# Measurement Operators in Conjugate Transformation Structure - Conjugate Hierarchy of Multiple Levels on Logic Constructions, Pairs of 0-1 Feature Vectors and Hamiltonian Dynamics

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**Research Article**

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**Posted Date:** September 16th, 2020

**DOI:** https://doi.org/10.21203/rs.3.rs-76545/v1

# Measurement Operators in Conjugate Transformation Structure

## - Conjugate Hierarchy of Multiple Levels on Logic Constructions, Pairs of 0-1 Feature Vectors and Hamiltonian Dynamics

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**Abstract** Hamiltonian dynamics play a key role in the foundation of modern physics and mathematics with wider applications in multiple advanced sciences and technologies.
This paper proposes a conjugate transformation structure and its measurement operators on a hierarchy of multiple levels to support intermediate transforming structures from pairs of logic states as micro-ensembles to feature vector transformations as global measurements.
Using logic equations and pairs of partitions on phase spaces, conjugate 0-1 vectors provide hypercomplex number systems. Multiple operators can be created and linked with Hamiltonian operators.
The main constructions of conjugate transformation structures are described and complex conjugate operators are discussed under a pair of symmetric and anti-symmetric parameters with $O(2^{2n} \times 2^{N}), 1 \leq n \leq 2^{m}$ structural complexity.
Using new operators, the Yang-Mills equations are briefly described as an example.
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## 1 Introduction

Quantum mechanics is one of the most important theories established in the 20 century, however, after nearly 100 developments, there is no a consistent logic framework to support it from micro-structures to global dynamics still associated with

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Funding Supports from the Key Project of Quantum Communication Technology of Yunnan (2018ZI002)

multiple paradoxes [7] (Schrodinger cat, quantum entanglement, EPR, Bell inequality, etc.) in a series of mysteries [8](Double slit, ghost image,quantum interference, etc.).

On modern theoretical frameworks, there are many existing works from discrete logic frameworks to continuous constructions such as complex analysis/complex operators [1, 2], multiple complex function theory [3], probability theory/measurement theory [4, 5], multiple statistical analysis [6], classical dynamics [9, 10], Hamiltonian dynamics [11, 12], Hilbert space/group theory/statistical mechanics/quantum mechanics [13], classical vector logic [15], quantum logic [14], information theory [18], cellular automata and complex theory [40, 42], conjugate transformation [48], variant construction [49].

In consequent sections, essential structures are briefly constructed & described: classical logic, complex number system, Hamiltonian dynamics, quantum logic, quantum information, cellular automaton interpretation of quantum mechanics, conjugate transformation and proposed structures, etc.

## *1.1 Classical Logic System and Complex Number Representation*

From a modern mathematical viewpoint, real number system is constructed from Euclid geometry under classical logic system via set theory and measure theory. Based on a pair of a real numbers and imaginary number $i$ on two dimensions, it is a natural way to extend complex number systems in advanced algebraic frameworks for many years.

From a foundational viewpoint, higher geometric construction via real numbers to complex numbers is a typical top-down approach that makes lower levels of refined structures extremely difficult to describe in detail. Intrinsic internal complexities may make complex stochastic properties in further analysis.

Forexample, in quantum mechanics, its foundation is Hilbert space, which is based on complex number systems to be a top-down construction. There is no direct linkage between modern logic systems and complex algebraic systems in their detailed levels on basic construction. A series of problems in quantum foundations cannot apply classical logic in consistent supports. It is easier to bring various uncertain paradoxes and mysteries into systems.

## *1.2 Measure Theory, Probability Theory and Statistical Mechanics*

Based on classical logic, real function theory and measure theory, modern probability theory and statistical mechanics are constructed from set theory and real number theory. Multiple variables of statistical analysis are applied matrix and complex analysis methodologies and toolkits.

From a foundation viewpoint, systematical operations are strongly relevant to complex geometry in corresponding measurements. These systems contain similar restrictions on detailed levels between classical logic and complex number systems.

## *1.3 1D Cellular Automata and Discrete Dynamics*

Based on switching logic algebra[15], Wolfram established 1D cellular automata by 0-1 logic functions in 1982. This construction provides modern complexity sciences useful methodologies and simulation tools with powerful visualization and dynamic simulation environments.

The basic components of a cellular automata contain a cell in $m$ bits that corresponds to $2^m$ states in a state space, and there are $2^{2^m}$ logic functions in its function space. Variation patterns are based on a 0-1 vector with $N$ bits in a total number of $2^N$ configurations as a configuration space. Selecting a function in repeat recursion, the whole system appears with time variation properties. Combining function space and configuration space, a united variation space has a structural complexity corresponding to $O(2^{2^m} \times 2^N)$[41].

It is interesting to note that this discrete dynamic system is constructed by the bottom-up principle to extend efficiently on discrete probability and multiple variable statistics. It provides a solid logic foundation on discrete dynamics.

## *1.4 Hamiltonian Dynamics*

From the least action viewpoint, key distinctions between the Lagrange and Hamilton formulas are shown in significantly symmetric differences. From basic levels of measurements, Hamilton equations were organized as a pair of symmetric structures, and Lagrange equations required additional conditions on symmetric properties. From a foundation viewpoint, this type of pair of symmetric structures is extremely important.

From a logic viewpoint, what types of Hamilton dynamics can satisfy both global and local logic systems with a hierarchical construction to support local and global transformations from micro-ensembles to global universals? It is a challenge and extremely hard problem in dynamic systems.

Considering Hamilton dynamics to play an essential role in classical dynamics and quantum dynamics, is it feasible to be an initial point using a logic framework to support its foundation? Further explorations are required.

## 1.5 Quantum Logic, Quantum Information and Cellular Automaton Interpretation of Quantum Mechanics

### 1.5.1 Quantum Logic

After Bohr and Heisenberg proposed the uncertainty principle, uncertainty was active as a new meta-parameter introduced into the logic system. Due to quantum logic associated with superior properties different from classical logic, 3-valued logic and multiple-valued logic become popular candidates as potential logic foundations for quantum logic in the direction[14].

Since either 3-valued or multiple-valued logic contains various paradoxes with intrinsic incomplete, it is extremely difficult for such logic systems to compare with modern switching theory[15] in foundation levels of classical logic.

### 1.5.2 Quantum Information

Quantum computing is an advanced approach based on quantum information using a matrix to generate quantum logic gates[17] to simulate quantum computing and quantum interferences etc.

From a quantum information viewpoint, this type of model and methodology is different from classical information theory[18]. Along this direction, advanced quantum technologies such as quantum communication and quantum encryptions have been developed.

### 1.5.3 Cellular Automaton Interpretation of Quantum Mechanics

Applying the results of 1D cellular automata [41], cellular automaton interpretation of quantum mechanics was proposed by G.'t Hooft (Nobel Prize Winner in Physics) in 2016 on a monograph published by Springer[19]. Different from other interpretations of quantum mechanics, such as Copenhagen, multiverse, and guide waves, this interpretation focuses attention on basic 0-1 logic as a foundation ([19], page 33) to establish proper computational structures supporting a complete framework of quantum mechanics.

In the book, a larger number of examples are listed to be shown that in the foundation of both classical 0-1 logic: 1D cellular automata and quantum mechanics, there are common parts in the bottom levels. Although advanced cellular automata cannot provide a hierarchical construction to support different applications, however, their excellent capacities in detailed simulation powers on the lowest levels are extremely important. From a measuring requirement, there are $10^{20} \leq 2^{70}$ gaps ([19], page 98) to describe huge energy gaps between Planck scale $10^{19}$ GeV and Nuclear scale 1 GeV.

To satisfy various scaling requirements, it is necessary for the foundation structure to have a hierarchy in multiple levels to support extra models from micro-ensembles to global unversals to meet various special requirements.

From an architectural viewpoint, how to apply the lowest levels of states establishes multiple levels of structures to support dynamic systems to be a basic and hardest challenge in cellular automaton interpretation of quantum mechanics.

## 1.6 Conjugate Transformation Structure CTS

Applying balanced organization and pairs of partitions on phase spaces, Jeffrey Z.J. Zheng proposed conjugate classification, conjugate transformation [45, 46] and reversible logic equations [47] in the 1990s. In his PhD thesis [48], four regular plane lattices for binary images were applied for analysis and processing, and a conjugate transformation structure was established (in the Section 5.4). In this direction, the newest progress is a monograph published by Springer [49] in 2019 to propose a 0-1 vector logic system from logic, measurement and visualization as key components to be variant construction.

To make a proper logic foundation, this paper is initially from 0-1 logic through multiple levels of conjugate constructions, consequently establishing a matrix algebraic system to support discrete dynamics.

## 1.7 Measurement Operators in Conjugate Transformation Structure - Paper's Structure

Based on the introduction in Section 1, in Section 2, basic logic constructions are described from variables and states to form conjugate states and function spaces. In Section 3, a sequence of state vectors is composed of configuration spaces. In Section 4, conjugate transformation is established by pairs of two sets of feature vectors to be a reversible logic expression. In Section 5, the conjugate transformation structure (CTS) is described. In Section 6, measurement operators of CTS were proposed to form an 8 element group in an algebraic construction. In Section 7, complex-conjugate operators are compared in classical dynamics and CTS. In Section 8, applying complex-conjugate operators described discrete measurement operators and classical Hamilton differential operators in comparison, and in Section 9, the conclusion of the paper and future explorations are discussed.

Phase spaces of discrete dynamics and their measurements are restricted with Hamilton operators. In this paper, discrete vectors and transformation rules are provided to link with conjugate transformation structures with Hamilton dynamics via various operators.

## 2 Transforming Structure

Conjugate logic construction is composed of hierarchical levels on 0-1 vector structures. Using $m+1$ logic variables to make a kernel form, multiple invariants are proposed to partition pairs of vector states into various clusters to construct multiple levels of phase spaces.

Applying a pair of balanced foreground and background feature clusters, conjugate classification organizes multiple invariants to partition its phase spaces as $2n$ classes. For a given configuration, it is possible to define a feature class to partition the relevant configuration into $2n$ representatives to express this possible phase space consistently. Under transformation, $2n$ classes are played in the most important role. Using this type of meta feature expression, it is feasible to establish an elementary equation of conjugate transformation structure.

### *2.1 Kernel Form*

Let $K$ be a kernel form; it has a certain geometric-logic shape with $m+1$ vector logic variables: $K = \{x_0, x_1, \cdots, x_k, \cdots, x_m\}, x_k \in \{0,1\}, 0 \leq k \leq m$.

Let $x$ be a corresponding position of the kernel form $K$, $\overrightarrow{x}$ be a vector with $m+1$ elements and $\overrightarrow{x} = (x, x_1, \cdots, x_k, \cdots, x_m)$, $K(\overrightarrow{x})$ be the kernel form of the $m+1$ elements.

For multiple variables, a kernel form has a special geometric structure, and multiple geometric and topologic invariants, such as translation, reflection and rotation, can be identified.

### *2.2 Phase Spaces and Conjugate Phase Spaces*

Let $\Omega$ be a phase space - a state space of a kernel form $K(\overrightarrow{x})$. For a given $K(\overrightarrow{x})$, there are distinguished $m+1$ logic variables. For a given $x$, there are two values (0,1), using this value as a reference, it is feasible to organize other $m$ variables to form a pair of two sets with conjugate vector states: $S(\overrightarrow{x})$ and its conjugation $\tilde{S}(\overrightarrow{x})$, $x_k \in \{0,1\}, 1 \leq k \leq m$.

$$S(\overrightarrow{x}) = K(x = 1, x_1, \cdots, x_k, \cdots, x_m) \quad (1)$$

$$\tilde{S}(\overrightarrow{x}) = K(x = 0, x_1, \cdots, x_k, \cdots, x_m) \quad (2)$$

Let $\Theta$ and $\tilde{\Theta}$ express a pair of conjugate phase spaces associated with two set operators: $\{\wedge, \vee\}$,

$$\Theta = \{S(\overrightarrow{x})|x = 1, x_k \in \{0,1\}, 1 \le k \le m\} \quad (3)$$

$$\tilde{\Theta} = \{S(\overrightarrow{x})|x = 0, x_k \in \{0,1\}, 1 \le k \le m\} \quad (4)$$

$$\Omega = \Theta \vee \tilde{\Theta} \quad (5)$$

$$\emptyset = \Theta \wedge \tilde{\Theta} \quad (6)$$

Let $|R|$ be the number of states in the set $R$.

