

Inside a Schwarzschild Black Hole Approach

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INSIDE A SCHWARZSCHILD BLACK HOLE APPROACH

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ABSTRACT

Our goal in this paper is to explain the internal properties of the black hole by considering its density as a function of the reciprocal of its radius and the temperature as a function of the reciprocal of the density. We then set the temperature to the Hawking temperature. This gives all the macroscopic quantities of the black hole, such as heat capacity, pressure, surface gravity, and equation of state. In this work, we only consider a black hole with mass M , radius r_+ and vacuum. Two internal forces and the corresponding potential are obtained.

Keywords Black hole density. Entropy. Heat capacity. Surface gravity. Pressure. Forces. Potential. Work.

1 Introduction

If we have no way of directly observing a black hole and knowing its internal properties, then we must imagine or guess its internal structure. Experience shows that the mass density is by no means a constant quantity, in general it is not homogeneous or isotropic. Under certain conditions we can assume that objects have a constant mass density, but for large objects, such as planets or stars, this fact is not true, the density increases in the depth of the large objects. For this reason, we consider density to be proportional to $1/r^n$, where n is a specific integer yet to be determined. On the other hand, as we know, the temperature is a consequence of the motion of the atoms or particles, that is, the temperature has to do with the kinetic energy of the atoms or particles. If the freedom of the particles or atoms to move in the container is reduced, we hope that the temperature will decrease. For this reason, we consider that the temperature is the inverse of the density.

Dimensional analysis is used to obtain the density, temperature and entropy of the black hole. Considering the equation of state of the black hole, we obtain the surface gravity, the head capacity, the temperature as a function of area, two forces that determine the pressure and motion in the black hole, and the potential energy associated with these forces.

2 Mass density

We assume that the density of the black hole is proportional to $1/r^n$, i. e:

$$\rho = \frac{ak}{r^n}, \quad (1)$$

We assume that the density of the black hole is proportional to $1/r^n$, i. e:

$$\rho = \frac{ak}{r^n}, \quad (2)$$

where k is dimensionless, n is a positive integer, and a is a certain function $a = a(G, c)$, where G and c are the gravitational and speed-of-light constants, respectively.

Let a be

$$a = G^x k_B^y, \quad (3)$$

where k_B is the Boltzmann constant.

In the dimensional analysis, we find the exponents x , y and n . From Eq. 2 and Eq. 3 we have that,

$$[\rho][r]^n = [G]^x [c]^y \text{ or} \quad (4)$$

$$[ML^{-3}][L]^n = [L^3 M^{-1} T^{-2}]^x [LT^{-1}]^y, \quad (5)$$

where

$$\begin{aligned} [G] &= L^3 M^{-1} T^{-2} \text{ and} \\ [c] &= LT^{-1} \end{aligned}$$

If we solve the system of equations, we have from Eq. , that $x = -1$, $y = 2$, and $n = 2$, then

$$\rho = k \frac{c^2}{Gr^2}. \quad (6)$$

Let M and r_+ be the mass and the Schwarzschild radius of the black hole. The integral of Eq. 6 lets us determine the k constant.

$$M = k \int_0^{r_+} \frac{c^2}{Gr^2} 4\pi r^2 dr \quad (7)$$

where $r_+ = 2GM/c^2$.

Then we find

$$k = \frac{1}{8\pi}, \quad (8)$$

and finally

$$\rho = \frac{1}{8\pi} \frac{c^2}{Gr^2}, \quad (9)$$

where $0r \leq r_+$.

ρ is the volumetric mass density, but can be seen as the surface mass density if we take Eq. 9 as

$$\rho = \frac{M}{4\pi r^2} \frac{1}{r_+} \quad (10)$$

In this way the mass m enclosed by volume $4/3\pi r^3$ is given by

$$m = \frac{M}{r_+} r, \quad (11)$$

or

$$m = \frac{c^2}{2G} r. \quad (12)$$

This is the mass inside of black hole. It increase lineally as r does.

