Decision-Making Models Based on Satisfaction Degree with Incomplete Hesitant Fuzzy Preference Relation

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Decision-making models based on satisfaction degree with incomplete hesitant fuzzy preference relation

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Abstract To address the situation where the incomplete hesitant fuzzy preference relation (IHFPR) is necessary, this paper develops decision-making models based on decision makers' satisfaction degree with IHFPR. First, the consistency measures from the perspectives of additive and multiplicative consistent IHFPR are defined based on the relationships between the IHPFRs and their corresponding priority weight vector, respectively. Second, two decision-making models are developed in view of the proposed additive and multiplicative consistency measures. The main characteristic of the constructed models are they taking into account the decision makers' satisfaction degree. The objective functions of the models are developed by maximizing the parameter of satisfaction degree. Third, a square programming model is developed to obtain the decision makers' weights by utilizing the optimal priority weight vectors information, the solution of the model is obtained by solving the partial derivatives of Lagrange function. Finally, a procedure for multi-criteria decision-making (MCDM) problems with IHFPRs is given, and an illustrative example in conjunction with comparative analysis is used to demonstrate the proposed models are feasible and efficiency for practical MCDM problems.

Keywords Multi-criteria decision-making, incomplete hesitant fuzzy preference relation, additive consistency, multiplicative consistency, satisfaction degree

1. Introduction

Group decision making (GDM) is a specific type of decision problem where several/many decision makers cooperate with each other and choose the best solution from a set of possible alternatives (Rabiee, Aslani, & Rezaei, 2021). In the GDM process, decision makers are invited to provide evaluation information by pairwise comparisons of alternatives, and at the end, a collective decision is reached by utilizing the predetermined criteria (Rodríguez, Labella, Dutta, & Martínez, 2021). Due to the complexity of decision making environments, and the limitations of decision makers' knowledge, experience, and ability, decision makers are hesitant about certain evaluation values during assessments (Yazdani, Mohammed, Bai, & Labib, 2021). To deal with this, Torra and Narukawa (2009) introduced the concept of the hesitant fuzzy set (HFS). Since that time, an increasing amount of research on the study of HFS has been published (Gong, Liu, You, & Yin, 2021; X. Liu, Wang, Zhang, & Garg, 2021; Mishra, Rani, Krishankumar, Ravichandran, & Kar, 2021). Later, M. Xia and Xu (2013) found the advantages of hesitant fuzzy element (HFE) and introduced the concept of hesitant fuzzy preference relation (HFPR). Following the original work of M. Xia and Xu (2013), many multi-criteria decision-making (MCDM) approaches based on HFPR have been developed. A concise literature review of these approaches is presented as follows and summarized in Table 1.

According to number of fuzzy preference relation (FPR) was used to derive priority weight vector, all these MCDM approaches can be classified into four categories (Meng, Chen, & Tang, 2020): (1) Only considers one FPR derived from HFPR (Song & Li, 2019; Bin Zhu & Xu, 2014a; B Zhu, Xu, & Xu, 2014). This method also named optimistic consistency, that is, a reduced FPR with the highest consistency degree is derived from HFPR. The optimistic consistency method can reflect the highest consistency degree of HFPR, but it cannot reflect the hesitancy of decision makers. It leads to substantial information loss. (2) Based on ordered FPRs derived from normalized HFPR (H. Liu, Xu, & Liao, 2016; Z. Zhang, Wang, & Tian, 2015a, 2015b). This method also named normalized consistency. The normalized consistency requires that any two HFEs have an equal number of elements, if two HFEs have an unequal number of elements, a normalized process is needed. Therefore, the normalized consistency method may distort the original information provided by decision makers. (3) Based on all possible FPRs including in HFPR (Z. Zhang, Kou, & Dong, 2018; Z. Zhang, Kou, Yu, & Guo, 2018). This method also named average consistency. The method defines the concept of consistent HFPR used all possible FPRs, this seems too restricted. It is difficult for decision makers to provide such pairwise judgement in the actual decision-making process. (4) Based on the derived FPRs for each value in HFEs (Meng & An, 2017; Tang, An, Meng, & Chen, 2017). This method also named partial average consistency. The main feature of this method is that it considers all the evaluation information, and neither adds values into HFEs nor removes values from HFEs. Compared with (3), this method only used some possible FPRs including in HFPR.

Table 1 A summary of different consistent HFPRs

The category of different consistent HFPRs	Main characteristic	Representative literature
Optimistic consistency	Only considers one FPR derived from HFPRs	(Song & Li, 2019; B Zhu, et al., 2014)
Normalized consistency	Based on ordered FPRs derived from normalized HFPRs	(H. Liu, et al., 2016; Z. Zhang, et al., 2015a)
Average consistency	Based on all possible FPRs including in HFPRs	(Z. Zhang, X. Kou, & Q. Dong, 2018; Z. Zhang, X. Kou, W. Yu, et al., 2018)
Partial average consistency	Based on the derived FPRs for each value in HFEs	(Meng & An, 2017; Tang, et al., 2017)

Due to the lack of knowledge and decision makers' limited expertise, it may be difficult for decision makers to provide complete preference relations over alternatives (J. Liu, Li, Huang, Liu, & Liu, 2021; Wan, Yuan, & Dong, 2021; Z. Zhang & Chen, 2021a). At the end, lots of MCDM approaches have been developed to managing incomplete information (Dong, Liu, Chiclana, Kou, & Herrera-Viedma, 2019; Meng & Chen, 2021; Xie, Xu, Ren, & Herrera-Viedma, 2020). According to their principles of derived priority weight vector, all these MCDM approaches can be classified into two categories: (1) deriving the priority weight vector based on complete FPR (Z. Zhang, 2016; Z. Zhang, et al., 2015b). This method firstly obtained the missing values based on certain rules, and then derived priority weight vector from complete FPR. However, this method only apply to the situations where each alternative is compared at least once (Ding, et al., 2020). (2) Deriving the priority weight vector based on incomplete FPR (Xu, Chen, Rodríguez, Herrera, & Wang, 2016; Z. Zhang, X. Kou, W. Yu, et al., 2018). When implementing this method, the priority weight vector can be derived by using some programming models. It does not need to derive the missing values, and have ability to handle the case where ignored alternatives exist (Tang, Chen, & Meng, 2019).

Since the development of HFPRs and decision makers may provide incomplete preference relations over alternatives. It is necessary to develop some approaches to managing incomplete information for HFPRs. For that, several MCDM approaches based on incomplete HFPR (IHFPR) have been proposed (Khalid & Beg, 2017; Xu, et al., 2016; Z. Zhang, 2016; Z. Zhang, X. Kou, W. Yu, et al., 2018; Z. Zhang, et al., 2015b). For example, Z. Zhang, et al. (2015b) developed two methods to estimate the missing elements in an IHFPR based on the properties of additive consistent HFPR, while Z. Zhang (2016) in a similarly way to estimate the missing elements based on the properties of multiplicative consistent IHFPR. Xu, et al. (2016) developed two goal programming models to derive the priority weights from an IHFPR based on additive and multiplicative consistency, respectively. Z. Zhang, X. Kou, W. Yu, et al. (2018) proposed an approach to deriving a priority weight vector from an IHFPR using the logarithmic least squares method.

The concept of IHFPR has been introduced, and several scholars have studied some MCDM methods under incomplete hesitant evaluation environments. However, there are still some important issues need to be solved. (1) The concept of additive and multiplicative consistent IHFPR. As HFPR, additive and multiplicative consistent IHFPR develops in considering one FPR derived from IHFPR may lead to information loss (Z. Zhang, 2016), develops in considering ordered FPRs derived from normalized IHFPR may distort the preference information (Z. Zhang, 2016; Z. Zhang, et al., 2015b), and develops in considering all possible FPRs in IHFPR seems too restriction (Z. Zhang, X. Kou, W. Yu, et al., 2018). (2) Almost MCDM methods with IHFPR focus on checking and improving the consistency and consensus (Meng, et al., 2020). The priority vector follow the consistent IHFPR can obtain a reason ranking. However, for improving the consistency and consensus level may lead to destroy the original evaluation information (Peijia Ren, Xu, Wang, & Zeng, 2021). Moreover, the disobedience and non-cooperation behaviors may be ignored in above mentions MCDM methods (H. Zhang, Palomares, Dong, & Wang, 2018). Besides, these methods seldom consider the satisfaction degree of decision makers.

To eliminate above mention defects, the consistency measures from the perspectives of additive and multiplicative consistent IHFPR are defined based on the relationships between the IHPFRs and their corresponding weight vector, respectively. And two decision-making models are developed in view of the proposed additive and multiplicative consistency measures. The primary contributions of this study are summarized as follows.