**Lemma 1.** *For a given $K(\overrightarrow{x})$ and its phase space $\Omega$, $|\Theta| = |\tilde{\Theta}| = 2^m$,*

$$|\Omega| = |\Theta| + |\tilde{\Theta}| = 2^{m+1}$$

Two conjugate phase spaces partition the phase space in balance, and two conjugate phase spaces contain conjugate clusters as one to one pairs to organize all possible states as a balanced hierarchical structure.

### *2.3* $2n$ *Feature Clusters of Conjugate Classification)*

Due to $\Theta$ and $\tilde{\Theta}$ with a pair of two vector state sets, each set contains $2^m$ states. Using a specific partition, it is feasible to define relevant clusters as $n$ groups. This classification is a conjugate classification if each group has a conjugate group and all groups are composed of the whole states.

Let $\Gamma_j(\tilde{\Gamma_j})$ be the $j$-th cluster of $\Theta(\tilde{\Theta})$, for any state $S(\overrightarrow{x}) \in \Gamma_j$, iff, there is one corresponding state $\tilde{S}(\overrightarrow{x}) \in \tilde{\Gamma_j}, y = \neg x = 0, y_k = \neg x_k; x_k, y_k \in \{0,1\}, 0 < k \le m$, and vise versa.

Using a conjugate pair of 1-1 mapping, equations of $|\Gamma_j| = |\tilde{\Gamma_j}| \ge 1$ are true on the following equations.

$$\Theta = \bigvee_{j=1}^{n} \Gamma_j \quad (7)$$

$$\tilde{\Theta} = \bigvee_{j=1}^{n} \tilde{\Gamma_j} \quad (8)$$

$$\emptyset = \Gamma_k \wedge \Gamma_j = \tilde{\Gamma_k} \wedge \Gamma_j = \Gamma_k \wedge \tilde{\Gamma_j} = \tilde{\Gamma_k} \wedge \tilde{\Gamma_j}, 1 \le j \ne k \le n \quad (9)$$

**Theorem 1.** *For a given conjugate classification, $2n$ feature classes $\{\{\Gamma_j\}_{j=1}^n, \{\tilde{\Gamma_j}\}_{j=1}^n\}$ partition a pair of two conjugate phase spaces $\Theta$ and $\tilde{\Theta}$ respectively.*

## 3 Configuration

Let $X$ be a configuration that is composed of a sample set with $N$ elements, and each element is a state.

$$X = (X_1, \cdots, X_l, \cdots, X_N), X_l \in \Omega . 1 \le l \le N \quad (10)$$

$$X_l = \begin{cases} S(\overrightarrow{x^l}), X_l \in \Theta, \\ \tilde{S}(\overrightarrow{x^l}), X_l \in \tilde{\Theta}; \end{cases} \quad (11)$$

$$\overrightarrow{x^l} = (x^l, x^l_1, \cdots, x^l_k, \cdots, x^l_m), x^l, x^l_k \in \{0,1\}, 1 \le k \le m \quad (12)$$

From a representing viewpoint, $X$ may be equivalent to a $(m+1) \times N$ 0-1 sequence to be composed of $N$ states, and each $X_l$ state contains $m+1$ bits.

i.e. When consequential states have overlapped parts in $m$ bits, $X$ could be represented as a 1D cellular automata with a total of $N$ cells and each cell with $m+1$ bits.

### *3.1 Two Sets of Feature Vectors*

For convenient analysis, suppose two states can be identified, iff, they are in two distinguished groups. Under this condition, the configuration vector can be mapped into $2n$ feature vectors.

Using defined $2n$ classes, their feature values are:

$$\Gamma_j(X_l) = \begin{cases} 1, X_l \in \Gamma_j; \\ 0, X_l \notin \Gamma_j \end{cases} \quad (13)$$

$$\tilde{\Gamma}_j(X_l) = \begin{cases} 0, X_l \in \tilde{\Gamma}_j; \\ 1, X_l \notin \tilde{\Gamma}_j \end{cases} \quad (14)$$

Under this paired mapping rule, all $X_l, 1 \le l \le N$ correspond to $2n$ 0-1 feature vectors.

$$\Gamma_j(X) = (\Gamma_j(X_1), \cdots, \Gamma_j(X_l), \cdots, \Gamma_j(X_N)), 1 \le l \le N, 1 \le j \le n \quad (15)$$

$$\tilde{\Gamma}_j(X) = (\tilde{\Gamma}_j(X_1), \cdots, \tilde{\Gamma}_j(X_l), \cdots, \tilde{\Gamma}_j(X_N)), 1 \le l \le N, 1 \le j \le n; \quad (16)$$

### *3.2 Two Constant Vectors*

In addition to these feature vectors, there are two constant vectors.

$$\Gamma_0(X_l) = 0, 1 \le l \le N; \quad (17)$$

$$\tilde{\Gamma}_0(X_l) = 1, 1 \le l \le N \quad (18)$$

Let two constant vectors be $\overrightarrow{0}$ and $\overrightarrow{1}$,

$$\overrightarrow{0} = \Gamma_0(X) = (\Gamma_0(X_1)\cdots\Gamma_0(X_l)\cdots\Gamma_0(X_N)) = (0\cdots0\cdots0), 1 \le l \le N; \quad (19)$$

$$\overrightarrow{1} = \tilde{\Gamma}_0(X) = (\tilde{\Gamma}_0(X_1)\cdots\tilde{\Gamma}_0(X_l)\cdots\tilde{\Gamma}_0(X_N)) = (1\cdots1\cdots1), 1 \le l \le N \quad (20)$$

Two constant vectors in configuration select $\emptyset$ feature groups.

### 3.3 Feature Counting

In a 1D cellular automata, the $l$-th position of feature vector $\Gamma_j(X)$ has a 1 value if $X^l \in \Gamma_j\}$ and the $l$-th position of conjugate feature vector $\tilde{\Gamma}_j(X)$ has 0 values if $X^l \in \tilde{\Gamma}_j$. Both feature vectors are 0-1 vectors with $N$ bits to be a feature projection from the configuration $X$ to be composed of two sets of $2n$ feature vectors.

Counting the number of 1 elements in $\Gamma_j(X)$ and the number of 0 elements in $\tilde{\Gamma}_j(X)$, two coefficients $a_j, b_j$ can be calculated.

$$a_j(X) = \sum_{l=1}^{N}(\Gamma_j(X_l) == 1) \quad (21)$$

$$b_j(X) = \sum_{l=1}^{N}(\tilde{\Gamma}_j(X_l) == 0) \quad (22)$$

In general, the feature vector length $N$ satisfies the following equation for $2n$ coefficients.

$$N = \sum_{j=1}^{n}(a_j(X) + b_j(X)) \quad (23)$$

## 4 Basic Logic Operations on Feature Vectors

### 4.1 Two Canonical Feature Vectors

Using two logic operators $\{\cap, \cup\}$, combinations of whole feature vectors to be represented as two canonical feature vectors

$$\Gamma(X) = \bigcup_{j=0}^{n} \Gamma_j(X); \quad (24)$$

$$\tilde{\Gamma}(X) = \bigcap_{j=0}^{n} \tilde{\Gamma}_j(X) \quad (25)$$

**Theorem 2.** *For a given configuration $X$, two canonical feature vectors satisfy $\Gamma(X) = \tilde{\Gamma}(X)$.*

*Proof.* For a state in a feature group, its feature value is 1 and a state in a conjugate feature group, its special value is 0. When two types of feature vectors are merged together only positions in feature groups keep values 1 under OR operations and conjugate positions in conjugate groups keep values 0 under AND operations that make two integrated feature vectors be equally valued.

### 4.2 Special Logic Operators

Using two logic operators $\{\neg, \tilde{}\}$, the following equations are satisfied.

**Theorem 3.** *For two feature vectors $\{\Gamma_j(X), \tilde{\Gamma}_j(X)\}$, conjugate operator $\tilde{}$ reverse feature vector and original configuration.*

$$\tilde{\Gamma}_j(X) = \neg\Gamma_j(\neg X); \tag{26}$$

$$\neg\tilde{\Gamma}_j(\tilde{X}) = \Gamma_j(\neg X); \tag{27}$$

$$\Gamma_j(X) = \neg\tilde{\Gamma}_j(\neg X) = \tilde{\tilde{\Gamma}}_j(X); \tag{28}$$

$$\neg\Gamma_j(X) = \tilde{\Gamma}_j(\neg X) \tag{29}$$

*Proof.* For the first equation, the $l$-th 0 element of $\tilde{\Gamma}_j(X)$ describes the $j$-th group of $X_l = \tilde{S}(\overrightarrow{y^l}) \in \tilde{\Gamma}_j$, conjugate operator $\neg X_l = \tilde{\tilde{S}}(\overrightarrow{y^l}) = (1, x_1^l, \cdots, x_k^l, \cdots, x_m^l) = S(\overrightarrow{y^l}), x_k^l = \neg y_k^l, \tilde{S}(\overrightarrow{y^l}) \in \tilde{\Gamma}_j \to \neg X_l \in \Gamma_j$. One 1-element of $\Gamma_j(\neg X)$ corresponds to one 0-element of $\tilde{\Gamma}_j(X)$ undertaken $\neg$ operations. For other equations, similar operations can be performed in the proof procedure.

### 4.3 Irreducible Expression

When two feature vectors are on a logic expression, if the expression cannot be reduced as a constant vector or simplified as a single feature vector, then this logic expression is an irreducible expression; otherwise it is a reducible expression.

**Theorem 4.** *For two distinguished feature vectors under $\{\neg, \cap, \cup, \tilde{}\}$ operators, there are only 6 groups of irreducible expressions.*

$$\neg\Gamma_j(X)\cap\neg\Gamma_k(X) = \tilde{\Gamma}_j(\neg X)\cap\tilde{\Gamma}_k(\neg X); \tag{30}$$
$$\Gamma_j(X)\cup\Gamma_k(X) = \neg\tilde{\Gamma}_j(\neg X)\cup\neg\tilde{\Gamma}_k(\neg X); \tag{31}$$
$$\neg\tilde{\Gamma}_j(X)\cup\neg\tilde{\Gamma}_k(X) = \Gamma_j(\neg X)\cup\Gamma_k(\neg X); \tag{32}$$
$$\tilde{\Gamma}_j(X)\cap\tilde{\Gamma}_k(X) = \neg\Gamma_j(\neg X)\cap\neg\Gamma_k(\neg X); \tag{33}$$
$$\neg\Gamma_j(X)\cap\tilde{\Gamma}_k(X) = \neg\Gamma_j(X)\cap\neg\Gamma_k(\neg X); \tag{34}$$
$$\Gamma_j(X)\cup\neg\tilde{\Gamma}_k(X) = \Gamma_j(X)\cup\Gamma_k(\neg X) \tag{35}$$

## 5 Conjugate Transformation

For convenient description of a given configuration, an elementary equation of conjugate transformation can be established.

### 5.1 A Pair of Parameter Sets

Let $I=\{1,\cdots,n\}$ be an index set of feature vectors, and $A,B\subseteq I$ be two subsets of the index set.

$$\langle A,B\rangle = \{\{\Gamma_j\}_{j\in A},\{\tilde{\Gamma}_k\}_{k\in B}\} \tag{36}$$

In $\langle A,B\rangle$, the first parameter is the index set of the phase space $\Theta$ and the second parameter is the index set of the conjugate phase space $\tilde{\Theta}$.

e.g., if $A=\{2,4\},B=\{3,4\}$, then $\langle A,B\rangle=\{\Gamma_2,\Gamma_4,\tilde{\Gamma}_3,\tilde{\Gamma}_4\}$.