3 Temperature

We now consider that the dependence of temperature on density is given by a certain function of G , \hbar , k_B and c , i.e. $a = a[G, \hbar, c, k_B]$, where \hbar is the Planck constant.

Inside the black hole, the density is very high, which reduces the distance between the particles and the kinetic energy of the particles. Since temperature is a measure of kinetic energy, it decreases when you are deep inside the black hole. Then we can consider that the temperature to be,

$$T = c_0 \frac{a}{\rho} \quad (13)$$

where c_0 is a constant that has no dimension

$$T = c_0 \frac{8\pi G}{c^2} r^2 \quad (14)$$

In the dimensional analysis, we find the unknowns x , y , z and w from the dimensional relation

$$[T] = [G]^x [k_B]^y [\hbar]^z [c]^w [G][r]^2 [c]^{-2} \quad (15)$$

Solving the corresponding set of equations, we get $x = -5/2$, $y = -1$, $z = -1/2$, and $w = 15/2$.

Substituting these values into Eq. 14, one obtains

$$T = c_0 \frac{8\pi c^{11/2}}{k_B \hbar^{1/2} G^{3/2}} r^2. \quad (16)$$

To find the c_0 constant, we consider that $T[r_+] = T_H$, where T_H is the Hawking temperature of the black hole. Here we find

$$c_0 = \frac{1}{256\pi^2} \left[\frac{(\hbar c)}{G} \right]^{1/2} \frac{1}{M}^3. \quad (17)$$

Substituting this expression into Eq. 16 leads to

$$T = \frac{\hbar c}{4\pi k_B r_+^3} r^2. \quad (18)$$

If $r = r_+$, we get the Hawking temperature [2].

For a given black hole of mass M , this temperature increases with r and is zero at the center. This is because r is the radius of the sphere of mass m Eq. 11, which is zero when $r = 0$, which also corresponds to zero volume.

The maximum value of T occurs at $r = r_+$, the Hawking temperature T_H . This temperature, Eq. 18, states that the entire black hole is not in thermal equilibrium, each surface of area $A = 4\pi r^2$ has its own temperature T , this causes continuous heat transport as the inner surfaces become cooler with decreasing r and tend towards zero Kelvin which cannot be reached. Thermal equilibrium is only reached on any surface that lies between r and $r + dr$.

4 Entropy

The next important physical quantity of the black hole is entropy. Experience tells us that if the system is orderly, the entropy is low. If the physical system has a high density, then it is ordered, to this the entropy is proportional to the reciprocal of the density, but not only to the mass. Density plays a fundamental role in this analysis. Then

$$S = b \frac{c_1}{\rho} k_B, \quad (19)$$

where b is a number and c_1 is a certain function of G , \hbar and c , so $c_1 = c_1(G, \hbar, c)$. Since the unit of entropy is $[S] = [k_B]$, then $[c_1] = [\rho]$.

In a similar way as before, we consider a dimensional analysis

$$[G]^x[\hbar]^y[c]^z = [\rho] \quad (20)$$

or

$$[L^3M^{-1}T^{-2}]^x[ML^2T^{-2}]^y[LT^{-1}]^z = [ML^{-3}]. \quad (21)$$

Solving the resulting system, we have $x = -2$, $y = -1$ and $z = 5$. Then it follows from Eqs. 19 and 9 that,

$$S = a \frac{8\pi k_B c^3}{\hbar G} r^2. \quad (22)$$

Now we have to analyze the coefficient a . We are dealing with an isolated system, a black hole. Except a , all terms occurring in 22 are positive. The entropy of an isolated system increases, so a must also be positive.

Since $0 < r \leq r_+$, the maximum value of S occurs at the event horizon. If $r = r_+$, we have then,

$$S = a \frac{8k_B c^3}{4\hbar G} A, \quad (23)$$

that is the entropy of the black hole surface, which differs from the Hawking entropy, S_H , by a factor $8a$, which coefficient a will be determined in the next section. Eq. 23 refers to the entropy of the respective inner surface of the black hole.

5 Heat Capacity

Now we determine the heat capacity C in two ways and compare these results.