(1) To overcome the shortcoming of additive consistent and multiplicative consistent IHFPR develops in considering a FPR derived from IHFPR, ordered FPRs derived from normalized IHFPR, and all possible FPRs including in IHFPR, a new concept of additive and multiplicative consistent IHFPR is proposed, respectively.

(2) To consider the satisfaction degree of decision makers, two decision-making models are developed based on the proposed additive and multiplicative consistency measures. The main characteristic of the constructed models is that the objective functions of the models are obtained by maximizing the parameter of satisfaction degree.

The remainder of the paper is organized as follows. In Section 2, basic concepts and operations related to FPR, HFS and HFPR are reviewed. In Section 3, the concepts of additive and multiplicative consistent IHFPR are presented, and two decision-making models are developed in view of the proposed additive and multiplicative consistency measures. In Section 4, a square programming model is developed to obtain the decision makers' weights, and a procedure for MCDM problems with IHFPR is provided. In Section 5, the proposed method is illustrated by an example, and a comparative analysis is provided. Finally, conclusions are presented in Section 6.

2. Preliminaries

To carry out the following research, this part briefly reviews some basic concepts, including the concepts of FPR, HFS, and HFPR.

2.1 FPR

Let $X=\{x_1,x_2,\cdots,x_n\}$ denotes a finite set of alternatives, where x_i represents the *i* th alternate. Orlovsky (1978) introduced the concept of FPR to represent fuzzy judgement value.

Definition 1 (Orlovsky, 1978). A FPR on a set of alternatives X is represented by a matrix $R=\left(r_{ij}\right)_{n\times n}\subset X\times X$, where r_{ij} is interpreted as the degree to which alternative x_i is preferred to x_j. Furthermore, r_{ij} should satisfy the following conditions:

$$r_{ij}+r_{ji}=1,\ \ r_{ii}=0.5 \ \text{ for all } \ i,j\in N. \tag{1}$$

To measure the rationality of FPR, the concepts of additive and multiplicative consistent FPR were developed.

Definition 2 (Tanino, 1984). For a FPR $R=\left(r_{ij}\right)_{n\times n}$, suppose that $W=\left(w_1,w_2,\cdots,w_n\right)$ is the priority weight vector derived from R, where $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$. Then FPR is called to be additive consistency if

$$r_{ij}=\frac{1}{2}\left(w_i-w_j\right)+0.5 \ \text{ for all } \ i,j\in N. \tag{2}$$

And FPR is multiplicative consistency if

$$r_{ij} = \frac{w_i}{w_i + w_j} \text{ for all } i, j \in N . \quad (3)$$

2.2 HFS

To express the hesitant preference information, Torra and Narukawa (2009) introduced the concept of HFS.

Definition 3 (Torra & Narukawa, 2009). Let X be a fixed set. Accordingly, a HFS E on X is defined in terms of a function $h_E(x)$ that when applied to X returns a finite subset of [0, 1].

To be easily understood, M. M. Xia and Xu (2011) expressed the HFS as the following mathematical symbol:

$$E = \{ < x, h_E(x) > | x \in X \} , \quad (4)$$

where $h_E(x)$ is a set of values in [0, 1] representing the possible membership degrees of element x in X to E, and $h_E(x)$ is named hesitant fuzzy element (HFE) and denoted as $h = \{ \gamma^s | s = 1, 2, \ldots, \#h \}$, $\#h$ is the number of elements including in h.

The number of elements in different HFEs may be different. Any two HFEs are required to have the same length when develop the MCDM methods which the evaluation values with HFEs. To this end, a normalization process is necessary.

Definition 4 (Bin Zhu & Xu, 2014b). Let $h = \{ \gamma^s | s = 1, 2, \ldots, \#h \}$ be a HFE, γ^- and γ^+ denote the minimum and maximum values in h, respectively. Then, $(1-\xi)\gamma^- + \xi\gamma^+$ is called an adding value in h, where ξ is an optimized parameter determined by decision makers' risk preference. Especially, the decision makers are pessimistic when $\xi = 0$ and decision makers are optimistic when $\xi = 1$.

Evidently, different forms of normalized HFEs (NHFEs) will be derived with respect to decision makers with different risk preferences. That is, the normalization process is influenced by the subjectivity of the decision makers, and different ranking values will be derived with respect to MCDM methods with different forms of NHFEs. These shortcomings have been studied by several scholars. For more detail, the readers turn to (Li, Wang, & Hu, 2019; Meng & An, 2017; Z. Zhang, X. Kou, W. Yu, et al., 2018). Meanwhile, an issue on obtaining ranking values that are not relied on NHFEs arises. It will be discussed in Section 3.

2.3 HFPR

M. Xia and Xu (2013) first proposed the concept of HFPR. However, the sequence relationships of the elements including in HFPR are needed, this leads to some complexity in actual application. To overcome this weaknesses, Xu, Cabrerizo, and Herrera-Viedma (2017) developed a new definition of HFPR that does not need to arrange the elements in descending or ascending sequence.

Definition 5 (Xu, et al., 2017). Let $X=\{x_1,x_2,\ldots,x_n\}$ be a fixed set, HFPRs on X is represented by a matrix $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$, where $h_{ij}=\left\{\gamma_{ij}^{s}\,\middle|\,s=1,2,\ldots,\#h_{ij}\right\}$ is a HFE indicating the possible values of the preference degrees to which alternative x_i is preferred to alternative x_j. For all $i,j\in N$, h_{ij} should satisfy:

$$\gamma_{ij}^{s}+\gamma_{ji}^{\#h_{ij}-s+1}=1,\ \gamma_{ii}=0.5,\ \#h_{ij}=\#h_{ji}, \tag{5}$$

where γ_{ij}^{s} refers to the s th element in h_{ij}.

Incomplete evaluations sometimes occur for many reasons, including time pressure or lack of decision maker background knowledge. Xu, et al. (2016) developed the concept of IHFPR as follows.

Definition 6 (Xu, et al., 2016). Let $X=\{x_1,x_2,\ldots,x_n\}$ be a fixed set, then an IHFPR on X is represented by a matrix $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$, where all known HFEs $h_{ij}=\left\{\gamma_{ij}^{s}\,\middle|\,s=1,2,\ldots,\#h_{ij}\right\}$ indicating the possible values of the preference degrees to which alternative x_i is preferred to x_j. For all $i,j\in N$, h_{ij} should satisfy the following conditions:

$$\gamma_{ij}^{s}+\gamma_{ji}^{\#h_{ij}-s+1}=1,\ \gamma_{ii}=0.5,\ \#h_{ij}=\#h_{ji}, \tag{6}$$

where γ_{ij}^{s} refers to the s th element in h_{ij}.

Integrating the concepts of HFPR with additive and multiplicative consistency into IHFPR, Xu, et al. (2016) developed the concepts of additive and multiplicative consistent IHFPR.

Definition 7 (Xu, et al., 2016). Let $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$ be an IHFPR, where $h_{ij}=\left\{\gamma_{ij}^{s}\,\middle|\,s=1,2,\ldots,\#h_{ij}\right\}$. If R satisfies the following condition:

$$\frac{1}{2}\left(w_i-w_j\right)+0.5=\gamma_{ij}^{1}\ or\ \gamma_{ij}^{2}\ or\ldots\ or\ \gamma_{ij}^{\#h_{ij}}, \tag{7}$$

then R is called additive consistent IHFPR. Where $W=\left(w_1,w_2,\ldots,w_n\right)$ is the priority weight vector derived from

R.

And IHFPR is multiplicative consistency if

$$\frac{w_i}{w_i + w_j} = \gamma_{ij}^1 \ or \ \gamma_{ij}^2 \ or \ldots \ or \ \gamma_{ij}^{\# h_{ij}} \ . \tag{8}$$

3. Deriving priority weight vectors from IHFPRs

In this section, we first introduce the concepts of additive and multiplicative consistent IHFPR, and then introduce several programming models for deriving priority weight vectors from IHFPRs, which considers the satisfaction degrees of decision makers.

3.1 Additive and multiplicative consistent IHFPR

To further consider Definition 7, the concepts of additive and multiplicative consistent IHFPR are respectively defined on the basis of the relationships between the formula consisting of priority weights and the values including in HFEs. However, the relationships present in Eqs. (7) and (8) only consider the relationships between one priority weight formula and all the values including in IHFPR but cannot reflect the hesitancy of decision makers. It is reasonable that for every value including in IHFPR has a relationship to one priority weight formula. That is to say, the additive and multiplicative consistent IHFPR are in accordance with the derived FPRs with respect to each fixed values. On the basis of this consideration, new concepts for additive and multiplicative consistent IHFPR are defined as follows.