### 5.2 Pair Sets of Feature Vectors

For any configuration $X,\langle A,B\rangle X$ represents a pair set of feature vectors.

$$\langle A,B\rangle X = \{\{\Gamma_j(X)\}_{j\in A},\{\tilde{\Gamma}_k(X)\}_{k\in B}\} \tag{37}$$

To express two sets of selected feature vectors, let $\Gamma_A(X)$ be a merged set of the index set $A$ under OR operations and $\tilde{\Gamma}_B(X)$ be a joined set of the index set $B$ of feature vectors under AND operations.

$$\Gamma_A(X) = \bigcup_{j\in A}\Gamma_j(X) \tag{38}$$
$$\tilde{\Gamma}_B(X) = \bigcap_{k\in B}\tilde{\Gamma}_k(X) \tag{39}$$
$$\langle A,B\rangle X = \{\Gamma_A(X),\tilde{\Gamma}_B(X)\} \tag{40}$$

$\langle A,B\rangle X$ is composed of two feature vectors.

While $A = B = \emptyset$, let $\mathit{\Gamma}_{\emptyset}(X) = \mathit{\Gamma}_0(X) = \overrightarrow{0}$ and $\tilde{\mathit{\Gamma}}_{\emptyset}(X) = \tilde{\mathit{\Gamma}}_0(X) = \overrightarrow{1}$ be two constant vectors.

## 5.3 Elementary Equation of Conjugate Transformation

Let $\natural$ be a reversing operator; it reverses selected parts of the configuration to satisfy the following equations.

Let $F(\langle A,B\rangle X)$ be an elementary equation of Conjugate Transformation

$$F(\langle A,B\rangle X) = \mathit{\Gamma}(X)\natural\langle A,B\rangle X \tag{41}$$

$$= (\mathit{\Gamma}(X) \cap (\neg\mathit{\Gamma}_A(X))) \cup (\neg\tilde{\mathit{\Gamma}}_B(X)) \tag{42}$$

Operator $\natural$ has the following meaning: selected components $\langle A,B\rangle X$, the operator changes selected sets of feature vectors and retains other unselected sets of feature vectors be invariant.

**Theorem 5.** *The elementary equation is a vector logic equation to reverse the selected sets of feature vectors and keep other unselected sets of feature vectors invariant.*

**Theorem 6.** *Under the $\neg$ operator, the elementary equation is a self-conjugate equation to exchange index sets of feature vectors and to be reversed on the configuration vector.*

$$\neg F(\langle A,B\rangle X) = F(\langle B,A\rangle \neg X) \tag{43}$$

*Proof.*

$$\begin{aligned}
\neg F(\langle A,B\rangle X) &= \neg((\mathit{\Gamma}(X) \cap (\neg\mathit{\Gamma}_A(X))) \cup (\neg\tilde{\mathit{\Gamma}}_B(X))) \\
&\quad \text{de Morgan Rule} \\
&= \neg(\mathit{\Gamma}(X) \cap (\neg\mathit{\Gamma}_A(X))) \cap \tilde{\mathit{\Gamma}}_B(X) \\
&\quad \text{Distribution Rule} \\
&= (\neg\mathit{\Gamma}(X) \cup \mathit{\Gamma}_A(X)) \cap \tilde{\mathit{\Gamma}}_B(X) \\
&\quad \because \mathit{\Gamma}_A(X) \cap \tilde{\mathit{\Gamma}}_B(X) = \mathit{\Gamma}_A(X) \\
&= (\neg\mathit{\Gamma}(X) \cap \tilde{\mathit{\Gamma}}_B(X)) \cup (\mathit{\Gamma}_A(X) \cap \tilde{\mathit{\Gamma}}_B(X)) \\
&\quad \because \tilde{\mathit{\Gamma}}_B(X) = \neg\mathit{\Gamma}_B(\neg X), \mathit{\Gamma}_A(X) = \neg\tilde{\mathit{\Gamma}}_A(\neg X) \\
&= (\neg\mathit{\Gamma}(X) \cap \tilde{\mathit{\Gamma}}_B(X)) \cup \mathit{\Gamma}_A(X) \\
&\quad \because \neg\mathit{\Gamma}(X) = \tilde{\mathit{\Gamma}}(\neg X) = \mathit{\Gamma}(\neg X) \\
&= (\mathit{\Gamma}(\neg X) \cap \neg\mathit{\Gamma}_B(\neg X)) \cup \neg\tilde{\mathit{\Gamma}}_A(\neg X) \\
&= F(\langle B,A\rangle \neg X)
\end{aligned}$$

**Theorem 7.** *For a given $\langle A,B\rangle$, if $A=\emptyset$ or $B=\emptyset$, then $F(\langle A,B\rangle X)$ for components $\langle A,B\rangle X$ is an equation of dilation or erosion.*

*Proof.*

$$\begin{aligned}
F(\langle \emptyset,B\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_0(X)))\cup(\neg\tilde{\Gamma}_B(X))\\
&= \Gamma(X)\cup(\neg\tilde{\Gamma}_B(X))(\text{ Dilation})\\
F(\langle A,\emptyset\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_A(X)))\cup(\neg\tilde{\Gamma}_0(X))\\
&= \Gamma(X)\cap(\neg\Gamma_A(X))(\text{ Erosion})
\end{aligned}$$

**Theorem 8.** *The elementary equation of conjugate transformation generates four extreme vectors $\{\Gamma(X),\overrightarrow{0},\overrightarrow{1},\neg\Gamma(X)\}$; four extreme vectors are $\langle A,B\rangle=\{\langle\emptyset,\emptyset\rangle,\langle\emptyset,I\rangle,\langle I,\emptyset\rangle,\langle I,I\rangle\}$.*

*Proof.*

$$\begin{aligned}
F(\langle \emptyset,\emptyset\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_0(X)))\cup(\neg\tilde{\Gamma}_0(X))\\
&= \Gamma(X)\rightarrow\text{ An Invariant Vector}\\
F(\langle \emptyset,I\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_0(X)))\cup(\neg\tilde{\Gamma}_I(X))\\
&= \Gamma(X)\cup\neg\tilde{\Gamma}_I(X)\\
&= \Gamma(X)\cup\neg\tilde{\Gamma}(X)\\
&= \Gamma(X)\cup\neg\Gamma(X)\\
&= \overrightarrow{1}\rightarrow\text{ A Constant 1 Vector}
\end{aligned}$$

$$\begin{aligned}
F(\langle I,\emptyset\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_I(X)))\cup(\neg\tilde{\Gamma}_0(X))\\
&= \Gamma(X)\cap(\neg\Gamma_I(X))\\
&= \Gamma(X)\cap(\neg\Gamma(X))\\
&= \overrightarrow{0}\rightarrow\text{ A Constant 0 Vector}\\
F(\langle I,I\rangle X) &= (\Gamma(X)\cap(\neg\Gamma_I(X)))\cup(\neg\tilde{\Gamma}_I(X))\\
&= (\Gamma(X)\cap(\neg\Gamma(X)))\cup(\neg\tilde{\Gamma}(X))\\
&= \neg\tilde{\Gamma}(X)\\
&= \neg\Gamma(X)\rightarrow\text{ A Reversed Vector}
\end{aligned}$$

**Theorem 9.** *For a configuration $X$ and its two sets of feature vectors $\{\Gamma_j(X)\}_{j=0}^{n},\{\tilde{\Gamma}_k(X)\}_{k=0}^{n}$, the elementary equation of conjugate transformation can generate $2^{2n}$ vector functions.*

*Proof.* Each selection of $\langle A,B\rangle$ determines a vector logic function. Two sets are composed of all possible combinations of $A,B\subseteq I=\{1,\cdots,n\}$, each index set has $2^n$ selections, and a total of $2^{2n}$ vector logic functions can be generated.

# 6 Conjugate Transformation Structure (CTS)

## 6.1 Transformation Structure

Feature vectors are 0-1 vectors. Using defined $2 \times (n+1)$ logic vectors, it is feasible to define a new transformation structure. Let $\Pi$ be a set of feature vectors.

$$\Pi(X) = \{\Gamma(X), \tilde{\Gamma}(X), \{\Gamma_j(X)\}_{j=0}^{n}, \{\tilde{\Gamma}_k(X)\}_{k=0}^{n}\} \tag{44}$$

Let the conjugate transformation structure be CTS.

$$CTS(X) = \{\Pi(X), \neg, \cap, \cup, \sim\} \tag{45}$$

Each feature vector is a logic vector variable, and two sets of conjugate feature vectors make each $CTS(X)$ be a vector logic algebra - conjugate vector algebra. To explore the global properties of this structure, it is helpful for comparison to use examples of vector logic algebra in [S. Lee 1978, Knuth 2008] to count the total number of logic functions in a vector logic system.

## 6.2 Transformation Structures

Let $\{Y_k\}_{k=1}^{2n}$ be $2n$ Boolean vectors. Using these vectors under three logic operators, it is possible to establish Boolean vector algebra $BVA(Y)$.

$$BVA(Y) = \{\{Y_k\}_{k=1}^{2n}, \neg, \cap, \cup\} \tag{46}$$

Boolean expressions of $BVA$ can be formed as a canonical sum-of-products form (Disjunctive Normal Form DNF) or a canonical product-of-sums form (Conjunctive Normal Form CNF).

$$\bigcup_{J \in \Lambda} \left( \bigcap_{i=1}^{2n} Y_i^{J_i} \right) \text{ (DNF)} \tag{47}$$

$$\bigcap_{J \notin \Lambda} \left( \bigcup_{i=1}^{2n} Y_i^{\neg J_i} \right) \text{ (CNF)} \tag{48}$$

where $\Lambda$ expresses a subset of $\{0, 1, 2, \cdots, 2^{2n} - 1\}$.

$$J = \sum_{i=1}^{2n} J_i \times 2^{i-1}, J_i \in \{0, 1\} \tag{49}$$

$$Y_i^{J_i} = \begin{cases} Y_i, J_i = 1; \\ \neg Y_i, J_i = 0. \end{cases} \tag{50}$$

**Lemma 2.** *For a BVA$(Y)$ with $2n$ distinct vector variables, there are $2^{2n}$ states and $2^{2^{2n}}$ logic functions.*

*Proof.* Using a selected or unselected role to handle each vector as a bit, it is natural for $2n$ bit variables in a Boolean algebra system to contain $2^{2n}$ states and $2^{2^{2n}}$ functions. Either DNF or CNF provide the same number.

**Theorem 10.** *For a $CTS(X)$ on $1 \leq n \leq 2^m$, there are $2^{2n}$ functions.*

*Proof.* Using a *DNF* form, a simplifying process can be described as follows.

$$(\bigcap_{i=1}^{n} \Gamma_i(X)^{J_i}) \cap (\bigcup_{k=1}^{n} \tilde{\Gamma}_k(X)^{J_k})$$

In the first part, each item can be reduced.

| Type | Simplified | Condition |
|---|---|---|
| 1 | $\neg\Gamma(X)$ | $\forall i, J_i = 0$ |
| 2 | one of $\{\Gamma_1(X), \cdots, \Gamma_n(X)\}$ | $\exists! i, J_i = 0$ |
| 3 | $\overrightarrow{0}$ | $\exists i \neq k, J_i = J_k = 0$ |

In the second part, each item can also be reduced.

| Type | Simplified | Condition |
|---|---|---|
| 1 | $\neg\tilde{\Gamma}(X)$ | $\forall k, J_k = 0$ |
| 2 | One of $\{\neg\tilde{\Gamma}_1(X), \cdots, \neg\tilde{\Gamma}_n(X)\}$ | $\exists! k, J_k = 1$ |
| 3 | $\overrightarrow{0}$ | $\exists k \neq i, J_k = J_i = 1$ |

因为$\neg\Gamma(X) \cap \neg\tilde{\Gamma}_j(X) = \neg\tilde{\Gamma}_j(X)$, $\neg\Gamma_i(X) \cap \neg\tilde{\Gamma}_k(X) = \overrightarrow{0}$。
Since $\neg\Gamma(X) \cap \neg\tilde{\Gamma}_j(X) = \neg\tilde{\Gamma}_j(X)$, $\neg\Gamma_i(X) \cap \neg\tilde{\Gamma}_k(X) = \overrightarrow{0}$.
Only items in *DNF* are

$$\{\Gamma_i(X)\}_{i=1}^{n}, \{\neg\tilde{\Gamma}_k(X)\}_{k=1}^{n}$$

Using two feature vector sets, there are $2^{2n}$ combinations as functions.
For a *CNF* form, it can be simplified in the same way.