By definition,

$$C = \left(\frac{dU}{dT} \right)_P. \quad (24)$$

The internal energy of the black hole, U , is given by,

$$U = mc^2 \quad (25)$$

where m is given by Eq. 12 and T is given by Eq. 18. Then

$$C_P = \left(\frac{dU}{dr} \frac{dr}{dT} \right)_P \quad (26)$$

and we obtain

$$C_P = \frac{\pi k_B c^3 r_+^3}{G \hbar r}. \quad (27)$$

For $r = r_+$ we get,

$$C_P = \frac{Mc^2}{2T_H} \quad (28)$$

showing that $C_P > 0$, which differ in sign from [3].

On the other hand,

$$C_P = T \left(\frac{\partial S}{\partial T} \right)_P. \quad (29)$$

Using the Eqs. 18 and 22 and considering that

$$C_P = T \left(\frac{\partial S}{\partial r} \frac{\partial r}{\partial T} \right)_P \quad (30)$$

we have that

$$C_P = \frac{8a\pi k_B c^3}{G\hbar} r^2, \quad (31)$$

which for $r = r_+$ becomes,

$$C_P = \frac{4aMc^2}{T_H}. \quad (32)$$

From this analysis we can see that the comparison of these two results, Eqs. 28 and 32 yield that $a = 1/80$, as we hope, then,

$$C_P = \frac{Mc^2}{2T_H} \theta, \quad (33)$$

which show us that there is not local instability.

Taking into account that $a = 1/8$, Eq. 23 can be written as,

$$S = \frac{k_B c^3}{4\hbar G} A, \quad (34)$$

a desired result, the Hawking entropy for the black hole.

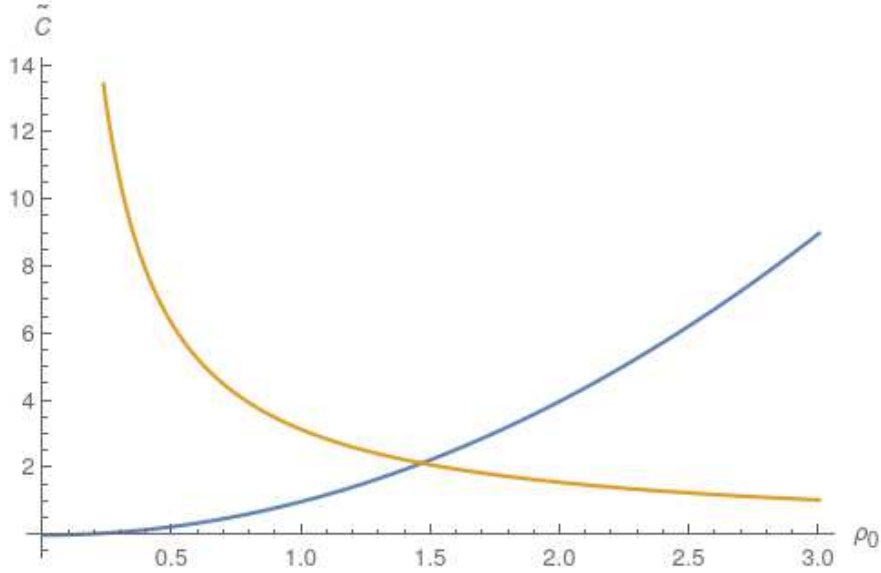


Figure 1: Two heat capacities according to the model, $0 < \rho_0 \leq 1$ and $\rho_0 = r/r_+$

In the above figure, $\tilde{C} = C/C_0$, where

$$C_0 = \frac{\pi k_B c^3 r_+^2}{G\hbar}.$$

We will consider the equation of state of the black hole to find the surface gravity κ , that is,

$$dm = \frac{\kappa}{8\pi} dA. \quad (35)$$

If we now use Eq. 12 and take into account that $\mathcal{A} = 4\pi r^2$, we obtain

$$\frac{c^2}{2G} dr = \kappa r dr, \quad (36)$$

then the surface gravity is given by

$$\kappa = \frac{c^2}{2Gr}. \quad (37)$$

We see from Eq. 37 that only if $r \rightarrow \infty$, then $\kappa \rightarrow 0$. If $r = r_+$, κ reaches the minimum value

$$\kappa = \frac{c^4}{4MG} \quad (38)$$

the surface gravity at the event horizon.