Definition 8. Let $R = \left(h_{ij}\right)_{n\times n} \subset X \times X$ be an IHFPR, where $h_{ij} = \left\{\gamma_{ij}^s \middle| s = 1,2,\ldots,\# h_{ij}\right\}$. Then, R is called additive consistent IHFPR if all known elements including in R satisfying the following condition:

$$\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5\right) = \delta_{ij}\left(\sum_{s=1}^{\# h_{ij}} \alpha_{ij}^s \gamma_{ij}^s\right), \tag{9}$$

for all $i, j = 1,2,\ldots,n$, with $i < j$, where w_i^k, $k = 1,2,\ldots,\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\# h_{ij}\right)^{\delta_{ij}}$ and $i = 1,2,\ldots,n$ are the priority weights such that $w_i^k \geq 0$ and $\sum_{i=1}^{n} w_i^k = 1$, $i = 1,2,\ldots,n$ for all $k = 1,2,\ldots,\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\# h_{ij}\right)^{\delta_{ij}}$. In addition, α_{ij}^s, $s = 1,2,\ldots,\# h_{ij}$ are $\# h_{ij}$ list of 0-1 indicator variables, which satisfy $\sum_{s=1}^{\# h_{ij}} \alpha_{ij}^s = 1$. To make sure for each possible value $\gamma_{i_o j_o}^s \in h_{ij}$, $i_0 < j_0$ and $s = 1,2,\ldots,\# h_{ij}$ has a relationship to one priority weight formula, we set $\alpha_{i_o j_o}^{s}=1$ for each possible value. And δ_{ij} is an indicator variable, where $\delta_{ij}=\begin{cases}1, \gamma_{ij} \text{ is not a missing HFE} \\ 0, \gamma_{ij} \text{ is a missing HFE}\end{cases}$.

Obviously, there are $\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\#h_{ij}\right)^{\delta_{ij}}$ FPRs corresponding to R. For convenience in following discussion, let $l=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\#h_{ij}\right)^{\delta_{ij}}$.

In a similarly way, the concept of multiplicative consistent IHFPR is developed as follows.

Definition 9. Let $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$ be an IHFPR, where $h_{ij}=\left\{\gamma_{ij}^{s}\middle|s=1,2,\cdots,\#h_{ij}\right\}$. Then, R is called multiplicative consistent IHFPR if all known elements including in R satisfying the following condition:

$$\delta_{ij}\left(\frac{w_i^k}{w_i^k+w_j^k}\right)=\delta_{ij}\left(\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^{s}\right)^{\alpha_{ij}^{s}}\right), \tag{10}$$

for all $i,j=1,2,\cdots,n$, with $i<j$. The meanings of symbols w_i^k, δ_{ij} and α_{ij}^{s} are the same as those shown in Eq. (9).

Remark 1. From Definition 8 and Definition 9 can be easily found that, there are $\#h_{ij}$ equations including in Eq. (9) or Eq. (10), whereas there is only one equation including in Eq. (7) or Eq. (8). This is the difference between Xu, et al. (2016)'s concept and proposed definitions from the view of mathematical symbol. In addition, Xu, et al. (2016)'s concept is also named optimistic consistency, a reduced FPR with the highest consistency degree is obtained, whereas the proposed definitions derived $\#h_{ij}$ possible FPRs including in HFPR.

3.2 Deriving priority weight vectors from IHFPRs

Consistency of preference relations is related to rationality. By comparison, inconsistent preference relations often lead to misleading solutions. Therefore, developing some approaches to obtain the expected consistency level is necessary. However, only few scholars focus on optimization-based method to obtain the expected consistent IHFPR at present. Therefore, in this section, several mathematical programming models are proposed to obtain acceptable consistent IHFPR which considering the satisfaction degrees of decision makers. There are two cases including, namely deriving priority weight vectors from IHFPRs based on the additive consistency and multiplicative consistency, respectively.

Case 1. Deriving priority weight vectors from IHFPRs with additive consistency

According to the definition of additive consistent IHFPR, we obtain $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5\right)=\delta_{ij}\left(\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)$, where δ_{ij}, $i,j=1,2,\ldots,n$ indicates whether γ_{ij} is a missing value or not. The priority weights of complete additive consistent IHFPR can be derived by solving a list of equations $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5\right)=\delta_{ij}\left(\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)$, $i,j=1,2,\ldots,n$, $i<j$, $k=1,2,\ldots,l$. However, the above mentioned equations do not constantly hold in general given a deviation between $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5\right)$ and $\delta_{ij}\left(\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)$ for each possible value $\gamma_{i_o j_o}^s \in h_{ij}$, $i_0<j_0$ and $s=1,2,\ldots,\#h_{ij}$. Moreover, the more $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)$ approaches to 0, the more valid and reasonable the priority weights are.

In this regard, we try to obtain the priority weights of IHFPR by constructing a satisfaction degree function for the decision makers with respect to the priority weights. If the IHFPR is complete consistency then the decision maker completely satisfies the priority weights w_i^k and w_j^k, and the satisfaction degree of the decision maker is 1; otherwise, the satisfaction degree of the decision maker reduces. Motivated by these studies (P. Ren, Hao, Wang, Zeng, & Xu, 2020; P. Ren, Zhu, & Xu, 2018; Bin Zhu & Xu, 2014a), a membership function for the satisfaction degree in incomplete hesitant fuzzy environment can be constructed as:

$$M_{ij}\left(w^k\right)=\begin{cases}1+\dfrac{\delta_{ij}\left(\frac{1}{2}\left(w_i^k-w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)}{\varsigma_{ij}}, & if\ \frac{1}{2}\left(w_i^k-w_j^k\right)+0.5\le\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s \\ 1-\dfrac{\delta_{ij}\left(\frac{1}{2}\left(w_i^k-w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)}{\varsigma_{ij}}, & if\ \frac{1}{2}\left(w_i^k-w_j^k\right)+0.5>\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\end{cases}, \quad (11)$$

where $k=1,2,\ldots,l$, ς_{ij}, $\varsigma_{ij}>0$ is the parameter to present the decision makers' acceptable deviation between judgement values $\delta_{ij}\left(\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)$ and priority weights $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5\right)$ for alternative x_i and alternative x_j. The satisfaction degree value of $M_{ij}\left(w^k\right)$ varies within $\left(-\infty,1\right]$. If $M_{ij}\left(w^k\right)=1$, means the equation $\delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right)=0$ hold, indicates that the decision makers completely satisfy the priority weights; If $0<M_{ij}\left(w^k\right)<1$, means the absolute value inequality $\delta_{ij}\left|\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right|<\varsigma_{ij}$ hold, indicates that the decision makers partly satisfy the priority weights; If $M_{ij}\left(w^k\right)\le 0$, means the absolute value inequality $\delta_{ij}\left|\frac{1}{2}\left(w_i^k - w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right|\ge \varsigma_{ij}$ hold, indicates that the decision makers does not satisfy the priority weights.

We can utilize a regular $n-1$-simplex to present the membership functions of $M_{ij}\left(w^k\right)$, $i,j=1,2,\cdots,n$, which can be denoted as follows:

$$SX^{n-1}=\left\{w_1^k,w_2^k,\cdots,w_n^k\,\middle|\,\sum_{i=1}^{n}w_i^k=1 \text{ and } w_i^k\ge 0 \text{ for all } i\right\},\ \ k=1,2,\cdots,l. \tag{12}$$

A function can be provided to synthesize all satisfaction degrees of the priority weights w_i^k and w_j^k, $i,j=1,2,\cdots,n$ in SX^{n-1} simplex

$$M=\min_{w^k\in SX^{n-1}}\left\{M_{ij}\left(w^k\right)\middle|\, i,j=1,2,\cdots,n\right\},\ \ k=1,2,\cdots,l. \tag{13}$$

The theory of maximum minimization is utilized to guarantee the minimum satisfaction degree be not too low. Therefore, the objective function is denoted as: $\max_{i,j=1,2,\cdots,n}\min_{w^k\in SX^{n-1}}\left\{M_{ij}\left(w^k\right)\middle|\, i,j=1,2,\cdots,n\right\}$, for all $k=1,2,\cdots,l$, which can be represented by the following programming model:

$$\begin{aligned}&\max\quad z^k=\varphi\\ &s.t.\begin{cases}M_{ij}\left(w^k\right)\ge\varphi\\ w^k\in SX^{n-1}\\ i,j=1,2,\cdots,n,\ i\ne j\end{cases}\end{aligned}\ . \tag{14}$$