It is important to note that $2n$ vector variables in *BVA* have $2^{2^{2n}}$ logic functions, however $2n$ feature vectors in *CTS*, there are at most $2^{2^{m+1}}$ functions. From a numeric viewpoint, it is feasible to compare two numbers of functions between *BVS* and *CTS* on their largest ones.

**Theorem 11.** *For $2n$ vector variables in BVA and $2n$ feature vectors on $m+1$ kernel form in CTS, if $n \leq \lceil \frac{m+1}{2} \rceil$, then the function number of CTS is systematically much larger than the function number of BVA.*

*Proof.* Compared with the worst condition on $2^{2^{2n}}$ and $2^{2^{m+1}}$, if $2n \leq m+1$, then *CTS* contains larger numbers than *BVA*.

**Corollary 1.** *In general, CTS provides a hierarchy of flexible function representatives in wider regions from $2^2, \cdots, 2^{2n}$ to $2^{2^{m+1}}$ instead of $2^{2^{2n}}$ on BVA.*

In other words, when selected clusters are less than half of state length, the phase spaces of $CTS$ have much simpler properties significantly superior to $BVA$ even on the worst condition.

Since each configuration has a $CTS$ structure, combining possible configurations and feature functional spaces of $CTS$ can be integrated.

Let $\Sigma$ be a number of combining spaces on $CTS$.

**Theorem 12.** *For a configuration with $N$ elements under a kernel form with $m+1$ bits in $2n$ clusters, a total number of combinations $\Sigma(N,m,n)$ is determined by*

$$\Sigma(N,m,n) = 2^N \times 2^{2n}, 1 \leq n \leq 2^m \tag{51}$$

*Proof.* For each configuration, the distinguished number is $2^N$, and the number of functions on $2n$ clusters is $2^{2n}$, as shown in Corollary 1.

**Corollary 2.** *Combining various configurations and feature functions, their controlable spaces have numeric regions from $\Sigma(N,m,1) = 2^{N+2}$ to $\Sigma(N,m,2^m) = 2^N \times 2^{2^{m+1}}$.*

*Proof.* Selecting $n = 1$ and $2^m$, two numbers for minimal and maximal ones are determined.

### *6.3 Three Set Operators*

Let $\{', \vee, \wedge\}$ be {complement, union, intersection} operators of index sets. For an index set $I$, $A$ a subset and $A'$ a complement subset, $A \subset I$ and $A' \subset I$ satisfy the following equation.

$$A \vee A' = I \tag{52}$$

$$A \wedge A' = \emptyset \tag{53}$$

$$\emptyset' = I \tag{54}$$

$$I' = \emptyset \tag{55}$$

Applying this expression, a canonical vector $\Gamma(X)$ or $\tilde{\Gamma}(X)$ can be described as two parts linked with a $\cup$ or $\cap$ operator.

$$\Gamma(X) = \bigcup_{j=0}^{n} \Gamma_j(X) = \bigcup_{j=1}^{n} \Gamma_j(X) = \Gamma_A(X) \cup \Gamma_{A'}(X) \tag{56}$$

$$\tilde{\Gamma}(X) = \bigcap_{k=0}^{n} \tilde{\Gamma}_k(X) = \bigcap_{k=1}^{n} \tilde{\Gamma}_k(X) = \tilde{\Gamma}_B(X) \cap \tilde{\Gamma}_{B'}(X) \tag{57}$$

**Theorem 13.** *For a $\langle A,B \rangle X \in CTS(X)$, the elementary equation of $CTS$ has four equivalent expressions.*

$$
\begin{aligned}
F(\langle A,B\rangle X) &= (\Gamma(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X) &(58)\\
&= \Gamma_{A'}(X))\cup\neg\tilde{\Gamma}_B(X) &(59)\\
&= \neg\Gamma_A(X))\cap\tilde{\Gamma}_{B'}(X) &(60)\\
&= (\Gamma(X)\cup\neg\tilde{\Gamma}_B(X))\cap\neg\Gamma_A(X) &(61)
\end{aligned}
$$

*Proof.* From the first one to the second one, we have

$$
\begin{aligned}
F(\langle A,B\rangle X) &= (\Gamma(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X)\\
&\because \Gamma(X) = (\Gamma_A(X)\cup\Gamma_{A'}(X))\\
&= ((\Gamma_A(X)\cup\Gamma_{A'}(X))\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X)\\
&= (\Gamma_A(X)\cap\neg\Gamma_A(X))\cup(\Gamma_{A'}(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X)\\
&\because \overrightarrow{0} = (\Gamma_A(X)\cap\neg\Gamma_A(X))\\
&= (\overrightarrow{0})\cup(\Gamma_{A'}(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X)\\
&= (\Gamma_{A'}(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X)\\
&= \Gamma_{A'}(X)\cup\neg\tilde{\Gamma}_B(X)
\end{aligned}
$$

The other two expressions can be deduced in a similar way.

From the four equivalent expressions, symmetric properties can be observed in the formula.

## 6.4 Four Conjugate-Logic Operators

Applying refined set operators, four conjugate-logic operators $\{\neg,\cap,\cup,\sim\}$ {NOT, AND, OR, Conjugate} are used to explore symmetric properties of the elementary equation.

For the $\neg$ operator, the elementary equation can be described in two forms.

**Theorem 14.** *For an elementary equation, its $\neg$ description has two forms.*

$$
\begin{aligned}
\neg F(\langle A,B\rangle X) &= F(\langle A',B'\rangle X) &(62)\\
&= F(\langle B,A\rangle \tilde{X})
\end{aligned}
$$

*Proof.* Checking the new form of the equation, we have

$$
\begin{aligned}
\neg F(\langle A,B\rangle X) &= \neg((\Gamma(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X))\\
&= \neg(\Gamma(X)\cap\neg\Gamma_A(X))\cap\tilde{\Gamma}_B(X)\\
&\quad \because \Gamma(X)=\tilde{\Gamma}(X)\\
&= \neg(\tilde{\Gamma}(X)\cap\neg\Gamma_A(X))\cap\tilde{\Gamma}_B(X)\\
&\quad \because \tilde{\Gamma}(X)=\tilde{\Gamma}_B(X)\cap\tilde{\Gamma}_{B'}(X)\\
&= \neg(\tilde{\Gamma}_B(X)\cap\tilde{\Gamma}_{B'}(X)\cap\neg\Gamma_A(X))\cap\tilde{\Gamma}_B(X)\\
&= (\neg\tilde{\Gamma}_B(X)\cup\neg\tilde{\Gamma}_{B'}(X)\cup\Gamma_A(X))\cap\tilde{\Gamma}_B(X)\\
&= \neg\tilde{\Gamma}_B(X)\cap\tilde{\Gamma}_B\cup\neg\tilde{\Gamma}_{B'}(X)\cap\tilde{\Gamma}_B\cup\Gamma_A(X)\cap\tilde{\Gamma}_B\\
&= \overrightarrow{0}\cup\neg\tilde{\Gamma}_{B'}(X)\cup\Gamma_A(X)\\
&= \neg\tilde{\Gamma}_{B'}(X)\cup\Gamma_A(X)\\
&= \Gamma_A(X)\cup\neg\tilde{\Gamma}_{B'}(X)\\
&= \Gamma_{(A')'}(X)\cup\neg\tilde{\Gamma}_{B'}(X)\\
&= F(\langle A',B'\rangle X)
\end{aligned}
$$

For processing more complex operations in $CTS$, let $Y_1 = F(\langle A_1,B_1\rangle X)$ and $Y_2 = F(\langle A_2,B_2\rangle X)$ be two vectors in $CTS(X)$.

**Theorem 15.** *For any $Y_1,Y_2 \in CTS(X)$ under logic operators $\{\cap,\cup\}$, two combined vectors are in $CTS(X)$.*

$$Y_1\cap Y_2 = F(\langle A_1\vee A_2, B_1\wedge B_2\rangle X) \tag{63}$$

$$Y_1\cup Y_2 = F(\langle A_1\wedge A_2, B_1\vee B_2\rangle X) \tag{64}$$

*Proof.*

$$
\begin{aligned}
Y_1\cap Y_2 &= ((\Gamma(X)\cap\neg\Gamma_{A_1}(X))\cup\neg\tilde{\Gamma}_{B_1}(X))\cap((\Gamma(X)\cap\neg\Gamma_{A_2}(X))\cup\neg\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap\neg\Gamma_{A_1}(X)\cap\neg\Gamma_{A_2}(X))\cup(\neg\tilde{\Gamma}_{B_1}(X))\cap\neg\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap\neg(\Gamma_{A_1}(X)\cup\Gamma_{A_2}(X))\cup\neg(\tilde{\Gamma}_{B_1}(X)\cup\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap\neg(\Gamma_{A_1\vee A_2}(X))\cup\neg(\tilde{\Gamma}_{B_1\wedge B_2}(X))\\
&= F(\langle A_1\vee A_2, B_1\wedge B_2\rangle X)\\
Y_1\cup Y_2 &= ((\Gamma(X)\cap\neg\Gamma_{A_1}(X))\cup\neg\tilde{\Gamma}_{B_1}(X))\cup((\Gamma(X)\cap\neg\Gamma_{A_2}(X))\cup\neg\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap(\neg\Gamma_{A_1}(X)\cup\neg\Gamma_{A_2}(X))\cup(\neg\tilde{\Gamma}_{B_1}(X))\cup\neg\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap\neg(\Gamma_{A_1}(X)\cap\Gamma_{A_2}(X))\cup\neg(\tilde{\Gamma}_{B_1}(X)\cap\tilde{\Gamma}_{B_2}(X))\\
&= (\Gamma(X)\cap\neg(\Gamma_{A_1\wedge A_2}(X))\cup\neg(\tilde{\Gamma}_{B_1\vee B_2}(X))\\
&= F(\langle A_1\wedge A_2, B_1\vee B_2\rangle X)
\end{aligned}
$$

For any feature vector, the conjugate operator $\sim$ exchanges a pair of two feature vectors and two logic operators.

$$\sim: \begin{cases} \Gamma_j(X)\leftrightarrow\tilde{\Gamma}_j(X)\\ \cap\leftrightarrow\cup \end{cases} \tag{65}$$

**Theorem 16.** *For an elementary equation, the conjugate operator exchanges two index sets of feature vectors.*

$$\tilde{F}(\langle A,B\rangle X) = F(\langle B,A\rangle X) \tag{66}$$

*Proof.*

$$\begin{aligned}\tilde{F}(\langle A,B\rangle X) &= \sim((\Gamma(X)\cap\neg\Gamma_A(X))\cup\neg\tilde{\Gamma}_B(X))\\ &= (\tilde{\Gamma}(X)\cup\neg\tilde{\Gamma}_A(X))\cap\neg\Gamma_B(X)\\ &= (\Gamma(X)\cap\neg\Gamma_B(X))\cup\neg\tilde{\Gamma}_A(X)\\ &= F(\langle B,A\rangle X)\end{aligned}$$

**Lemma 3.** *For an elementary equation, the conjugate operator and ¬ operator generate different results.*

$$\tilde{F}(\langle A,B\rangle X) \neq \neg F(\langle A,B\rangle X) \tag{67}$$

## *6.5 Conjugate Twist Operators*

Let $\zeta$ be a conjugate twist operator or a twistor, let $E$ be a unit operator.

$$\zeta\langle A,B\rangle = \langle B',A\rangle \tag{68}$$

$$\zeta : \langle A,B\rangle \Rightarrow \{\langle A,B\rangle,\langle B',A\rangle,\langle A',B'\rangle,\langle B,A'\rangle\} \tag{69}$$

$$\zeta(F(\langle A,B\rangle X)) = F(\zeta\langle A,B\rangle X) = F(\langle B',A\rangle X) \tag{70}$$

$$= (\Gamma(X)\cap\neg\Gamma_{B'}(X))\cup\neg\tilde{\Gamma}_A(X) \tag{71}$$

**Theorem 17.** *For an elementary equation, an NOT operation can be performed by twice twistors.*

$$\zeta^2 F(\langle A,B\rangle X) = F(\zeta^2\langle A,B\rangle X) = F(\langle A',B'\rangle X) = \neg F(\langle A,B\rangle X) \tag{72}$$