6 Equation of State

The black hole equation of state is analysed in [1], in terms of volume and pressure. In this work, the pressure P will be considered as a function of \mathcal{A} , since thermodynamic equilibrium exists only on the surface of the black hole. Equation of state, temperature and entropy are related to \mathcal{A} .

We first consider the first law of thermodynamics,

$$U = TS - PV. \quad (39)$$

Solving for P and considering the Eqs. 18, 22-25, $V = (1/6\sqrt{\pi})(\mathcal{A})^{3/2}$ and $r = 1/2(\mathcal{A}/\pi)^{1/2}$ we have

$$P = \frac{3c^4}{2\pi G} \left[\frac{\left(\frac{\mathcal{A}}{\pi}\right)^{1/2}}{16r_+^3} - \left(\frac{\mathcal{A}}{\pi}\right)^{-1} \right] \quad (40)$$

This is the pressure on every surface inside the black hole. If $\mathcal{A} = 4\pi r_+^2$ (the event horizon),

$$P = -\frac{3c^2}{16\pi G r_+^2}. \quad (41)$$

That is, at the event horizon, the pressure P is negative. A closer look at the brackets in Eq. 40 shows us that for $\mathcal{A} \leq 4\pi r_+^2$ this term is negative,

$$\frac{\left(\frac{\mathcal{A}}{\pi}\right)^{1/2}}{16r_+^3} - \left(\frac{\mathcal{A}}{\pi}\right)^{-1} = \frac{1}{4r_+^2} \left[\frac{1}{2} \left(\frac{r}{r_+}\right)^3 - 1 \right] \leq 0,$$

where $\mathcal{A} = 4\pi r^2$.

This means that even inside the black hole $P < 0$, and for $r = \sqrt[3]{2}r_+$, outside the black hole, it is zero. In the next section we explain why this is so.

Let us now consider the Eqs. 22, 25 and solving 39 for P , we get.

$$P(\mathcal{A}, T) = \frac{3\sqrt{\pi}k_B c^3 T}{2G\hbar} \mathcal{A}^{-1/2} - \frac{3c^4}{2G} \mathcal{A}^{-1} \quad (42)$$

At constant T P has a maximum value at

$$\mathcal{A}_0 = \frac{1}{\pi} \left(\frac{2c\hbar}{k_B T} \right)^2,$$

equal to

$$P(\mathcal{A}_0, T) = \frac{3\pi}{8G} \left(\frac{k_B c T}{\hbar} \right)^2, \quad (43)$$

but this value does not correspond to the interior of the black hole, as we saw earlier.

Eq. 42 represents the pressure of the black hole as a function of area \mathcal{A} and temperature T , but we can also rewrite this by solving for T to get

$$T(\mathcal{A}, P) = \frac{c\hbar}{\pi k_B} \left(\frac{\mathcal{A}}{\pi} \right)^{-1/2} + \frac{2\hbar G P}{3c^3 k_B} \left(\frac{\mathcal{A}}{\pi} \right)^{1/2}, \quad (44)$$

which correspond to the equation of state of the black hole in terms of \mathcal{A} and P .

For a constant pressure P , T has a minimum value at

$$\mathcal{A}_1 = \frac{3c^4}{2GP}$$

equal to

$$T(\mathcal{A}_1, P) = \frac{2\hbar}{k_B c} \sqrt{\frac{2PG}{3\pi}}. \quad (45)$$

The area of the event horizon is equal to,

$$\mathcal{A}_{eh} = 16\pi \frac{M^2 G^2}{c^4},$$

and $\mathcal{A}_1 = 2\mathcal{A}_{eh}$.

Then, Eq. 45 does not match the temperature of the black hole inside.

