as following:

$$
\begin{aligned}
d\eta_2 &= -\frac{1}{L} l_1 \hat{x}_2^{\frac{11}{7}} dt, \ \ \hat{x}_2 = (\eta_2 + l_1 y)^{\frac{r_2}{r_1}}, \\
d\eta_3 &= -\frac{1}{L} l_2 \hat{x}_3^3 dt, \ \ \hat{x}_3 = (\eta_3 + l_2 \hat{x}_2)^{\frac{r_3}{r_2}}, \\
v &= -\frac{\beta_3}{L^{q_4}} \left(\hat{x}_3^{\frac{\sigma}{r_3}} + \beta_2^{\frac{\sigma}{r_3}} (\hat{x}_2^{\frac{\sigma}{r_2}} + \beta_1^{\frac{\sigma}{r_2}} y^{\frac{\sigma}{r_1}})\right)^{\frac{r_4}{\sigma}}.
\end{aligned} \tag{86}
$$

Choose the observer and controller parameters as $l_1 = 8, l_2 = 6, \beta_1 = 3.5, \beta_2 = 2.6, \beta_3 = 2$. The initial value of system is set as $\left(z_1(0), z_2(0), z_3(0), \eta_2(0), \eta_3(0)\right)^T = (-1, 2, 0.5, 0, 0)^T$. From Figs 5, 6, 7, 8, the efficiency of the controller is illustrated.

6 Conclusion

In this paper, a global adaptive control scheme is proposed for the stabilization of a family of high-order SNSs with uncertain output function and unknown homogeneous growth rates. With the help of the proposed controller, we ensure the convergence of system states and the boundedness of dynamic gain in probability. Future work will be extended to time-delay SNSs and study the design of controller.

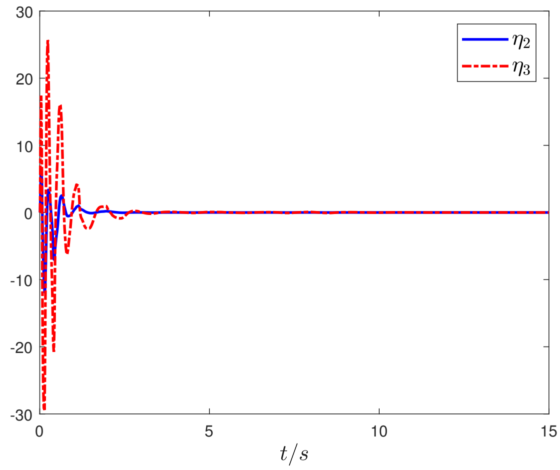

Fig. 6 The trajectories of η in upper-triangular SNSs.

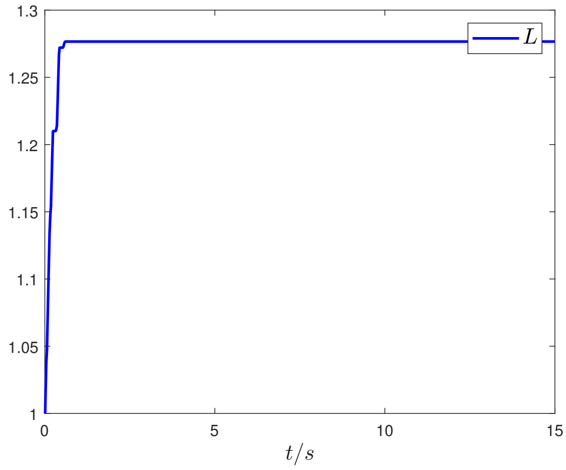

Fig. 7 The trajectory of L in upper-triangular SNSs.

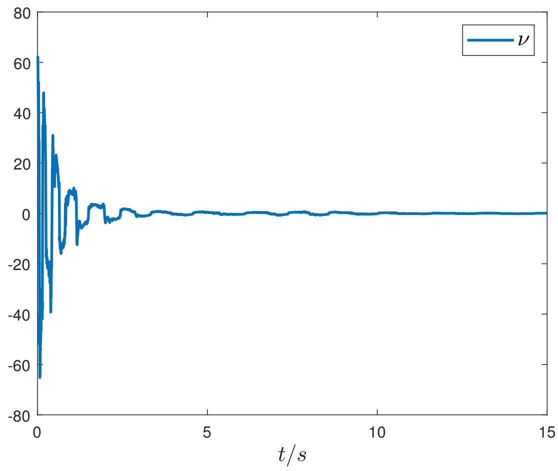

Fig. 8 The trajectory of ν in upper-triangular SNSs.

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Data availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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