Substituting equation (11) into equation (14), for each possible value $\gamma_{i_o j_o}^s\in h_{ij}$, with $i_0<j_0$ for each $s=1,2,\cdots,\#h_{ij}$, the mathematical programming model can be expressed in detail as:

$$\max \quad z^k = \varphi$$

$$s.t.\begin{cases} \varsigma_{ij} + \delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s\right) \geq \varphi\varsigma_{ij} \\ \varsigma_{ij} - \delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s\right) \geq \varphi\varsigma_{ij} \\ \sum_{i=1}^{n} w_i^k = 1, \quad k = 1,2,\ldots,l \\ \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1 \\ \alpha_{ij}^s = 0 \vee 1, \quad s = 1,2,\ldots,\#h_{ij} \\ \delta_{ij} = 0 \vee 1 \\ \alpha_{i_0 j_0}^s = 1, \quad i_0 < j_0 \\ i,j = 1,2,\ldots,n, i \neq j \end{cases} . \tag{15}$$

Suppose the decision makers provide the preference degree ς_{ij} with the equal significance, that is, $\varsigma_{ij} = \varsigma$, for all $i, j = 1,2,\ldots,n$, in this case, the mathematical programming model can be further simplified to:

$$\max \quad z^k = \varphi$$

$$s.t.\begin{cases} \varsigma + \delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s\right) \geq \varphi\varsigma \\ \varsigma - \delta_{ij}\left(\frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s\right) \geq \varphi\varsigma \\ \sum_{i=1}^{n} w_i^k = 1, \quad k = 1,2,\ldots,l \\ \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1 \\ \alpha_{ij}^s = 0 \vee 1, \quad s = 1,2,\ldots,\#h_{ij} \\ \delta_{ij} = 0 \vee 1 \\ \alpha_{i_0 j_0}^s = 1, \quad i_0 < j_0 \\ i,j = 1,2,\ldots,n, i \neq j \end{cases} . \tag{16}$$

Substituting deviation variable value ς into the above model, it can be easily proved that Eq. (16) is a 0-1 programming model. Therefore, at least one priority weight vector can be derived with the maximum satisfaction degree φ.

Solving Eq. (16), a list of priority weight vectors w^k, $k = 1,2,\ldots,l$ can be derived. Since w^k can be viewed as the possible priority weight vector of R. And based on the ideas of Z. Zhang, X. Kou, W. Yu, et al. (2018) and Wu, Zhu, Zhou, and Chen (2019). The distance between w^k and R is developed to select the best priority weight vector of R.

Definition 10. Let $R=\left(h_{ij}\right)_{n\times n}\subset X\times X$ be an IHFPR, and $w^k=\left(w_1^k,w_2^k,\ldots,w_n^k\right)$, $k=1,2,\ldots,l$ be a list of priority weight vectors derived from Eq. (16). Then, the distance between w^k and R is developed as follows:

$$d_1\left(w^k,R\right)=\frac{1}{l}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}\left|\frac{1}{2}\left(w_i^k-w_j^k\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s\right|, \tag{17}$$

where $l=\prod_{i=1}^{n-1}\prod_{j=i+1}^{n}\left(\#h_{ij}\right)^{\delta_{ij}}$ is the number of known value in upper triangle of IHFPR. It can be easily found that the distance $d_1\left(w^k,R\right)$ reflects the average of total the square deviation between $\frac{1}{2}\left(w_i^k-w_j^k\right)+0.5$ and $\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s$ for all known elements including in R. It is natural that the optimal priority weight vector is the one that minimizes the deviation $d_1\left(w^k,R\right)$.

For the distance measure presents in Eq. (17), it can be proven that it satisfy the axiom of the distance measure.

Property 1. For the distance measure presents in Eq. (17), we have:

(1) non-negativity: $0\le d_1\left(w^k,R\right)\le 1$;

(2) reflexivity: $d_1\left(w^k,R\right)=0 \Leftrightarrow \frac{1}{2}\left(w_i^k-w_j^k\right)+0.5=\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s\gamma_{ij}^s$;

(3) commutativity: $d_1\left(w^k,R\right)=d_1\left(R,w^k\right)$ and

(4) triangle inequality:

if $d_1^{1,2}\left(w^k,R\right)=\frac{1}{l}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}\left|\frac{1}{2}\left(w_i^{k,1}-w_j^{k,1}\right)+0.5-\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,1}\gamma_{ij}^{s,1}-\frac{1}{2}\left(w_i^{k,2}-w_j^{k,2}\right)-0.5+\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,2}\gamma_{ij}^{s,2}\right|$ and

$d_1^{2,3}\left(w^k,R\right)=\frac{1}{l}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}\left|\frac{1}{2}\left(w_i^{k,2}-w_j^{k,2}\right)-0.5+\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,2}\gamma_{ij}^{s,2}-\frac{1}{2}\left(w_i^{k,3}-w_j^{k,3}\right)-0.5+\sum_{s=1}^{\#h_{ij}}\alpha_{ij}^{s,3}\gamma_{ij}^{s,3}\right|$. Then $d_1^{1,3}\left(w^k,R\right)\le d_1^{1,2}\left(w^k,R\right)+d_1^{2,3}\left(w^k,R\right)$.

Proof: The proof of (2) and (3) is obvious, they do not appear in this study. In the following section, we only provide the proof of (1) and (4).

(1) non-negativity: for each known elements including in R, we have $0\le\left|\frac{1}{2}\left(w_i^k-w_j^k\right)+0.5-\alpha_{ij}^s\gamma_{ij}^s\right|\le 1$, and

$0 \le \left| \frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s \right| \le \#h_{ij}$. Moreover, $0 \le \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s \right| \le l$, then

$0 \le \frac{1}{l} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \frac{1}{2}\left(w_i^k - w_j^k\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s \gamma_{ij}^s \right| \le 1$, that is $0 \le d_1\left(w^k, R\right) \le 1$.

(4) triangle inequality:

$$d_1^{1,3}\left(w^k, R\right) = \frac{1}{l} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \frac{1}{2}\left(w_i^{k,1} - w_j^{k,1}\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,1} \gamma_{ij}^{s,1} - \frac{1}{2}\left(w_i^{k,3} - w_j^{k,3}\right) - 0.5 + \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,3} \gamma_{ij}^{s,3} \right|$$

$$= \frac{1}{l} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \begin{array}{l} \frac{1}{2}\left(w_i^{k,1} - w_j^{k,1}\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,1} \gamma_{ij}^{s,1} - \frac{1}{2}\left(w_i^{k,2} - w_j^{k,2}\right) - 0.5 + \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,2} \gamma_{ij}^{s,2} \\ + \frac{1}{2}\left(w_i^{k,2} - w_j^{k,2}\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,2} \gamma_{ij}^{s,2} - \frac{1}{2}\left(w_i^{k,3} - w_j^{k,3}\right) - 0.5 + \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,3} \gamma_{ij}^{s,3} \end{array} \right|$$

$$\le \frac{1}{l} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \frac{1}{2}\left(w_i^{k,1} - w_j^{k,1}\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,1} \gamma_{ij}^{s,1} - \frac{1}{2}\left(w_i^{k,2} - w_j^{k,2}\right) - 0.5 + \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,2} \gamma_{ij}^{s,2} \right| + \frac{1}{l} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \delta_{ij} \left| \frac{1}{2}\left(w_i^{k,2} - w_j^{k,2}\right) + 0.5 - \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,2} \gamma_{ij}^{s,2} - \frac{1}{2}\left(w_i^{k,3} - w_j^{k,3}\right) - 0.5 + \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^{s,3} \gamma_{ij}^{s,3} \right|$$

. Then

$d_1^{1,3}\left(w^k, R\right) \le d_1^{1,2}\left(w^k, R\right) + d_1^{2,3}\left(w^k, R\right)$.

As a consequence, the priority weight vector of R is developed as follows.

Definition 11. Let $R = \left(h_{ij}\right)_{n \times n} \subset X \times X$ be an IHFPR, and $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, $k = 1, 2, \ldots, l$ be a list of priority weight vectors derived from Eq. (16). Then, the priority weight vector of R is developed as follows:

$$w^{k*} = \left(w_1^{k*}, w_2^{k*}, \ldots, w_n^{k*}\right) = \arg\min_{w^k} d_1\left(w^k, R\right). \tag{18}$$

The symbol arg presents in Eq. (18) means the priority weight vector is derived from the minimum distance calculate from $d_1\left(w^k, R\right)$.