*Proof.*

$$\begin{aligned}\zeta^2 F(\langle A,B\rangle X) &= \zeta(\zeta(F(\langle A,B\rangle X)))\\ &= \zeta(F(\zeta\langle A,B\rangle X))\\ &= \zeta F(\langle B',A\rangle X)\\ &= F(\zeta\langle B',A\rangle X)\\ &= F(\langle A',B'\rangle X)\\ &= \neg F(\langle A,B\rangle X)\end{aligned}$$

**Corollary 3.** *For an elementary equation, four twistors are performed to be a unit operator; it is back to be itself.*

$$\zeta^4 F(\langle A,B\rangle X) = E(F(\langle A,B\rangle X)) = F(\langle A,B\rangle X) \tag{73}$$

### 6.6 Twist and Conjugate Operators

**Corollary 4.** *Applying two operators $\{\sim,\zeta\}$ repeat, an elementary equation has eight expressions.*

$$\{\sim,\zeta\} : \langle A,B\rangle \Rightarrow \left\{ \begin{array}{c} \langle A,B\rangle, \langle B',A\rangle, \langle A',B'\rangle, \langle B,A'\rangle \\ \langle B,A\rangle, \langle A,B'\rangle, \langle B',A'\rangle, \langle A',B\rangle \end{array} \right\} \tag{74}$$

From various corollaries on $\{\sim,\zeta\}$ operators, the conjugate operator and twistor play a more essential role than a NOT operator in $CTS$. From the viewpoint of extending logic transformations, twistor is a new logic operator, and no corresponding operators exist in classical logic systems.

Applying Corollary 4, transforming expressions are in eight forms; for a given configuration, selected feature sets can be determined by $\langle A,B\rangle X$.

## 7 Measurement Structure

For a given $\langle A,B\rangle X$, let $H$ be a measurement operator in a complex measurement mode as follows.

$$\begin{aligned} H(\langle A,B\rangle X) &= (a_A, b_B) = (a_A + i\cdot b_B) \qquad (75)\\ &= \begin{pmatrix} \delta_1^A\cdot a_1, \delta_1^B\cdot b_1 \\ \cdots \\ \delta_j^A\cdot a_j, \delta_j^B\cdot b_j \\ \cdots \\ \delta_n^A\cdot a_n, \delta_n^B\cdot b_n \end{pmatrix} = \begin{pmatrix} \delta_1^A\cdot a_1 + i\cdot\delta_1^B\cdot b_1 \\ \cdots \\ \delta_j^A\cdot a_j + i\cdot\delta_j^B\cdot b_j \\ \cdots \\ \delta_n^A\cdot a_n + i\cdot\delta_n^B\cdot b_n \end{pmatrix} \end{aligned}$$

where $\delta_j^A = \begin{cases} 1, & j\in A; \\ 0, & j\notin A; \end{cases}$ $\delta_j^B = \begin{cases} 1, & j\in B; \\ 0, & j\notin B; \end{cases}$ $a_j, b_j \in \mathscr{R}, j\in\{1,\cdots,n\}, i=\sqrt{-1}$ be an imaginary number, $\mathscr{R}$ be a real number field.

Each $\delta_j^A$ and $\delta_j^B$ are extracted non-trivial measurements to satisfy relevant pairs of combinations in feature spaces.

### 7.1 Various Equations

Using $H(\langle A,B\rangle X)$ expressions, four special measurements can be produced.

$$H(\langle\emptyset,\emptyset\rangle X) = (a_0, b_0) = (a_0 + i \cdot b_0) \quad (76)$$
$$H(\langle I,\emptyset\rangle X) = (a_I, b_0) = (a_I + i \cdot b_0) \quad (77)$$
$$H(\langle\emptyset,I\rangle X) = (a_0, b_I) = (a_0 + i \cdot b_I) \quad (78)$$
$$H(\langle I,I\rangle X) = (a_I, b_I) = (a_I + i \cdot b_I) \quad (79)$$

From a measurement viewpoint, selecting a measurement and its complement measurement have opposite values.

$$H(\langle\emptyset,\emptyset\rangle X) + H(\langle I,I\rangle X) = H(\langle I,\emptyset\rangle X) + H(\langle\emptyset,I\rangle X) = 0$$
$$\Rightarrow a_0 = -a_I, b_0 = -b_I$$

For a selection $H(\langle A,B\rangle X)$,

$$H(\langle A,B\rangle X) + H(\langle A',B'\rangle X) = 0$$
$$\Rightarrow a_A = -a_{A'}, b_B = -b_{B'}$$

## 7.2 Hypercomplex Numbers and Octonion in CTS

Using such corresponding relationships, a list of formulas can be established and shown in Table 1.

**Table 1** Eight Expressions in Complex Equation, Matrix and Conjugate Transformation Structure

| Complex Op. | Complex Exp. | Complex Formula | Conjugate Op. | Matrix Base | Conjugate Formula |
|---|---|---|---|---|---|
| $1$ | $1 \cdot H(.)$ | $H(\langle A,B\rangle X)$ | $E = \zeta^0 = \zeta^4$ | $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $H(\langle A,B\rangle X)$ |
| $i$ | $i \cdot H(.)$ | $i \cdot H(\langle A,B\rangle X)$ | $\zeta$ | $e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $H(\langle B',A\rangle X)$ |
| $i^2 = -1$ | $i^2 \cdot H(.)$ | $i^2 \cdot H(\langle A,B\rangle X)$ | $\zeta^2 = \neg E$ | $e_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $H(\langle A',B'\rangle X)$ |
| $i^3 = -i$ | $i^3 \cdot H(.)$ | $i^3 \cdot H(\langle A,B\rangle X)$ | $\zeta^3 = \neg\zeta$ | $e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $H(\langle B,A'\rangle X)$ |
| $(\leftrightarrow)$ | $(i^3 \cdot H(.))^*$ | $(i^3 \cdot H(\langle A,B\rangle X))^*$ | $\sim$ | $e_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $H(\langle B,A\rangle X)$ |
| $i \cdot (\leftrightarrow)$ | $i \cdot (i^3 \cdot H(.))^*$ | $i \cdot (i^3 \cdot H(\langle A,B\rangle X))^*$ | $\zeta \sim$ | $e_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ | $H(\langle A',B\rangle X)$ |
| $i^2 \cdot (\leftrightarrow)$ | $i^2 \cdot (i^3 \cdot H(.))^*$ | $i^2 \cdot (i^3 \cdot H(\langle A,B\rangle X))^*$ | $\zeta^2 \sim$ | $e_6 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ | $H(\langle B',A'\rangle X)$ |
| $i^3 \cdot (\leftrightarrow)$ | $i^3 \cdot (i^3 \cdot H(.))^*$ | $i^3 \cdot (i^3 \cdot H(\langle A,B\rangle X))^*$ | $\zeta^3 \sim$ | $e_7 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $H(\langle A,B'\rangle X)$ |

where abbreviations are used: Op - Operator; Complex Exp - Complex Expression; Complex Comp - Complex Computation; Conj Op - Conjugate Operator; Conj Exp - Conjugate Expression; Conj Comp - Conjugate Computation.

Using the eight base matrices, two representations of Octonion groups can be identified, as shown in Table 2.

**Table 2** Octonion Groups in two representations (a) Octonion bases (b) $\pm$ four bases

(a):

| $e_i e_j$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | $e_2$ | $e_3$ | $e_0$ | $e_7$ | $e_4$ | $e_5$ | $e_6$ |
| $e_2$ | $e_2$ | $e_3$ | $e_0$ | $e_1$ | $e_6$ | $e_7$ | $e_4$ | $e_5$ |
| $e_3$ | $e_3$ | $e_0$ | $e_1$ | $e_2$ | $e_5$ | $e_6$ | $e_7$ | $e_4$ |
| $e_4$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_0$ | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_5$ | $e_6$ | $e_7$ | $e_4$ | $e_3$ | $e_0$ | $e_1$ | $e_2$ |
| $e_6$ | $e_6$ | $e_7$ | $e_4$ | $e_5$ | $e_2$ | $e_3$ | $e_0$ | $e_1$ |
| $e_7$ | $e_7$ | $e_4$ | $e_5$ | $e_6$ | $e_1$ | $e_2$ | $e_3$ | $e_0$ |

(b):

| $e_i e_j$ | $e_0$ | $e_1$ | $-e_0$ | $-e_1$ | $e_4$ | $e_5$ | $-e_4$ | $-e_5$ |
|---|---|---|---|---|---|---|---|---|
| $e_0$ | $e_0$ | $e_1$ | $-e_0$ | $-e_1$ | $e_4$ | $e_5$ | $-e_4$ | $-e_5$ |
| $e_1$ | $e_1$ | $-e_0$ | $-e_1$ | $e_0$ | $-e_5$ | $e_4$ | $e_5$ | $-e_4$ |
| $-e_0$ | $-e_0$ | $-e_1$ | $e_0$ | $e_1$ | $-e_4$ | $-e_5$ | $e_4$ | $e_5$ |
| $-e_1$ | $-e_1$ | $e_0$ | $e_1$ | $-e_0$ | $e_5$ | $-e_4$ | $-e_5$ | $e_4$ |
| $e_4$ | $e_4$ | $e_5$ | $-e_4$ | $-e_5$ | $e_0$ | $e_1$ | $-e_0$ | $-e_1$ |
| $e_5$ | $e_5$ | $-e_4$ | $-e_5$ | $e_4$ | $-e_1$ | $e_0$ | $e_1$ | $-e_0$ |
| $-e_4$ | $-e_4$ | $-e_5$ | $e_4$ | $e_5$ | $-e_0$ | $-e_1$ | $e_0$ | $e_1$ |
| $-e_5$ | $-e_5$ | $e_4$ | $e_5$ | $-e_4$ | $e_1$ | $-e_0$ | $-e_1$ | $e_0$ |

## *7.3 Eight Corresponding Complex Number Formula*

Using complex operators $\{i, *, (r \leftrightarrow i)\}$ (imaginary number, complex conjugation and imaginary & real exchange) and conjugate operators $\{\zeta, \sim\}$ (twist and conjugate), eight equations are established as follows.

$$\begin{aligned}
H(\langle A,B\rangle X) &= a_A + i\cdot b_B = i^4\cdot(a_A + i\cdot b_B) = i^4\cdot H(\langle A,B\rangle X)\\
&= H(\zeta^4\langle A,B\rangle X) \qquad (80)\\
H(\langle B',A\rangle X) &= b_{B'} + i\cdot a_A = -b_B + i\cdot a_A = i\cdot(a_A + i\cdot b_B) = i\cdot H(\langle A,B\rangle X)\\
&= H(\zeta\langle A,B\rangle X) \qquad (81)\\
H(\langle A',B'\rangle X) &= a_{A'} + i\cdot b_{B'} = -a_A - i\cdot b_B = i^2\cdot(a_A + i\cdot b_B) = i^2\cdot H(\langle A,B\rangle X)\\
&= H(\zeta^2\langle A,B\rangle X) \qquad (82)\\
H(\langle B,A'\rangle X) &= b_B + i\cdot a_{A'} = b_B - i\cdot a_A = i^3\cdot(a_A + i\cdot b_B) = i^3\cdot H(\langle A,B\rangle X)\\
&= H(\zeta^3\langle A,B\rangle X) \qquad (83)\\
H(\langle B,A\rangle X) &= b_B + i\cdot a_A = (i^3\cdot H(\langle A,B\rangle X))^*\\
&= H(\sim\langle A,B\rangle X) \qquad (84)\\
H(\langle A',B\rangle X) &= a_{A'} + i\cdot b_B = -a_A + i\cdot b_B = i\cdot H(\langle B,A\rangle X)\\
&= H(\zeta\langle B,A\rangle X) \qquad (85)\\
H(\langle B',A'\rangle X) &= b_{B'} + i\cdot a_{A'} = -b_B - i\cdot a_A = i^2\cdot(b_B + i\cdot a_A) = i^2\cdot H(\langle B,A\rangle X)\\
&= H(\zeta^2\langle B,A\rangle X) \qquad (86)\\
H(\langle A,B'\rangle X) &= a_A + i\cdot b_{B'} = a_A - i\cdot b_B = i^3\cdot H(\langle B,A\rangle X)\\
&= H(\zeta^3\langle B,A\rangle X) \qquad (87)
\end{aligned}$$

**Theorem 18.** *For a selected $H(\langle A,B\rangle X)$, eight complex operators and eight conjugate measurement operators have 1-1 correspondence in equivalent relationships.*

*Proof.* Check Equations (81)-(87), they are 1-1 equivalent properties.

