Controllability of Nonlinear Deterministic Chaotic Systems with Control-Induced Delay

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Research Article

Keywords: Approximate controllability, Delay differential systems, Control delays, Local Lipschitz condition, Nonlinear chaos

Posted Date: September 28th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-686705/v1

Controllability of Nonlinear Deterministic Chaotic Systems with Control-Induced Delay

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Abstract

This paper presents the analysis of a class of retarded nonlinear chaotic systems with control-induced delay. The mild solution is obtained by using the local Lipschitz condition on nonlinearity and Banach contraction principle. The approximate controllability for linear and nonlinear control delay systems has been established by sequence method and using the Nemytskii operator. The application of results is explained through an example of a parabolic partial differential equation.

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1. Introduction

The aftereffect in the dynamics of a natural phenomena is very common or to say spontaneous. The control problems are prompt to understand the delay effect. This paper introduces the control-induced delay in a dynamical system that appears in various real life phenomena. It can be observed from the present pandemic situation. It is a drastic and challenging situation during its wave flow through a herd of dense population. The curative form of control interventions is still lacking which results in loss of many lives. It is a real life experience of control-induced delay exhibiting the chaotic dynamics. Almost all the natural disasters are chaotic and observe the control-induced delay. The delay in control inputs results in the retarded output which behaves nonlinearly too. The action of retarded controls is also nonlinear due to its natural practice to overcome and manage the phenomena to perform in a favorable way. Thus the control-induced retarded system is better representation to describe a system having natural control delay. Motivated by this, this work presents the existence, uniqueness and controllability results for a class of retarded nonlinear deterministic chaotic systems with control-induced delay.

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Let X and U be Hilbert spaces of state and control with norm $||\cdot||$ and $||\cdot||_U$, respectively. Define $\mathscr{C}_t := C([-\tau, t]; X)$ for $\tau > 0$, $Y := L^2([0,T]; X)$ and $V := L^2([0,T]; U)$. The set $\mathscr{C}_t$ is a Banach space equipped with the norm $||x||_t := \sup_{-\tau \le \xi \le t} ||x(\xi)||$, Y and V are standard Hilbert spaces. The norms of Y and V are denoted by $||\cdot||_Y$ and $||\cdot||_V$, respectively. The norm $||\cdot||_{op}$ denotes the operator norm between specified spaces of operators.

An abstract semilinear chaotic system in X is given by

$$\dot{x}(t) = Ax(t) + g(x) \tag{1.1a}$$
$$x(0) = x_0, \tag{1.1b}$$

where $x(t) \in X$, $A : D(A) \subset X \to X$ is system operator, $D(A)$ is dense subspace of X, $g : X \to X$ is a nonlinear function creating chaos. The existence and uniqueness for solution of (1.1) is well presented in books [1, 2]. The chaos in the system may be reduced and the system performs in feasible way by applying control. Thus, the abstract semilinear chaotic control system is

$$\dot{x}(t) = Ax(t) + Bu(t) + g(x) + h(u) \tag{1.2a}$$
$$x(0) = x_0, \tag{1.2b}$$

where $u(t) \in U$, $B : U \to X$ is bounded linear operator and $h : U \to X$ is nonlinear control function. The solution and controllability results of system (1.2) can be seen in [3, 4] and references therein.

This work introduces control-induced delay following the natural observation of delay in the application of control. Let $\alpha : [0,T] \to [-\tau, T]$ be a nondecreasing nonexpansive continuous function satisfying $\alpha(t) \le t$. To specify the aftereffect due to α, we denote $Y_\alpha = L^2(\mathcal{R}(\alpha); X)$ and $V_\alpha = \{u(\alpha(t)) : u \in L^2(\mathcal{R}(\alpha); U)\}$, where $\mathcal{R}(\alpha)$ is the range of α. Let us consider the following abstract semilinear retarded control system:

$$\dot{x}(t) = Ax(t) + (B_0 u)(t) + (B_1 u)(\alpha(t)) + f(t, x_{\alpha(t)}, u(\alpha(t))),\ t \in [0,T] \tag{1.3a}$$
$$x_0 = \phi \quad \text{on} \quad [-\tau, 0], \tag{1.3b}$$

where $x \in \mathscr{C}_T$ describes trajectory of the system (1.3), $x_{\alpha(t)} \in \mathscr{C}_0$ is the retarded state function, $B_0 : V \to Y$ and $B_1 : V \to Y_\alpha$ are linear control operators, $f : [0,T] \times \mathscr{C}_0 \times V_\alpha \to X$ is a nonlinear function, and $\phi \in \mathscr{C}_0$ is initial trajectory.