Remark 2. There are may be more than one priority weight vectors including in $\min_{w^k} d_1\left(w^k, R\right)$, that is to say, sometimes the solution of the Eq. (18) is not unique. In this case, the priority weight vector of R is developed as average of multiple priority weight vectors:

$$w^{k*} = \left(w_1^{k*}, w_2^{k*}, \ldots, w_n^{k*}\right) = \left(\frac{1}{l_0} \sum_{k=1}^{l_0} w_1^k, \frac{1}{l_0} \sum_{k=1}^{l_0} w_2^k, \ldots, \frac{1}{l_0} \sum_{k=1}^{l_0} w_n^k \right), \tag{19}$$

where w_i^k, $k=1,2,\ldots,n$, and it indicates that the number of priority weight vectors including in $\min_{w^k} d_1\left(w^k,R\right)$ is l_0.

Case 2. Deriving priority weight vectors from IHFPRs with multiplicative consistency

Similar to the idea of additive consistent IHFPR presented in case 1, the following membership function with the satisfaction degree can be constructed when we consider multiplicative consistent IHFPR.

$$F_{ij}\left(w^k\right)=\begin{cases}1+\dfrac{\delta_{ij}\left(\dfrac{w_i^k}{w_i^k+w_j^k}-\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right)}{\varepsilon_{ij}}, & if\ \dfrac{w_i^k}{w_i^k+w_j^k}\le\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\\ 1-\dfrac{\delta_{ij}\left(\dfrac{w_i^k}{w_i^k+w_j^k}-\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right)}{\varepsilon_{ij}}, & if\ \dfrac{w_i^k}{w_i^k+w_j^k}>\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\end{cases}, \tag{20}$$

where $k=1,2,\ldots,l$. The meanings of symbols w_i^k, ε_{ij}, α_{ij}^s and $F_{ij}\left(w^k\right)$ are the same as those given in Eq. (11).

Similar to the idea of additive consistent IHFPR presented in case 1, the mathematical programming model for deriving priority weight vectors can be expressed in detail as:

$$\max\quad z^k=\pi$$
$$s.t.\begin{cases}\varepsilon_{ij}+\delta_{ij}\left(\dfrac{w_i^k}{w_i^k+w_j^k}-\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right)\ge\pi\varepsilon_{ij}\\ \varepsilon_{ij}-\delta_{ij}\left(\dfrac{w_i^k}{w_i^k+w_j^k}-\prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right)\ge\pi\varepsilon_{ij}\\ \sum_{i=1}^{n}w_i^k=1,\quad k=1,2,\ldots,l\\ \sum_{s=1}^{\#h_{ij}}\alpha_{ij}^s=1\\ \alpha_{ij}^s=0\vee1,\quad s=1,2,\ldots,\#h_{ij}\\ \delta_{ij}=0\vee1\\ \alpha_{i_0j_0}^s=1,\quad i_0<j_0\\ i,j=1,2,\ldots,n,i\ne j\end{cases}. \tag{21}$$

Suppose $\varepsilon_{ij}=\varepsilon$, for all $i,j=1,2,\ldots,n$, then, the mathematical programming model can be further simplified to:

$$\max \quad z^k = \pi$$

$$s.t.\begin{cases} \varepsilon + \delta_{ij}\left(\dfrac{w_i^k}{w_i^k + w_j^k} - \prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right) \geq \pi\varepsilon \\ \varepsilon - \delta_{ij}\left(\dfrac{w_i^k}{w_i^k + w_j^k} - \prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right) \geq \pi\varepsilon \\ \sum_{i=1}^{n} w_i^k = 1, \quad k = 1,2,\ldots,l \\ \sum_{s=1}^{\#h_{ij}} \alpha_{ij}^s = 1 \\ \alpha_{ij}^s = 0 \vee 1, \quad s = 1,2,\ldots,\#h_{ij} \\ \delta_{ij} = 0 \vee 1 \\ \alpha_{i_0 j_0}^s = 1, \quad i_0 < j_0 \\ i,j = 1,2,\ldots,n, i \neq j \end{cases} . \tag{22}$$

Substituting deviation variable value ε into the above model, it can be easily proved that Eq. (22) is a nonlinear programming model. In this case, the optimal solution of it can be obtained by utilizing the optimization software, such as LINGO 11.0, Matlab and Mathematica.

Solving Eq. (22), a list of weight vectors w^k, $k = 1,2,\ldots,l$ can be derived. Since w^k can be viewed as the possible priority weight vector of R. Similarly to additive consistent IHFPR, the distance between w^k and R is developed to select the best priority weight vector.

Definition 12. Let $R = \left(h_{ij}\right)_{n\times n} \subset X \times X$ be an IHFPR, and $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, $k = 1,2,\ldots,l$ be a list of priority weight vectors derived from Eq. (22). Then, the distance between w^k and R is developed as follows:

$$d_2\left(w^k, R\right) = \frac{1}{l}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\delta_{ij}\left|\frac{w_i^k}{w_i^k + w_j^k} - \prod_{s=1}^{\#h_{ij}}\left(\gamma_{ij}^s\right)^{\alpha_{ij}^s}\right|. \tag{23}$$

The meanings of symbols w_i^k , δ_{ij} , α_{ij}^s , l and $d_2\left(w^k, R\right)$ are the same as those given in Eq. (17).

Remark 3. Similar to the idea of Eq. (17), the proof of the axiom of the distance measure presents in Eq. (23) can be developed in a similar way.

Similarly, the priority weight vector of R is developed as follows.

Definition 13. Let $R = \left(h_{ij}\right)_{n\times n} \subset X \times X$ be an IHFPR, and $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, $k = 1,2,\ldots,l$ be a list of priority weight vectors derived from Eq. (22). Then, the priority weight vector of R is developed as follows:

$$w^{k*} = \left(w_1^{k*}, w_2^{k*}, \cdots, w_n^{k*}\right) = \arg\min_{w^k} d_2\left(w^k, R\right). \tag{24}$$

Remark 4. Similar to the idea of additive consistent IHFPR presented in case 1, there are may be more than one priority weight vectors including in $\min_{w^k} d_2\left(w^k, R\right)$. In this case, the priority weight vector of R is developed according to Eq. (19).

4. Framework of MCDM procedure with IHFPRs

In this section, the MCDM problems with IHFPRs are firstly introduced, and then an optimization model is constructed for determining the weights of decision makers. Finally, a framework of MCDM procedure with IHFPRs is introduced.

4.1 The MCDM problems with IHFPRs

Hesitant MCDM problems involve m alternatives denoted as $A = \{a_1, a_2, \cdots, a_m\}$. Each alternative is assessed based on several feature criteria. $E = \{e_1, e_2, \cdots, e_n\}$ is a set of decision makers and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)^T$ is the decision makers' weight vector. We assume that the weights of decision makers are completely unknown. The evaluation of the alternative $a_i, i = 1, 2, \cdots, m$ with respect to the feature criterion is provided by decision maker e_j, $j = 1, 2, \cdots, n$, and denotes as $h = \{\gamma^s \mid s = 1, 2, \cdots, \#h\}$, which is an IHFPR. Suppose IHFPRs provide by decision makers with additive consistency, then the priority weight vectors of each individual IHFPRs are derived according to Eq. (18) or Eq. (19); otherwise, the priority weight vectors of each individual IHFPRs are derived according to Eq. (23) or Eq. (19) if IHFPRs with multiplicative consistency.

4.2 Calculate the weight vectors of the decision makers

In this subsection, an optimization model is constructed to derive the weights of decision makers with complete unknown information. Considering that decision makers in the MCDM process typically construct from different knowledge backgrounds and have varied expertise in the domain area, each decision makers has different judgment values, which influences the solution differently. Therefore, each decision makers has a different importance weight when collecting the priority weight vectors. Given that the decision maker whose judgment values are far away from the collect judgment values indicates that the judgment values he/she provides are the least reliable, and the decision maker should endow a smaller weight value. By comparison, the decision maker whose judgment values are close to the collect judgment values indicates that the judgment values he/she provides are the most reliable, and the decision maker should endow a larger weight value. On this basis, the optimization model is constructed as follows:

$$\min \quad z=\sum_{k=1}^{n}\sum_{i=1}^{m}\left(w_i^{k*}-\sum_{k=1}^{n}\lambda_k w_i^{k*}\right)^2$$
$$s.t.\begin{cases}\sum_{k=1}^{n}\lambda_k=1\\ 0\le\lambda_k\le 1\\ k=1,2,\ldots,n\end{cases}. \tag{25}$$

As seen, Eq. (25) is a square programming model. The optimal solution of it can be obtained by utilizing the optimization software. Moreover, it can be easily found that Eq. (25) is constructed from the algorithm of least square method, according to the Lagrange multiplier method, the Lagrange function of Eq. (25) can be constructed as follows,