7 Pressure, Force and Potential

In the previous section, it was established that inside the black hole $P < 0$. Now we return to analyze this fact. To do this, we again consider 39 and solve it for P , now taking into account that $V = 4/3\pi r^3$ and $\mathcal{A} = 4\pi r^2$.

$$P(\mathcal{A}, r) = \frac{3c^4}{4r_+^3 G} \frac{r^3}{\mathcal{A}} - \frac{3Mc^2}{r_+} \frac{1}{\mathcal{A}}. \quad (46)$$

The above expression makes us see two forces, one of which depends on mass and the other does not.

$$F_{bh} = \frac{3c^4}{4G} \left(\frac{r}{r_+} \right)^3 \quad (47)$$

$$F_v = \frac{3c^4}{2G} \quad (48)$$

As a result, a net force acts on the surface \mathcal{A} , which is given by

$$F_n = \frac{3c^4}{2G} \left[\frac{1}{2} \left(\frac{r}{r_+} \right)^3 - 1 \right]. \quad (49)$$

This force acts perpendicular to the surface \mathcal{A} and it is radial outward from the black hole because in it $F_v > F_{bh}$, then

$$\mathbf{F}_n = \frac{3c^4}{2G} \left[\frac{1}{2} \left(\frac{r}{r_+} \right)^3 - 1 \right] \hat{\mathbf{r}}. \quad (50)$$

We consider the Origin in the infinite. On the event horizon this force

$$\mathbf{F}_n(r_+) = -\frac{3c^4}{4G} \hat{\mathbf{r}} \quad (51)$$

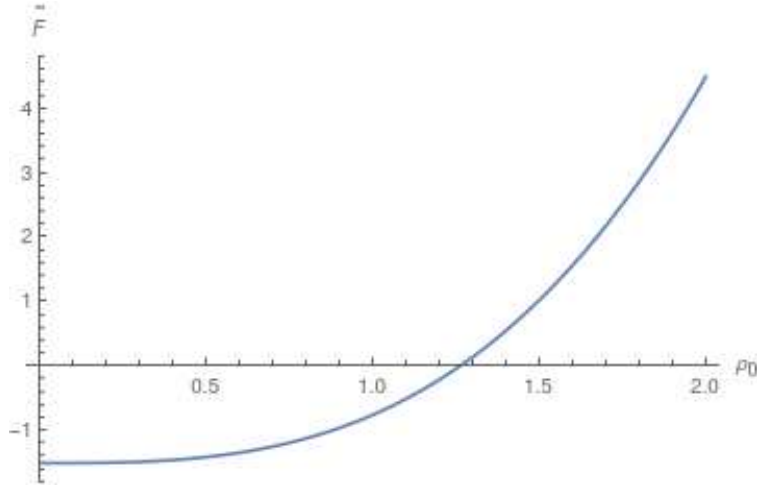


Figure 2: The net force \tilde{F} acting on the surface of the black hole

In the figure above, $\tilde{F} = F_n/F_0$, where.

$$F_0 = \frac{c^4}{G}.$$

Now it is clear why the pressure P at the event horizon is negative. No matter how big the black hole is, this force has the same value at the event horizon. This fact has an important consequence. If we assume the entire universe to be a black hole, in the sense that no matter can escape from it, then this force tends to expand it at the event horizon. The black hole force, F_{bh} , depends on the mass and corresponds to an attractive force and the vacuum force, F_v , to an attractive one in the opposite direction.

\mathbf{F}_n is of the form $\mathbf{F}_n = F(r)\hat{\mathbf{r}}$ and as we know, this type of force is conservative. This means that we can derive a corresponding potential energy $U(r)$. The integral of 50 from r_+ to a point r , inside the black hole, we obtain

$$U(r_+) - U(r) = \int_{r_+}^r F_n dr \quad (52)$$

$$U(r_+) - U(r) = \frac{3c^4}{2G} \int_{r_+}^r \left[\frac{1}{2} \left(\frac{r}{r_+} \right)^3 - 1 \right] dr,$$

$$U(r) = -\frac{3Mc^2}{8} \left[\left(\frac{r}{r_+} \right)^4 - 8 \left(\frac{r}{r_+} \right) + 7 \right] + U(r_+). \quad (53)$$

To determine the potential $U(r)$ at a point r inside the black hole, we need to know the potential at the surface of the black hole, $U(r_+)$.