Under this correspondence, eight operators satisfy both the complex and conjugate formulas in comparison listed in Table 3.

**Table 3** Eight operations in both the complex and conjugate fornumla in comparison

| Op | Complex Exp | Complex Comp | Conj Op | Conj Exp | Conj Comp | Notes |
|---|---|---|---|---|---|---|
| $1$ | $H(\langle A,B\rangle X)$ | $H(\langle A,B\rangle X)$ | $E$ | $H(\langle A,B\rangle X)$ | $H(\langle A,B\rangle X)$ | Original |
| $i$ | $H^i(\langle A,B\rangle X)$ | $i\cdot H(\langle A,B\rangle X)$ | $\zeta$ | $H(\langle B',A\rangle X)$ | $H(\zeta\langle A,B\rangle X)$ | Twist |
| $i^2$ | $H^{i^2}(\langle A,B\rangle X)$ | $i^2\cdot H(\langle A,B\rangle X)$ | $\zeta^2=\neg$ | $H(\langle A',B'\rangle X)$ | $H(\zeta^2\langle A,B\rangle X)$ | Negtive |
| $i^3$ | $H^{i^3}(\langle A,B\rangle X)$ | $i^3\cdot H(\langle A,B\rangle X)$ | $\zeta^3=\neg\zeta$ | $H(\langle B,A'\rangle X)$ | $H(\zeta^3\langle A,B\rangle X)$ | Negtive-Twist |
| $\leftrightarrow$ | $H^{\leftrightarrow}(\langle A,B\rangle X)$ | $(i^3\cdot H(\langle A,B\rangle X))^*$ | $\sim$ | $H(\langle B,A\rangle X)$ | $\tilde{H}(\langle A,B\rangle X)$ | Conjugate |
| $r$ | $H^r(\langle A,B\rangle X)$ | $i\cdot(i^3\cdot H(\langle A,B\rangle X))^*$ | $\zeta\sim$ | $H(\langle A',B\rangle X)$ | $\tilde{H}(\zeta\langle A,B\rangle X)$ | Real-Conjugate |
| $i^2\leftrightarrow$ | $H^{i^2\leftrightarrow}(\langle A,B\rangle X)$ | $(i\cdot H(\langle A,B\rangle X))^*$ | $\zeta^2\sim$ | $H(\langle B',A'\rangle X)$ | $\tilde{H}(\zeta^2\langle A,B\rangle X)$ | Negtive-Conjugate |
| $*$ | $H^*(\langle A,B\rangle X)$ | $i\cdot(i\cdot H(\langle A,B\rangle X))^*$ | $\zeta^3\sim$ | $H(\langle A,B'\rangle X)$ | $\tilde{H}(\zeta^3\langle A,B\rangle X)$ | Complex-Conjugate |

Where abbreviations are used: Op - Operator; Complex Exp - Complex Expression; Complex Comp - Complex Computation; Conj Op - Conjugate Operator; Conj Exp - Conjugate Expression; Conj Comp - Conjugate Computation.

## 8 Variations and Measurements under Complex-Conjugate Operators in CTS

For convenient comparison, measuring equations are investigated mainly by applying $*$ the complex-conjugate operator to observe various variations of a pair of complex-conjugate expressions between selected feature vectors.

Let $X_j, 0 \leq j \leq 7$ be eight sets of feature vectors related to $\langle A,B\rangle X$ on conjugate and complementary operators on eight index combinations.

$$\begin{aligned}
X_0 &= \langle A,B\rangle X\\
X_1 &= \langle A,B'\rangle X\\
X_2 &= \langle A',B\rangle X\\
X_3 &= \langle A',B'\rangle X\\
X_4 &= \langle B,A\rangle X\\
X_5 &= \langle B,A'\rangle X\\
X_6 &= \langle B',A\rangle X\\
X_7 &= \langle B',A'\rangle X
\end{aligned}$$

Let $Y_j = X_0|X_j$ be the $j$-th pair of variables. $Y_j$ is initially from $X_0$ to applying $X_j$ as a selected internal variable to generate a pair of symmetric and anti-symmetric forms.

Its reversed form is expressed as $Y_j^{-1} = X_j|X_0$.

Let $\{f, f^*\}$ be a pair of complex-conjugate measuring operators, and each operator contains a pair of two operators $\{u,v\}$ to satisfy the following equations using pairs of complex brackets $(,)$.

$$f = u+iv = (u,v) \tag{88}$$
$$f^* = u-iv = (u,-v) \tag{89}$$

Let $\{z,z^*\}$ be a pair of complex-conjugate measurements,

$$z = \alpha+i\beta = (\alpha,\beta) \tag{90}$$
$$z^* = \alpha-i\beta = (\alpha,-\beta) \tag{91}$$

Let $\Delta f, \Delta f^*, \Delta u, \Delta u^*, \Delta v, \Delta v^*$ be pairs of variations for measurements.

$$\Delta u(Y_j) = \tfrac{1}{2}[H(X_0)+H(X_j)] \quad \text{Symmetric Form} \tag{92}$$
$$\Delta v(Y_j) = \tfrac{1}{2}[H(X_0)-H(X_j)] \quad \text{Anti-Symmetric Form} \tag{93}$$
$$\Delta f(Y_j) = (\Delta u(Y_j), \Delta v(Y_j)) \quad \text{Measurement} \tag{94}$$
$$\Delta u^*(Y_j) = \Delta u(Y_j) \quad \text{Complex-Conjugate Symmetric Form} \tag{95}$$
$$\Delta v^*(Y_j) = -\Delta v(Y_j) \quad \text{Complex-Conjugate Anti-Symmetric Form} \tag{96}$$
$$\Delta f^* = (\Delta f)^* = (\Delta u^*, -\Delta v^*) \quad \text{Complex-Conjugate Measurement} \tag{97}$$

Let $\frac{\Delta f(Y_j)}{\Delta z}, \frac{\Delta u(Y_j)}{\Delta z}, \frac{\Delta v(Y_j)}{\Delta z}$ be derivative of complex measurements,

$$\frac{\Delta u(Y_j)}{\Delta z} = \left(\frac{\Delta u(Y_j)}{\Delta \alpha}, -\frac{\Delta u(Y_j)}{\Delta \beta}\right) \tag{98}$$

$$\frac{\Delta v(Y_j)}{\Delta z} = \left(\frac{\Delta v(Y_j)}{\Delta \alpha}, -\frac{\Delta v(Y_j)}{\Delta \beta}\right) \tag{99}$$

$$\frac{\Delta f(Y_j)}{\Delta z} = \left(\frac{\Delta u(Y_j)}{\Delta z}, -\frac{\Delta v(Y_j)}{\Delta z}\right) \tag{100}$$

Let $\frac{\Delta f^*(Y_j)}{\Delta z^*}, \frac{\Delta u^*(Y_j)}{\Delta z^*}, \frac{\Delta v^*(Y_j)}{\Delta z^*}$ be derivative of complex-conjugate measurements,

$$\frac{\Delta u^*(Y_j)}{\Delta z^*} = \left(\frac{\Delta u^*(Y_j)}{\Delta \alpha}, -\frac{\Delta u^*(Y_j)}{\Delta \beta}\right) = \left(\frac{\Delta u(Y_j)}{\Delta \alpha}, -\frac{\Delta u(Y_j)}{\Delta \beta}\right) \tag{101}$$

$$\frac{\Delta v^*(Y_j)}{\Delta z^*} = \left(\frac{\Delta v^*(Y_j)}{\Delta \alpha}, -\frac{\Delta v^*(Y_j)}{\Delta \beta}\right) = \left(-\frac{\Delta v(Y_j)}{\Delta \alpha}, \frac{\Delta v(Y_j)}{\Delta \beta}\right) \tag{102}$$

$$\frac{\Delta f^*(Y_j)}{\Delta z^*} = \left(\frac{\Delta u^*(Y_j)}{\Delta z^*}, \frac{\Delta v^*(Y_j)}{\Delta z^*}\right) \tag{103}$$

Let $\left[\frac{\Delta f(Y_j)}{\Delta z}, \frac{\Delta f^*(Y_j)}{\Delta z^*}\right]$ be derivative of an anti-symmetric operator,

$$\left[\frac{\Delta f(Y_j)}{\Delta z}, \frac{\Delta f^*(Y_j)}{\Delta z^*}\right] = \frac{\Delta f(Y_j)}{\Delta z} - \frac{\Delta f^*(Y_j)}{\Delta z^*} \tag{104}$$

Let $\left\{\frac{\Delta f(Y_j)}{\Delta z}, \frac{\Delta f^*(Y_j)}{\Delta z^*}\right\}$ be derivative of a symmetric operator,

$$\left\{\frac{\Delta f(Y_j)}{\Delta z}, \frac{\Delta f^*(Y_j)}{\Delta z^*}\right\} = \frac{\Delta f(Y_j)}{\Delta z} + \frac{\Delta f^*(Y_j)}{\Delta z^*} \tag{105}$$

Using this set of equations, a case of $Y_0$ on their differences and derivative are evaluated.

$$
\begin{aligned}
\Delta u(Y_0) &= (a_A, b_B) \\
\Delta v(Y_0) &= (0,0) \\
\frac{\Delta u}{\Delta z}(Y_0) &= (a_A, 0) - i(0, b_B) = (a_A + b_B, 0) \\
\frac{\Delta v}{\Delta z}(Y_0) &= (0,0) \\
\frac{\Delta f}{\Delta z}(Y_0) &= (a_A + b_B, 0) \\
\Delta u^*(Y_0) &= (a_A, -b_B) \\
\Delta v^*(Y_0) &= (0,0) \\
\frac{\Delta u^*}{\Delta z^*}(Y_0) &= (a_A, 0) + i(0, -b_B) = (a_A + b_B, 0) \\
\frac{\Delta v^*}{\Delta z^*}(Y_0) &= (0,0) \\
\frac{\Delta f^*}{\Delta z^*}(Y_0) &= (a_A + b_B, 0) \\
\left[\frac{\Delta f(Y_0)}{\Delta z}, \frac{\Delta f^*(Y_0)}{\Delta z^*}\right] &= (0,0) \\
\left\{\frac{\Delta f(Y_0)}{\Delta z}, \frac{\Delta f^*(Y_0)}{\Delta z^*}\right\} &= 2(a_A + b_B, 0)
\end{aligned}
$$

when $a_A = -b_B$ then $\left\{\frac{\Delta f(Y_0)}{\Delta z}, \frac{\Delta f^*(Y_0)}{\Delta z^*}\right\} = (0,0)$

## 8.1 Classical Hamilton Operators and Complex Conjugate Operators

In convenient comparison, classical Hamilton operators in differential equations [9, 13] and complex-conjugate operators in CTS on discrete derivatives are listed in Table 4. It is possible to check their similarities from expressions.

For various cases on both $\{Y_0, \cdots, Y_7\}$ and $\{Y_0^{-1}, \cdots, Y_7^{-1}\}$, sixteen measurements of variations and derivatives are listed in Table 5.

**Theorem 19.** *There are four cases of* $\{Y_0, Y_4, Y_0^{-1}, Y_4^{-1}\}$*, and symmetric expressions of their* $\{,\}$ *measurements satisfy classical Hamiltonian properties in complex-conjugate systems as discrete systems.*

*Proof.* Checking Table 5, each classical Hamiltonian operator has one by one a corresponding difference or derivative form one by one.