$L(\lambda,\mu)=\sum_{k=1}^{n}\sum_{i=1}^{m}\left(w_i^{k*}-\sum_{k=1}^{n}\lambda_k w_i^{k*}\right)^2+2\mu\left(\sum_{k=1}^{n}\lambda_k-1\right)$, where μ is a Lagrange multiplier. The solution of Eq. (25) can be obtained by solving the partial derivatives of Lagrange function $L(\lambda,\mu)$, and the result is listed as follows:

$$\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n)^T=A^{-1}\left(B-\frac{e^TA^{-1}Be-e}{e^TA^{-1}e}\right), \tag{26}$$

where $A=\begin{bmatrix}\sum_{i=1}^{m}w_i^{1*}w_i^{1*} & \sum_{i=1}^{m}w_i^{2*}w_i^{1*} & \cdots & \sum_{i=1}^{m}w_i^{n*}w_i^{1*}\\ \sum_{i=1}^{m}w_i^{1*}w_i^{2*} & \sum_{i=1}^{m}w_i^{2*}w_i^{2*} & \cdots & \sum_{i=1}^{m}w_i^{n*}w_i^{2*}\\ \vdots & \vdots & \vdots & \vdots\\ \sum_{i=1}^{m}w_i^{1*}w_i^{n*} & \sum_{i=1}^{m}w_i^{2*}w_i^{n*} & \cdots & \sum_{i=1}^{m}w_i^{n*}w_i^{n*}\end{bmatrix}$, $B=\begin{bmatrix}\sum_{k=1}^{n}\sum_{i=1}^{m}w_i^{k*}w_i^{1*}\\ \sum_{k=1}^{n}\sum_{i=1}^{m}w_i^{k*}w_i^{2*}\\ \vdots\\ \sum_{k=1}^{n}\sum_{i=1}^{m}w_i^{k*}w_i^{n*}\end{bmatrix}$, $e=(1,1,\ldots,1)^T$ and A^{-1} is the inverse matrix of A, e^T is the transpose matrix of e.

4.3 Framework of MCDM procedure with IHFPRs

The proposed decision making procedure is summarized in the following steps.

Step 1: Form individual IHFPR matrices.

According to the determine criteria and alternatives, the decision makers respectively provide their judgement matrices, and denotes as $R_k=\left(h_{ij,k}\right)_{m\times m}\subset X\times X$, $k=1,2,\ldots,n$.

Step 2: Derive the priority weight vectors.

Utilize Eq. (16) or Eq. (22) to obtain the priority weight vectors $w^k = \left(w_1^k, w_2^k, \ldots, w_n^k\right)$, $k = 1, 2, \ldots, n$ from the individual IHFPR.

Step 3: Derive the optimal priority weight vector.

First, Utilize Eq. (17) or Eq. (23) to calculate the distance between w^k and R_k, if there is only one priority weight vector including in Eq. (18) or Eq. (24), then the optimal priority weight vector is determined according to Eq. (18) or Eq. (24), otherwise, the optimal priority weight vector is determined according to Eq. (19).

Step 4: Determine the weights of decision makers.

The weights of decision makers are determined according to Eq. (26).

Step 5: Compute the collective optimal priority weight vector.

The collective optimal priority weight vector is determined by the following formula:

$$\vartheta_i = \sum_{k=1}^{n} \lambda_k w_i^{k*}, \quad i = 1, 2, \ldots, m, \tag{27}$$

where λ_k is the weight of decision maker, and w_i^{k*} is the optimal priority weight vector determines in Step 3.

Step 6: Rank the alternatives.

The ranking order of all alternatives is obtained by the value of collective optimal priority weight vector ϑ_i, $i = 1, 2, \ldots, m$.

The proposed decision making procedure is depicted in Fig. 1.

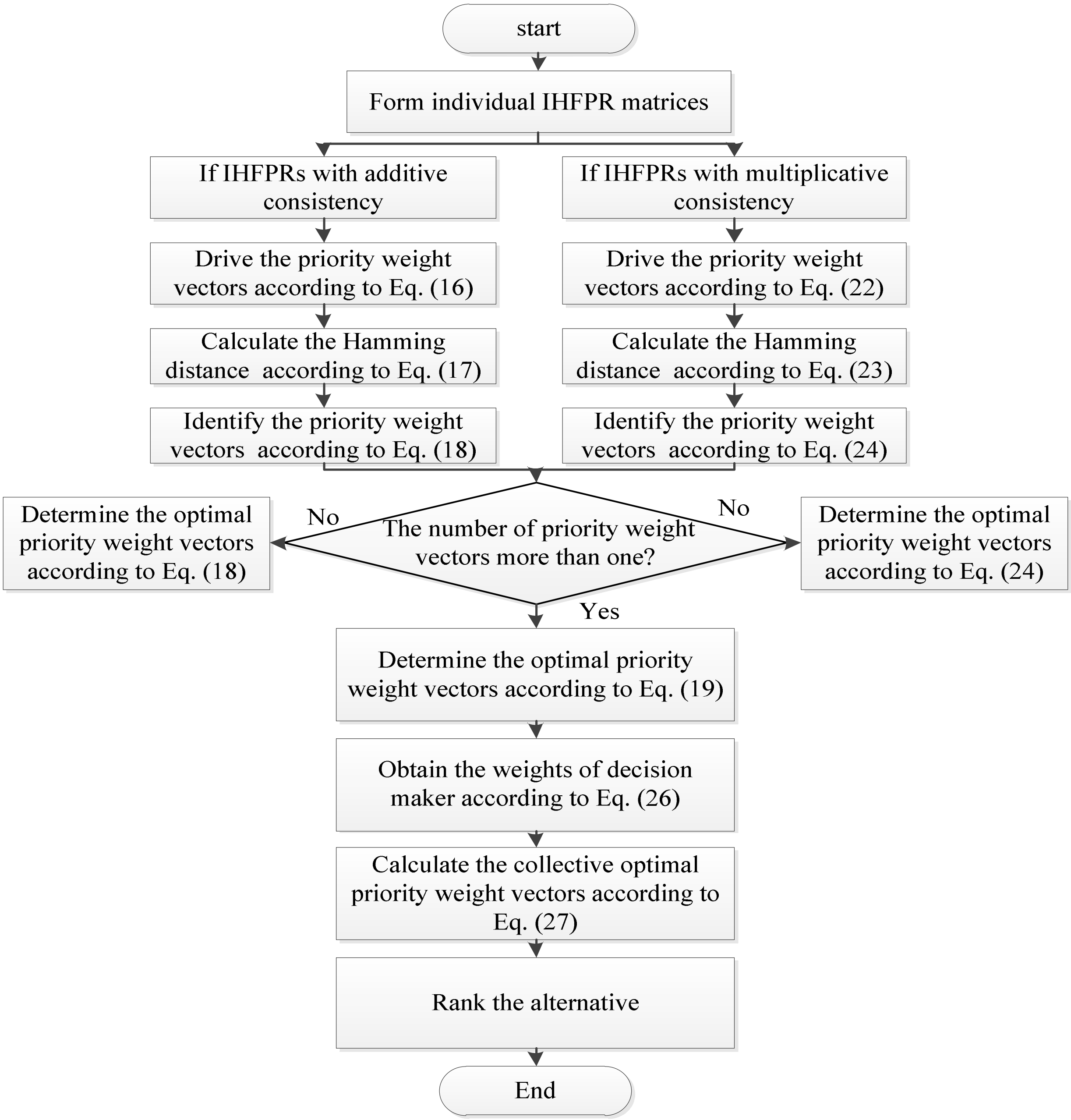

Fig.1. Framework of decision making process with IHFPRs

5. Illustrative example

In this section, selection of the most important project to invest problem (adapted from Xu, et al. (2016) and Parreiras, Ekel, Martini, and Palhares (2010)) is provided to illustrate the use of the proposed method, and conjunction with comparative analysis is conducted.

The enterprise's board of directors, which includes three members e_k, $k = 1, 2, 3$, named three decision makers, have to plan the development of strategy initiatives for the following several years. Suppose that there are three possible projects, denoted as: (1) a_1 medical intelligent logistics; (2) a_2 port intelligent logistics; and (3) a_3 cold chain intelligent logistics, to be evaluated. It is necessary to compare these projects in order to select the most important from the point view of their importance, taking into account four criteria suggested by the balanced scored methodology from the perspectives of: (1) learning and growth; (2) financial; (3) internal business process; and (4) the customer satisfaction. First, three decision makers are asked to give their opinion relative to each project. Because of the uncertainty of the criteria, it is difficult for the decision makers to use just one value to provide their evaluation values. To facilitate the elicitation of their evaluation values, HFPR is just an effective tool to deal with such situations. Furthermore, some decision makers may be have limited expertise and lacking in knowledge related to the problem domain, and thus, these decision makers provide their evaluation with IHFPRs, as demonstrated in matrices 1-3.