8 Motion Analyses

We now consider the potential given by Eq. 53 and will try to describe qualitatively the motion of a classical particle of mass m in this potential field, for the case when its energy is zero and $U(r_+) = 0$. In this case we do not consider the angular momentum of the particle.

$$U(r) = -\frac{3Mc^2}{8} \left[\left(\frac{r}{r_+} \right)^4 - 8 \left(\frac{r}{r_+} \right) + 7 \right]. \quad (54)$$

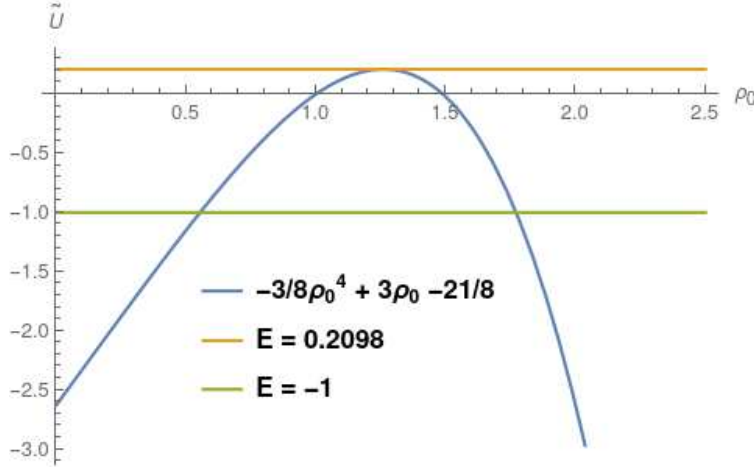


Figure 3: $U(\rho_0)$ is not symmetric with respect to $\rho_0 = \sqrt[3]{2}$. Here $\tilde{U} = U/Mc^2$.

The area $1 < r < 1.5r_+$ is classically forbidden for a particle, considering its energy E equal to zero. For photons, only the surface of the event horizon is allowed, in which case $U(r) = 0$, $r = r_+$ and $r \approx 1.5r_+$, the two spheres of the photon. If the particle has energy $E < \tilde{U}_{max} = 0.2098$, it cannot escape the attraction of the black hole. The repulsive force on the right side of $\rho_0 = \sqrt[3]{2}$ is stronger than on the left side. If the particle has energy $E < 0$ and comes from infinity, the hole exerts a strong repulsive force on it, perhaps this is the origin of the relativistic jet observed near the black hole. The opposite occurs when the particle comes from the left with energy $E < 0$. For the case when $E > \tilde{U}_{max}$ and the particle goes to the left, its position tends to infinity. For $E = 0.2098$, any perturbation tends to make the particle go to infinity or fall inside the black hole.

9 The Hawking entropy

At the event horizon, the net force \mathbf{F}_n and the Hawking temperature T_H are constants. We can show that $1/3$ of the ratio of the net work done for this force from $r = 0$ to the event horizon r_+ to the Hawking temperature T_H is equal to the Hawking entropy of the black hole, that is

$$S_H = \frac{1}{3} \frac{W}{T_H}. \quad (55)$$

The net work is

$$\begin{aligned}
W &= \int_0^{r_+} \frac{3c^4}{4G} dr \\
&= \frac{3c^4}{4G} r_+.
\end{aligned} \tag{56}$$

$$\begin{aligned}
S_H &= \frac{1}{3} \frac{3c^4}{4G} r_+ \frac{1}{T_H} \\
&= \frac{k_B c^3}{4\hbar G} \mathcal{A},
\end{aligned} \tag{57}$$

in this way we demonstrate the origin of the Hawking entropy.