**Table 4** Classical Hamiltonian Operators and Complex-Conjugate Operators in CTS

| H Op | H Formula | CC Op | CC Formula |
|---|---|---|---|
| $z,z^*$ | $z=x+iy, z^*=x-iy$ | $z,z^*$ | $z=\alpha+i\beta, z^*=\alpha-i\beta$ |
| $x,y$ | $x=\frac{z+z*}{2}, y=\frac{z-z*}{2}$ | $\alpha,\beta$ | $\alpha=\frac{z+z*}{2}, \beta=\frac{z-z*}{2}$ |
| $P$ | $P(x,y)=\frac{dx}{dt}$ | $\Delta u$ | $\Delta u(Y_j)=\frac{1}{2}(H(\langle A,B\rangle X_0)+H(\langle A,B\rangle X_j)$ |
| $P^*$ | $P^*(x,y)=P(x,y)$ | $\Delta u^*$ | $\Delta u^*=\Delta u$ |
| $Q$ | $Q(x,y)=\frac{dy}{dt}$ | $\Delta v$ | $\Delta v(Y_j)=\frac{1}{2}(H(\langle A,B\rangle X_0)-H(\langle A,B\rangle X_j)$ |
| $Q^*$ | $Q^*(x,y)=-Q(x,y)$ | $\Delta v^*$ | $\Delta v^*=-\Delta v$ |
| $F$ | $F(z,z^*)=P(x,y)+iQ(x,y)$ | $f$ | $f=u+iv$ |
| $F^*$ | $F^*(z,z^*)=P^*(x,y)-iQ^*(x,y)$ | $f^*$ | $f^*=u^*-iv^*$ |
| $dz,dz^*$ | $dz=dx+idy, dz^*=dx-idy$ | $\Delta z,\Delta z^*$ | $\Delta z=\Delta\alpha+i\Delta\beta, \Delta z^*=\Delta\alpha-i\Delta\beta$ |
| $dF,dF^*$ | $dF=dP+idQ, dF^*=dP-idQ$ | $\Delta f,\Delta f^*$ | $\Delta f=\Delta u+i\Delta v, \Delta f^*=\Delta u-i\Delta v$ |
| $\frac{dF}{dz},\frac{dF^*}{dz^*}$ | $\frac{dF}{dz}=\frac{dP}{dz}-i\frac{dQ}{dz}, \frac{dP^*}{dz^*}=\frac{dP^*}{dz^*}+i\frac{dQ^*}{dz^*}$ | $\frac{\Delta f}{\Delta z},\frac{\Delta f^*}{\Delta z^*}$ | $\frac{\Delta f}{\Delta z}=\frac{\Delta u}{\Delta z}-i\frac{\Delta v}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}=\frac{\Delta u^*}{\Delta z^*}+i\frac{\Delta v^*}{\Delta z^*}$ |
| $\frac{dP}{dz},\frac{dP^*}{dz^*}$ | $\frac{dP}{dz}=\frac{dP}{dx}-i\frac{dP}{dy}, \frac{dP^*}{dz^*}=\frac{dP^*}{dx}-i\frac{dP^*}{dy}$ | $\frac{\Delta u}{\Delta z},\frac{\Delta u^*}{\Delta z^*}$ | $\frac{\Delta u}{\Delta z}=\frac{\Delta u}{\Delta\alpha}-i\frac{\Delta u}{\Delta\beta}, \frac{\Delta u^*}{\Delta z^*}=\frac{\Delta u^*}{\Delta\alpha}-i\frac{\Delta u^*}{\Delta\beta}$ |
| $\frac{dQ}{dz},\frac{dQ^*}{dz^*}$ | $\frac{dQ}{dz}=\frac{dQ}{dx}-i\frac{dQ}{dy}, \frac{dQ^*}{dz^*}=\frac{dQ^*}{dx}-i\frac{dQ^*}{dy}$ | $\frac{\Delta v}{\Delta z},\frac{\Delta v^*}{\Delta z^*}$ | $\frac{\Delta v}{\Delta z}=\frac{\Delta v}{\Delta\alpha}-i\frac{\Delta v}{\Delta\beta}, \frac{\Delta v^*}{\Delta z^*}=\frac{\Delta v^*}{\Delta\alpha}-i\frac{\Delta v^*}{\Delta\beta}$ |
| $[\frac{dF}{dz},\frac{dF^*}{dz^*}]$ | $[\frac{dF}{dz},\frac{dF^*}{dz^*}]=\frac{dF}{dz}-\frac{dF^*}{dz^*}=0$ | $[\frac{\Delta f}{\Delta z},\frac{\Delta f^*}{\Delta z^*}]$ | $[\frac{\Delta f}{\Delta z},\frac{\Delta f^*}{\Delta z^*}]=\frac{\Delta f}{\Delta z}-\frac{\Delta f^*}{\Delta z^*}=0$ |
| $\{\frac{dF}{dz},\frac{dF^*}{dz^*}\}$ | $\{\frac{dF}{dz},\frac{dF^*}{dz^*}\}=\frac{dF}{dz}+\frac{dF^*}{dz^*}=0$ | $\frac{\Delta f}{\Delta z},\frac{\Delta f^*}{\Delta z^*}$ | $\{\frac{\Delta f}{\Delta z},\frac{\Delta f^*}{\Delta z^*}\}=\frac{\Delta f}{\Delta z}+\frac{\Delta f^*}{\Delta z^*}=0$ |

where H Op: Hamilton Operators; H Formula: Hamilton Formula; CC Op: Complex-conjugate Operator;CC Formula: Complex conjugate Formula.

## *8.2 Conjugate and Complex-Conjugate Operators*

From an operational viewpoint, the complex-conjugate operator is only one of eight conjugate operators. In Table, the conjugate operator and complex-conjugate operators are compared on discrete dynamics. It is possible to see their similarities and refined differences. Further investigations and applications are required.

## *8.3 One Example*

An interesting example is selected from [44] on page 951 equations (12-13), Professor Chern deduced the Yang-Mills equation in the $q=1$ condition.

**Table 5** Sixteen Sets of Variations under Complex-Conjugate Operators in CTS

| $Y$ | $\Delta u(Y)$ | $\Delta v(Y)$ | $\frac{\Delta u(Y)}{\Delta z}$ | $\frac{\Delta v(Y)}{\Delta z}$ | $\Delta u^*(Y)$ | $\Delta v^*(Y)$ |
|---|---|---|---|---|---|---|
| $Y_0$ | $(a_A, b_B)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A, -b_B)$ | $(0,0)$ |
| $Y_1$ | $(a_A, 0)$ | $(0, b_B)$ | $(a_A, 0)$ | $(b_B, 0)$ | $(a_A, 0)$ | $(0, -b_B)$ |
| $Y_2$ | $(0, b_B)$ | $(a_A, 0)$ | $(b_B, 0)$ | $(a_A, 0)$ | $(0, -b_B)$ | $(a_A, 0)$ |
| $Y_3$ | $(0,0)$ | $(a_A, b_B)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A, -b_B)$ |
| $Y_4$ | $\frac{1}{2}(a_A + b_B, a_A + b_B)$ | $\frac{1}{2}(a_A - b_B, b_B - a_A)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $\frac{1}{2}(a_A + b_B, -a_A - b_B)$ | $\frac{1}{2}(a_A - b_B, a_A - b_B)$ |
| $Y_5$ | $\frac{1}{2}(a_A + b_B, b_B - a_A)$ | $\frac{1}{2}(a_A - b_B, b_B + a_A)$ | $(b_B, 0)$ | $(a_A, 0)$ | $\frac{1}{2}(a_A + b_B, a_A - b_B))$ | $\frac{1}{2}(a_A - b_B, -a_A - b_B)$ |
| $Y_6$ | $\frac{1}{2}(a_A - b_B, b_B + a_A)$ | $\frac{1}{2}(a_A + b_B, b_B - a_A)$ | $(a_A, 0)$ | $(b_B, 0)$ | $\frac{1}{2}(a_A - b_B, -a_A - b_B)$ | $\frac{1}{2}(a_A + b_B, a_A - b_B)$ |
| $Y_7$ | $\frac{1}{2}(a_A - b_B, b_B - a_A)$ | $\frac{1}{2}(a_A + b_B, b_B + a_A)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $\frac{1}{2}(a_A - b_B, a_A - b_B))$ | $\frac{1}{2}(a_A + b_B, -a_A - b_B)$ |
| $Y_0^{-1}$ | $(a_A, b_B)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A, -b_B)$ | $(0,0)$ |
| $Y_1^{-1}$ | $(a_A, 0)$ | $(0, -b_B)$ | $(a_A, 0)$ | $(-b_B, 0)$ | $(a_A, 0)$ | $(0, b_B)$ |
| $Y_2^{-1}$ | $(0, b_B)$ | $(-a_A, 0)$ | $(b_B, 0)$ | $(-a_A, 0)$ | $(0, -b_B)$ | $(-a_A, 0)$ |
| $Y_3^{-1}$ | $(0,0)$ | $(-a_A, -b_B)$ | $(0,0)$ | $(-a_A - b_B, 0)$ | $(0,0)$ | $(-a_A, b_B)$ |
| $Y_4^{-1}$ | $\frac{1}{2}(a_A + b_B, a_A + b_B)$ | $\frac{1}{2}(b_B - a_A, a_A - b_B)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $\frac{1}{2}(a_A + b_B, -a_A - b_B)$ | $\frac{1}{2}(b_B - a_A, b_B - a_A)$ |
| $Y_5^{-1}$ | $\frac{1}{2}(a_A + b_B, b_B - a_A)$ | $\frac{1}{2}(a_A - b_B, -b_B - a_A)$ | $(b_B, 0)$ | $(-a_A, 0)$ | $\frac{1}{2}(a_A + b_B, a_A - b_B)$ | $\frac{1}{2}(b_B - a_A, a_A + b_B)$ |
| $Y_6^{-1}$ | $\frac{1}{2}(a_A - b_B, b_B + a_A)$ | $\frac{1}{2}(-a_A - b_B, a_A - b_B)$ | $(a_A, 0)$ | $(-b_B, 0)$ | $\frac{1}{2}(a_A - b_B, -a_A - b_B)$ | $\frac{1}{2}(-a_A - b_B, b_B - a_A)$ |
| $Y_7^{-1}$ | $\frac{1}{2}(a_A - b_B, b_B - a_A)$ | $\frac{1}{2}(a_A + b_B, -b_B - a_A)$ | $(0,0)$ | $(-a_A - b_B, 0)$ | $\frac{1}{2}(a_A - b_B, a_A - b_B))$ | $\frac{1}{2}(-a_A - b_B, a_A + b_B)$ |