Take the evaluation values $\{0.2, 0.3\}$ and φ from decision maker e_1 for example. The decision maker e_1 is hesitant two possible values 0.2 and 0.3 when assesses the alternatives a_2 to a_3, and cannot determine which one is the best. In such case, the evaluation value can be modeled by a HFE $\{0.2, 0.3\}$. And φ means decision maker e_1 cannot provide any evaluation value owing to lack of background knowledge when assesses the alternatives a_1 to a_3. Other entries, that is, HFEs, in matrices 1–3 are similarly explained.

$$R_1 = \begin{bmatrix} \{0.5\} & \{0.6\} & \varphi \\ \{0.4\} & \{0.5\} & \{0.2, 0.3\} \\ \varphi & \{0.7, 0.8\} & \{0.5\} \end{bmatrix}, \quad R_2 = \begin{bmatrix} \{0.5\} & \varphi & \{0.3, 0.4\} \\ \varphi & \{0.5\} & \{0.3\} \\ \{0.6, 0.7\} & \{0.7\} & \{0.5\} \end{bmatrix}, \text{ and}$$

$$R_3 = \begin{bmatrix} \{0.5\} & \{0.3, 0.4\} & \{0.4\} \\ \{0.6, 0.7\} & \{0.5\} & \varphi \\ \{0.6\} & \varphi & \{0.5\} \end{bmatrix}.$$

5.1 Illustration of the proposed method

The procedures for determining the most important project using the proposed method is discussed below.

Case 1: Suppose that all individual IHFPRs with additive consistency

Step 1: Form individual IHFPR matrices.

All individual IHFPR matrices have been provided, as demonstrated in matrices 1-3.

Step 2: Derive the priority weight vectors.

Suppose that the decision makers' acceptable deviation value $\varsigma=0.5$. According to Eq. (16), for decision maker e_1, we have:

$$\max \quad z^1=\varphi$$
$$s.t.\begin{cases}0.5+\frac{1}{2}\left(w_1^1-w_2^1\right)+0.5-0.6\geq 0.5\varphi\\ 0.5+\frac{1}{2}\left(w_2^1-w_3^1\right)+0.5-\left(0.2\alpha_{23}^1+0.3\alpha_{23}^2\right)\geq 0.5\varphi\\ 0.5\text{-}\frac{1}{2}\left(w_1^1-w_2^1\right)\text{-}0.5+0.6\geq 0.5\varphi\\ 0.5\text{-}\frac{1}{2}\left(w_2^1-w_3^1\right)\text{-}0.5+\left(0.2\alpha_{23}^1+0.3\alpha_{23}^2\right)\geq 0.5\varphi\\ w_1^1+w_2^1+w_3^1=1\\ \alpha_{23}^1+\alpha_{23}^2=1\\ \alpha_{23}^s=0\vee 1,\quad s=1,2\\ \alpha_{23}^1=1\end{cases}$$

By solving this optimization model, we obtain:

z^1=0.3; w_1^1=0.9, w_2^1=0 and w_3^1=0.1; $\alpha_{23}^1=1$.

By replacing $\alpha_{23}^2=1$ with $\alpha_{23}^1=1$ into above optimization model, we obtain:

z^2=0.4; w_1^2=0.8, w_2^2=0 and w_3^2=0.2; $\alpha_{23}^2=1$.

Similarly, for decision maker e_2, we can obtain

z^1=0.1; w_1^1=0.5, w_2^1=0.5 and w_3^1=0; $\alpha_{13}^1=1$.

z^2=0.2; w_1^2=0.6, w_2^2=0.4 and w_3^2=0; $\alpha_{13}^2=1$.

Moreover, for decision maker e_3, we can obtain

z^1=0.2; w_1^1=0, w_2^1=0.4 and w_3^1=0.6; $\alpha_{12}^1=1$.

z^2=0.3; w_1^2=0, w_2^2=0.5 and w_3^2=0.5; $\alpha_{12}^2=1$.

Step 3: Derive the optimal priority weight vector.

First, Utilize Eq. (17) to obtain the distance between w^k and R_k, for decision maker e_1, we have: $d_1\left(w^1, R_1\right)=\frac{1}{2}\left(\left|\frac{1}{2}(0.9-0)+0.5-0.6\right|+\left|\frac{1}{2}(0-0.1)+0.5-0.2\right|\right)=0.3$ and $d_1\left(w^2, R_1\right)=0.2$. Since there is only one priority weight vector including in Eq. (18), then the optimal priority weight vector can be determined as follows: $w^{1*}=(0.8,0,0.2)$.

Similarly, for decision maker e_2, we can obtain: $d_1\left(w^1, R_2\right)=0.45$ and $d_1\left(w^2, R_2\right)=0.4$. Since there is only one priority weight vector including in Eq. (18), then the optimal priority weight vector can be determined as follows: $w^{2*}=(0.6,0.4,0)$.

Moreover, for decision maker e_3, we can obtain: $d_1\left(w^1, R_3\right)=0.1$ and $d_1\left(w^2, R_3\right)=0.15$. Since there is only one priority weight vector including in Eq. (18), then the optimal priority weight vector can be determined as follows: $w^{3*}=(0,0.4,0.6)$.

Step 4: Determine the weights of decision makers.

According to Eq. (26), the weights of decision makers are determined as follows:

$\lambda_1=0.33$, $\lambda_2=0.33$ and $\lambda_3=0.33$.

Step 5: Compute the collective optimal priority weight vector.

According to Eq. (27), the collective optimal priority weight vector is determined as follows:

$\vartheta_1=0.47$, $\vartheta_2=0.27$ and $\vartheta_3=0.27$.

Step 6: Rank the alternatives.

Since $\vartheta_1>\vartheta_2=\vartheta_3$, the ranking order of all alternatives is obtained as $a_1 \succ a_2 \sim a_3$. Thus, the most important project to invest is medical intelligent logistics.

Case 2: Suppose that all individual IHFPRs with multiplicative consistency

Step $1'$**:** Form individual IHFPR matrices.

All individual IHFPR matrices have been provided, as demonstrated in matrices 1-3.

Step $2'$**:** Derive the priority weight vectors.

Suppose that the decision makers' acceptable deviation value $\varepsilon = 0.5$. According to Eq. (22), for decision maker e_1, we have:

$$\max \quad z^1 = \pi$$

$$s.t.\begin{cases} 0.5 + \dfrac{w_1^1}{w_1^1 + w_2^1} - 0.6 \ge 0.5\pi \\ 0.5 - \dfrac{w_1^1}{w_1^1 + w_2^1} + 0.6 \ge 0.5\pi \\ 0.5 + \dfrac{w_2^1}{w_2^1 + w_3^1} - 0.2^{\alpha_{23}^1} 0.3^{\alpha_{23}^2} \ge 0.5\pi \\ 0.5 - \dfrac{w_2^1}{w_2^1 + w_3^1} + 0.2^{\alpha_{23}^1} 0.3^{\alpha_{23}^2} \ge 0.5\pi \\ w_1^1 + w_2^1 + w_3^1 = 1 \\ \alpha_{23}^1 + \alpha_{23}^2 = 1 \\ \alpha_{23}^s = 0 \vee 1, \quad s = 1,2 \\ \alpha_{23}^1 = 1 \end{cases}$$

By solving this optimization model, we obtain:

$z^1 = 1$; $w_1^1 = 0.23$, $w_2^1 = 0.15$ and $w_3^1 = 0.62$; $\alpha_{23}^1 = 1$.

By replacing $\alpha_{23}^2 = 1$ with $\alpha_{23}^1 = 1$ into above optimization model, we obtain:

$z^2 = 1$; $w_1^2 = 0.31$, $w_2^2 = 0.21$ and $w_3^2 = 0.48$; $\alpha_{23}^2 = 1$.

Similarly, for decision maker e_2, we can obtain

$z^1 = 1$; $w_1^1 = 0.23$, $w_2^1 = 0.23$ and $w_3^1 = 0.54$; $\alpha_{13}^1 = 1$.

$z^2 = 1$; $w_1^2 = 0.32$, $w_2^2 = 0.2$ and $w_3^2 = 0.48$; $\alpha_{13}^2 = 1$.

Moreover, for decision maker e_3, we can obtain

$z^1 = 1$; $w_1^1 = 0.21$, $w_2^1 = 0.48$ and $w_3^1 = 0.31$; $\alpha_{12}^1 = 1$.