10 Expanding universe

Why is the universe expanding? We can answer this question by looking again at our simplest model, which consists of a Schwarchild black hole and the vacuum. A closer look at the net force F_n shows us that as the radius r_+ increases, i.e. the mass increases, the black hole force, F_{bh} , decreases.

$$F_n(r) = \frac{3}{2} \frac{c^4}{G} \left[\frac{1}{2} \left(\frac{r}{r_+} \right)^3 - 1 \right] \tag{58}$$

As r_+ increases, the black hole force F_{bh} decreases, but the net force F_n on any black hole surface increases. If we assume the early universe to be a black hole containing a finite amount of matter, then this force tends to expand it as the amount of matter increases. If the amount of matter increases by some process, the net force increases and approaches the maximum value $3/2c^4/G$ as the mass tends toward infinity. This is why the universe is expanding at an accelerated rate.

11 Concluding remarks

Treating the mass density of the black hole as a function of $\rho \propto \frac{1}{r^2}$ and its temperature as $T \propto \frac{1}{\rho}$, we investigate its intrinsic properties. We found for a given black hole that both the temperature and entropy have a parabolic dependence on the ratio r . Since the entropy of an isolated system is increasing, it is necessary that the coefficient a is positive, which implies that the head capacity is also positive. Considering the black hole equation, we obtain its equations of state as a function of area \mathcal{A} . A conservative force and the corresponding potential energy were also obtained. This net conservative force is composed of two forces, one corresponding to the black hole and the other to the vacuum. We show that in the black hole limit the vacuum force is larger than that of the black hole and that at the event horizon the pressure is negative. We make a qualitative analysis of the motion in the potential field and show the existence of a forbidden region for the total energy equal to zero, from the event horizon to 1.5 of this radius, both corresponding to the light spheres. We show that the Hawking entropy has its origin in the work done by the net force at the event horizon, from zero to the Schwarzschild radius over constant temperature. Finally, we can briefly explain the accelerated universe, taking into account the model studied and the two forces acting on the black hole.

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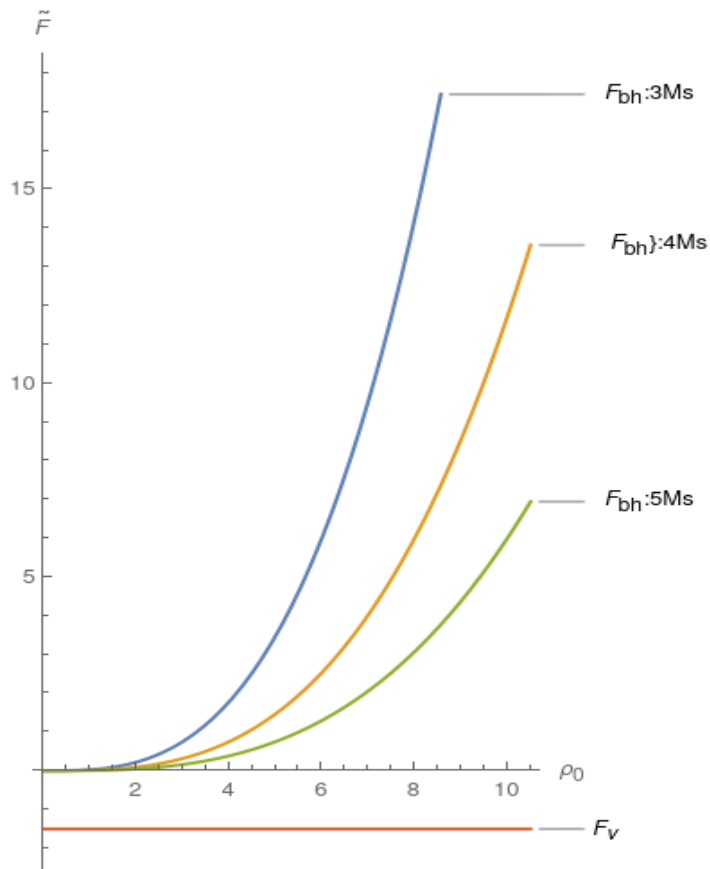


Figure 4: The black hole force F_{bh} (for three values of the mass M : $3M_S$, $4M_S$ and $5M_S$) and the vacuum force F_v , $\tilde{F} = F/c^4/G$.