| $Y$ | $\frac{\Delta u^*(Y)}{\Delta z^*}$ | $\frac{\Delta v^*(Y)}{\Delta z^*}$ | $\frac{\Delta f(Y)}{\Delta z}$ | $\frac{\Delta f^*(Y)}{\Delta z^*}$ | $[,]$ | $\{,\}$ |
|---|---|---|---|---|---|---|
| $Y_0$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $2(a_A + b_B, 0)$ |
| $Y_1$ | $(a_A, 0)$ | $(b_B, 0)$ | $(a_A, -b_B)$ | $(a_A, b_B)$ | $2(0, -b_B)$ | $2(a_A, 0)$ |
| $Y_2$ | $(b_B, 0)$ | $(a_A, 0)$ | $(b_B, -a_A)$ | $(b_B, a_A)$ | $2(0, -a_A)$ | $2(b_B, 0)$ |
| $Y_3$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(0, -a_A - b_B)$ | $(0, a_A + b_B)$ | $2(0, -a_A - b_B)$ | $(0,0)$ |
| $Y_4$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $2(a_A + b_B, 0)$ |
| $Y_5$ | $(b_B, 0)$ | $(a_A, 0)$ | $(b_B, -a_A)$ | $(b_B, a_A)$ | $2(0, -a_A)$ | $2(b_B, 0)$ |
| $Y_6$ | $(a_A, 0)$ | $(b_B, 0)$ | $(a_A, -b_B)$ | $(a_A, b_B)$ | $2(0, -b_B)$ | $2(a_A, 0)$ |
| $Y_7$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(0, -a_A - b_B)$ | $(0, a_A + b_B)$ | $2(0, -a_A - b_B)$ | $(0,0)$ |
| $Y_0^{-1}$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $2(a_A + b_B, 0)$ |
| $Y_1^{-1}$ | $(a_A, 0)$ | $(-b_B, 0)$ | $(a_A, b_B)$ | $(a_A, -b_B)$ | $2(0, b_B)$ | $2(a_A, 0)$ |
| $Y_2^{-1}$ | $(b_B, 0)$ | $(-a_A, 0)$ | $(b_B, a_A)$ | $(b_B, -a_A)$ | $2(0, a_A)$ | $2(b_B, 0)$ |
| $Y_3^{-1}$ | $(0,0)$ | $(-a_A - b_B, 0)$ | $(0, a_A + b_B)$ | $(0, -a_A - b_B)$ | $2(0, a_A + b_B)$ | $(0,0)$ |
| $Y_4^{-1}$ | $(a_A + b_B, 0)$ | $(0,0)$ | $(a_A + b_B, 0)$ | $(a_A + b_B, 0)$ | $(0,0)$ | $2(a_A + b_B, 0)$ |
| $Y_5^{-1}$ | $(b_B, 0)$ | $(-a_A, 0)$ | $(b_B, a_A)$ | $(b_B, -a_A)$ | $2(0, a_A)$ | $2(b_B, 0)$ |
| $Y_6^{-1}$ | $(a_A, 0)$ | $(-b_B, 0)$ | $(a_A, b_B)$ | $(a_A, -b_B)$ | $2(0, b_B)$ | $2(a_A, 0)$ |
| $Y_7^{-1}$ | $(0,0)$ | $(-a_A - b_B, 0)$ | $(0, a_A + b_B)$ | $(0, -a_A - b_B)$ | $2(0, a_A + b_B)$ | $(0,0)$ |

When $q = 1$, the Yang-Mills equations become the Maxwell equations, Ref[2]. In this case the group is $U(1)$, which is abelian.

Since $*^2 = 1, \Omega$ splits:

$$\Omega = \Omega^+ + \Omega^-, \tag{12}$$

where

$$*\Omega^+ = \Omega^+, *\Omega^- = -\Omega^-. \tag{13}$$

It follows that

$$YM = ||\Omega^+||^2 + ||\Omega^-||^2$$

...

Ref: [2] S.S. Chern, Vector bundles with a connection, Global Differential Geometry, Math. Ass.of America, 1989, 23-25.

**Table 6** Conjugate Operators and Complex-Conjugate Operators in CTS

| C Op | Formula | CC Op | Formula |
|---|---|---|---|
| $z, \tilde{z}$ | $z = (\alpha, \beta), \tilde{z} = (\beta, \alpha)$ | $z, z^*$ | $z = (\alpha, \beta), z^* = (\alpha, -\beta)$ |
| $X_j, \tilde{X}_j$ | $\tilde{X}_j = \sim (X_j) = (X_j)^\sim$ | $\alpha, \beta$ | $\alpha = \frac{z+z*}{2}, \beta = \frac{z-z*}{2}$ |
| $\Delta u$ | $u(X_j, \tilde{X}_j) = \frac{1}{2}[H(X_j) + H(\tilde{X}_j)]$ | $\Delta u$ | $\Delta u(Y_j) = \frac{1}{2}[H(X_0) + H(X_j)]$ |
| $\Delta \tilde{u}$ | $\Delta \tilde{u} = \Delta u$ | $\Delta u^*$ | $\Delta u^* = \Delta u$ |
| $\Delta v$ | $v(X_j, \tilde{X}_j) = \frac{1}{2}[H(X_j) - H(\tilde{X}_j)]$ | $\Delta v$ | $\Delta v(Y_j) = \frac{1}{2}[H(X_0) - H(X_j)]$ |
| $\Delta \tilde{v}$ | $\Delta \tilde{v} = -\Delta v$ | $\Delta v^*$ | $\Delta v^* = -\Delta v$ |
| $f$ | $f = (u, v)$ | $f$ | $f = (u, v)$ |
| $\tilde{f}$ | $\tilde{f} = (\tilde{v}, \tilde{u})$ | $f^*$ | $f^* = (u^*, -v^*)$ |
| $\Delta z, \Delta \tilde{z}$ | $\Delta z = (\Delta \alpha, \Delta \beta), \Delta \tilde{z} = (\Delta \beta, \Delta \alpha)$ | $\Delta z, \Delta z^*$ | $\Delta z = (\Delta \alpha, \Delta \beta), \Delta z^* = (\Delta \alpha, -\Delta \beta)$ |
| $\Delta f, \Delta \tilde{f}$ | $\Delta f = (\Delta u, \Delta v), \Delta \tilde{f} = (\Delta \tilde{v}, \Delta \tilde{u})$ | $\Delta f, \Delta f^*$ | $\Delta f = (\Delta u, \Delta v), \Delta f^* = (\Delta u, -\Delta v)$ |
| $\frac{\Delta f}{\Delta z}, \frac{\Delta \tilde{f}}{\Delta \tilde{z}}$ | $\frac{\Delta f}{\Delta z} = (\frac{\Delta u}{\Delta z}, -\frac{\Delta v}{\Delta z}), \frac{\Delta \tilde{f}}{\Delta \tilde{z}} = (\frac{\Delta \tilde{v}}{\Delta \tilde{z}}, -\frac{\Delta \tilde{u}}{\Delta \tilde{z}})$ | $\frac{\Delta f}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}$ | $\frac{\Delta f}{\Delta z} = \frac{\Delta u}{\Delta z} - i\frac{\Delta v}{\Delta z}, \frac{\Delta f^*}{\Delta z^*} = \frac{\Delta u^*}{\Delta z^*} + i\frac{\Delta v^*}{\Delta z^*}$ |
| $\frac{\Delta u}{\Delta z}, \frac{\Delta \tilde{v}}{\Delta \tilde{z}}$ | $\frac{\Delta u}{\Delta z} = (\frac{\Delta u}{\Delta \alpha}, -\frac{\Delta u}{\Delta \beta}), \frac{\Delta \tilde{v}}{\Delta \tilde{z}} = (-\frac{\Delta v}{\Delta \beta}, \frac{\Delta v}{\Delta \alpha})$ | $\frac{\Delta u}{\Delta z}, \frac{\Delta u^*}{\Delta z^*}$ | $\frac{\Delta u}{\Delta z} = \frac{\Delta u}{\Delta \alpha} - i\frac{\Delta u}{\Delta \beta}, \frac{\Delta u^*}{\Delta z^*} = \frac{\Delta u^*}{\Delta \alpha} - i\frac{\Delta u^*}{\Delta \beta}$ |
| $\frac{\Delta v}{\Delta z}, \frac{\Delta \tilde{u}}{\Delta \tilde{z}}$ | $\frac{\Delta v}{\Delta z} = (\frac{\Delta v}{\Delta \alpha}, -\frac{\Delta v}{\Delta \beta}), \frac{\Delta \tilde{u}}{\Delta \tilde{z}} = (\frac{\Delta u}{\Delta \beta}, -\frac{\Delta u}{\Delta \alpha})$ | $\frac{\Delta v}{\Delta z}, \frac{\Delta v^*}{\Delta z^*}$ | $\frac{\Delta v}{\Delta z} = \frac{\Delta v}{\Delta \alpha} - i\frac{\Delta v}{\Delta \beta}, \frac{\Delta v^*}{\Delta z^*} = \frac{\Delta v^*}{\Delta \alpha} - i\frac{\Delta v^*}{\Delta \beta}$ |
| $[\frac{\Delta f}{\Delta z}, \frac{\Delta \tilde{f}}{\Delta \tilde{z}}]$ | $[\frac{\Delta f}{\Delta z}, \frac{\Delta \tilde{f}}{\Delta \tilde{z}}] = \frac{\Delta f}{\Delta z} - \frac{\Delta \tilde{f}}{\Delta \tilde{z}}$ | $[\frac{\Delta f}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}]$ | $[\frac{\Delta f}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}] = \frac{\Delta f}{\Delta z} - \frac{\Delta f^*}{\Delta z^*}$ |
| $\{\frac{\Delta f}{\Delta z}, \frac{\Delta \tilde{f}}{\Delta \tilde{z}}\}$ | $\{\frac{\Delta f}{\Delta z}, \frac{\Delta \tilde{f}}{\Delta \tilde{z}}\} = \frac{\Delta f}{\Delta z} + \frac{\Delta \tilde{f}}{\Delta \tilde{z}}$ | $\frac{\Delta f}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}$ | $\{\frac{\Delta f}{\Delta z}, \frac{\Delta f^*}{\Delta z^*}\} = \frac{\Delta f}{\Delta z} + \frac{\Delta f^*}{\Delta z^*}$ |

where C Op: Conjugate Operator; CC Op: Complex Conjugate Operator.

In this case, the basic operator is complex-conjugate. The most important part of Equation (12) is separations to make two components in Equation (13) symmetric and anti-symmetric parts,; relevant operators can be listed in Table 7 for comparison.

**Table 7** Comparison of Chern operators and conjugate operators

| Chern Op | Chern Formula | Conj Op | Conj Formula | Notes |
|---|---|---|---|---|
| $*$ | $*^2 = 1$ | $*$ | $*^2 = 1$ | complex-conjugate twice |
| $\Omega$ | $\Omega = \Omega^+ + \Omega^-$ | $H$ | $H = \frac{1}{2}(\Delta u(H) + \Delta v(H))$ | Symmetric + Anti-symmetric |
| $\Omega^+$ | $*\Omega^+ = \Omega^+$ | $\Delta u(H)$ | $\Delta u^*(H) = \Delta u(H)$ | $*$ Symmetric = Symmetric |
| $\Omega^-$ | $*\Omega^- = -\Omega^-$ | $\Delta v(H)$ | $\Delta v^*(H) = -\Delta v(H)$ | $*$ Anti-symmetric = − Anti-symmetric |

where Chern Op: Chern Operator; Conj Op: Conjugate Operators.

## 9 Conclusion

Based on the requirements of cellular automaton interpretation of quantum mechanics using logic construction, advanced quantum and astranomical sciences need to use a hierarchy of multiple levels on logic systems.

Starting from 0-1 variables, pairs of conjugate states are applied to the phase spaces of the kernel form. Under a state sequence as a configuration, $2n$ classes of state groups are mapped as $2n$ 0-1 feature vectors. A reversible logic expression is established as the elementary equation for conjugate transformation. The conjugate transformation structure fully covers 1D cellular automata and its discrete dynamics as a typical example. From a global comparison viewpoint, the core component of the conjugate transformation structure is composed of a hypercomplex number system - a group with 8 elements. This group is equivalent to having 8 complex number operators. Using complex-conjugate operators, classical Hamilton operators on differential equations and conjugate operators on CTS are compared. Due to the much more efficient properties of the new structure, further explorations are required.

When a kernel form has $m = 5$, there is a function space on $2^{2^6} = 2^{64} \sim 10^{19}$ scales. Associated with configuration functions, conjugate transformation structures can provide huge ranges of hierarchical structures to cover from elementary particles to astranomic universals for various dynamics from the finest Planck scale to global universal components.

Twistor is a new type of logic operator, and both twistor and conjugate operators have more essential properties than NOT operators. The conjugate transformation structure provides a solid logic foundation on complex number systems. This type of support will be applied to quantum mechanics, quantum information and quantum gravity etc. to provide a foundation of logic systems with consistent properties to overcome various paradoxes in supporting systems.

Feature vectors are sufficient to support higher dimensional geometry, and further investigations are required. It is only a special case to use complex conjugate operators in demonstration. Other operators need to be explored. Combinatorial properties of feature vectors show time-invariant properties. It is necessary for the elementary equation to explore time variation problems. Further extending topics on dynamics will be discussed in future papers.

## Conflict Interest

No conflict of interest has been claimed.

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