$z^2 = 1$; $w_1^2 = 0.25$, $w_2^2 = 0.375$ and $w_3^2 = 0.375$; $\alpha_{12}^2 = 1$.

Step $3'$ **:** Derive the optimal priority weight vector.

First, Utilize Eq. (23) to obtain the distance between w^k and R_k, for decision maker e_1, we have: $d_2\left(w^1, R_1\right) = \frac{1}{2}\left(\left|\frac{0.23}{0.23+0.15} - 0.6\right| + \left|\frac{0.15}{0.15+0.62} - 0.2\right|\right) = 0.0052$ and $d_2\left(w^2, R_1\right) = 0.004$. Since there is only one priority weight vector including in Eq. (24), then the optimal priority weight vector can be determined as follows: $w^{1*} = \left(0.31, 0.21, 0.48\right)$.

Similarly, for decision maker e_2, we can obtain: $d_2\left(w^1, R_2\right) = 0.0013$ and $d_2\left(w^2, R_2\right) = 0.0029$. Since there is only one priority weight vector including in Eq. (24), then the optimal priority weight vector can be determined as follows: $w^{2*} = \left(0.23, 0.23, 0.54\right)$.

Moreover, for decision maker e_3, we can obtain: $d_2\left(w^1, R_3\right) = 0.004$ and $d_2\left(w^2, R_3\right) = 0$. Since there is only one priority weight vector including in Eq. (24), then the optimal priority weight vector can be determined as follows: $w^{3*} = \left(0.25, 0.375, 0.375\right)$.

Step $4'$ **:** Determine the weights of decision makers.

According to Eq. (26), the weights of decision makers are determined as follows:

$\lambda_1 = 0.33$, $\lambda_2 = 0.33$ and $\lambda_3 = 0.33$.

Step $5'$ **:** Compute the collective optimal priority weight vector.

According to Eq. (27), the collective optimal priority weight vector is determined as follows:

$\vartheta_1 = 0.26$, $\vartheta_2 = 0.27$ and $\vartheta_3 = 0.47$.

Step $6'$ **:** Rank the alternatives.

Since $\vartheta_3 > \vartheta_2 > \vartheta_1$, the ranking order of all alternatives is obtained as a_3 f a_2 f a_1. Thus, the most important project to invest is cold chain intelligent logistics.

5.2 Comparative analysis and discussion

To validate the feasibility of the proposed method, we conducted a comparative study with other method based on the same illustrative example.

Xu, et al. (2016) first proposed the concept of IHFPR, and then introduced the concept of additive consistent IHFPR and multiplicative consistent IHFPR. Moreover, to obtain the priority vector of an IHFPR, two goal programming models are developed based on additive consistency and multiplicative consistency respectively. Finally, these two goal programming models have been extended to obtain the collective priority vector of several IHFPRs. To better comparison, the results obtained by Xu, et al. (2016)'s methods and the proposed methods are summarized in Table 2. The detailed calculation process of Xu, et al. (2016)'s methods can be found in Xu, et al. (2016).

Table 2: The ranking results of the different methods

Methods		Ranking values			Ranking results
		ϑ_1	ϑ_2	ϑ_3	
Xu, et al. (2016)'s methods	additive consistency	0.3184	0.2045	0.4773	$a_3 \succ a_1 \succ a_2$
	multiplicative consistency	0.3333	0.2333	0.4333	$a_3 \succ a_1 \succ a_2$
The proposed methods	additive consistency	0.47	0.27	0.27	$a_1 \succ a_2 \sim a_3$
	multiplicative consistency	0.26	0.27	0.47	$a_3 \succ a_2 \succ a_1$

As shown in Table 2, it can be seen that there are some differences in the ranking results. In Xu, et al. (2016)'s methods, the ranking results are the same, but the ranking values are different when considering different consistency. In the proposed methods, both the ranking results and ranking values are different when considering different consistency. It can be easily found that the best alternative obtained from Xu, et al. (2016)'s methods are the same as the proposed method with multiplicative consistency, but the ranking values are different. The possibility reasons for the inconsistency are explained as follows. The consistency definitions are different. In Xu, et al. (2016)'s methods, the consistency definitions based on one FPR derived from IHFPR, that is, optimistic consistency, while the proposed methods the consistency definitions based on each value in HFEs, they considering some possible FPRs derived from IHFPR. The consistency considers all the evaluation information, and neither adds values into HFEs nor removes values from HFEs. It can then avoid information loss and distort. Compare to Xu, et al. (2016)'s method, they only consider some evaluation information, based on this fact, the ranking result obtained from the proposed method seems more reasonable. Moreover, the objective functions are different. In Xu, et al. (2016)'s methods, the objective functions constructed based on minimizing the deviation from the target of the goal. However, the proposed methods focus on maximizing the parameter of satisfaction degree. The different perspectives for solving the problems lead to different decision-making results, but the proposed methods take the decision makers' satisfaction degree into account, this is more suitable for solving decision-making problems in some backgrounds (Z. Zhang & Chen, 2021b).

Furthermore, we provide a possible reason why the different best alternatives are obtained by the proposed methods with different consistency. The additive consistency shows the additive transitivity of the three related judgement, while multiplicative consistency shows the multiplicative transitivity of the three related judgement. Different focus will result in different ranking results. In addition, for the proposed methods with different consistency, lots of weight vectors of criteria are derived. For a better comparison, the results are summarized in Table 3.

Table 3: The weights of criteria derived from the proposed methods with different consistency

The proposed methods with different consistency	Decision makers	Weight vectors			Objective function value
		w_1	w_2	w_3	
additive consistency	e_1	0.9	0.0	0.1	0.3
		0.8	0.0	0.2	0.4
	e_2	0.5	0.5	0.0	0.1
		0.6	0.4	0.0	0.2
	e_3	0.0	0.4	0.6	0.2
		0.0	0.5	0.5	0.3
multiplicative consistency	e_1	0.23	0.15	0.62	1
		0.31	0.21	0.48	1
	e_2	0.23	0.23	0.54	1
		0.32	0.2	0.48	1
	e_3	0.21	0.48	0.31	1
		0.25	0.375	0.375	1

As shown in Table 3, it can be seen that different weight vectors are obtained for the proposed methods with different consistency. This is consistent with our previous view. In practical decision-making problems, one can determine the method according to the satisfaction value. Furthermore, for different decision makers, since the evaluation information is different, weight vectors of criteria derived from them are different, it is necessary consider the weights of decision makers when collecting the priority weight vectors. Unfortunately, the method of determining the weights of decision makers does not consider in Xu, et al. (2016)'s method.

According to the comparison analysis, the method proposed in this study has the following advantages over other existing methods.

(1) Two decision-making models are developed in view of the proposed additive consistency and multiplicative consistency measures. These methods directly utilize the original judgement of evaluation information to make decision without adjusting them. It can then avoid information loss and distort, and the ranking result obtained from the proposed method seems more reasonable.

(2) The proposed methods take into account the decision makers' satisfaction degree. The proposed method is superior to the decision-making methods based on acceptably consistent preference relations because it can omit the procedure to repair the inconsistent preference relations.

(3) The method of determining the weights of decision makers is developed. This method makes full use of the evaluation information of decision-makers.

6. Conclusion

This paper develops decision-making models based on decision makers' satisfaction degree with IHFPR. First, the consistency measures from the perspectives of additive and multiplicative consistent IHFPR are defined. Then, two decision-making models are developed in view of the proposed additive and multiplicative consistency measures. Second, a square programming model is developed to obtain the decision makers' weights. Finally, a procedure for MCDM problems with IHFPR is given, and an illustrative example in conjunction with comparative analysis is conducted.

The present study provides several significant contributions for MCDM problems with IHFPR. They are summarized as follows: (1) A new concept of additive and multiplicative consistent IHFPR is proposed, respectively. The main feature of them is that they consider all the evaluation information including in HFEs, and neither add values into HFEs nor remove values from HFEs. They can avoid information loss and distort. (2) Two decision-making models are developed based on the proposed additive and multiplicative consistency measures. The main characteristic of the constructed models is that these methods consider the decision makers' satisfaction degree. (3) A square programming model is developed to obtain the decision makers' weights, which is utilized the optimal priority weight vectors information derived from individual IHFPR matrices. In our future research, the proposed methods are extended to hesitant fuzzy linguistic preference relation and applied the proposed methods to solve other practical MCDM problems.

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Compliance with Ethical Standards

Conflict of Interest: Authors Jian Li, Li-li Niu, Qiongxia Chen and Zhong-xing Wang declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

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