There are two major contributions in the study of the system (1.3). First, the control-induced delay and its influence on the chaotic nature of nonlinearity. Second, a two-steps delay in state, exhibited by $x_{\alpha(t)}(\eta) = x(\alpha(t) + \eta), \eta \in [-\tau, 0]$. It was introduced in the pioneer work of Tomar and Kumar [5] and later considered by Haq and Sukavanam [6]. It provides feedback information in the further decision of control inputs. For example, the present pandemic crisis, the patients are getting curative medical support later than its actual requirement due to unavailability and thus the recovery rate is delayed.

The study of delay differential equations in control theory has adjoined real world approach to mathematical sciences more efficiently. There are several mathematical contributions to the existence theory and controllability of semilinear delay differential systems since its inception. In the pioneering work of Dubey and Bahuguna [7], the existence and regularity of the solution for a class of retarded semilinear differential equations with nonlocal history conditions are obtained by fixed point theorem. Chukwu [8] presented local controllability results for nonlinear delay systems. Dauer and Mahmudov [9] established sufficient conditions for the approximate and complete controllability of semilinear functional differential systems in Hilbert spaces. Sukavanam and Tomar [10] introduced the inclusion range condition of nonlinearity and control operator. Jeong *et al.* [11, 12] weakened Lipschitz continuity and the uniform boundedness of the nonlinearity and proved the inclusion of reachable sets of semilinear retarded functional differential equations into that of the corresponding linear system. Recently, Kim *et al.* [13] considered Fredholm alternative for nonlinear operators and proved approximate controllability. Henríquez and Prokopczyk [14] studied the controllability and stabilizability for a time-varying linear abstract control system with distributed delay in the state variables. The pioneering work of Klamka [15] established that the constrained local relative controllability of the semilinear dynamical point delay system is implied by the constrained global relative controllability of the associated linear control system. Sakthivel *et al.* [16] presented the approximate controllability of deterministic and stochastic nonlinear impulsive differential equations with resolvent operators and unbounded delay. Shukla *et al.* [17] applied sequence method to establish the approximate controllability of state delay semilinear system. Hernandez and O'Regan [18] introduced state dependent nonlocal conditions, and proved the existence and uniqueness of solution. Hernandez *et al.* [19] extended the study to the approximate controllability of a general class of first order abstract control problems with state-dependent delay. Recently, Haq and Sukavanam [20] established mild solution and approximate controllability of retarded semilinear differential equations with control delays and nonlocal conditions.

The paper is organized as follows: Section 2 describes the existence and uniqueness of mild solution of the system (1.3). Section 3 presents the controllability results for the linear and the semilinear control delay system. In the last, Section 4 exhibits the application of established results.

2. Mild Solution

The theory of abstract differential equations has defined classical and mild solutions. The mild solution is feasible for the analysis of dynamical systems with chaos. Deterministic chaos is sensitive to the given initial condition. Therefore, the interval of existence of the mild solution, $[-\tau, T]$, depends upon the delay initial condition ϕ and control function u, say $[-\tau, T_{u,\phi}]$. We put the following fundamental assumptions:

(A1) The system operator A generates a C_0-semigroup $S(t), t \geq 0$, on X. Let $M_0 \geq 1$ be such that $||S(t)||_{op} \leq M_0$ for $t \in [0, T]$.

(A2) The control operators B_0 and B_1 are bounded linear operators. Let $M_B := \max\{||B_0||_{op}, ||B_1||_{op}\}$.

(A3) The nonlinear map f is measurable in t and satisfies local Lipschitz condition, *i.e.* for every $R \geq 0$, there is a constant $N_R > 0$ such that

$$||f(t, x_1, u_1) - f(s, x_2, u_2)|| \leq N_R\big(|t-s| + ||x_1 - x_2||_0 + ||u_1 - u_2||_U\big)$$

for all $t, s \in [0, T_{u,\phi}]$ and $(x_1, u_1),\ (x_2, u_2) \in B_R((\phi, 0); \mathscr{C}_0 \times U)$, where

$$B_R((\phi, 0); \mathscr{C}_0 \times U) := \{(x, u) : ||x - \phi||_0 + ||u||_U \leq R\}$$

is a ball in $\mathscr{C}_0 \times U$. Moreover, $f(t, \phi, 0) = 0$.

Remark 2.1. The chaotic behavior of nonlinearity f may exhibit drastically varying amplitude of oscillations. So local Lipschitz condition is a feasible assumption for the existence of solution.

Definition 2.2. *The mild solution of* (1.3) *on* $[-\tau, T]$ *is a function* $x \in \mathscr{C}_T$ *given by*

$$x(t) = \begin{cases} \phi(t), \ \ t \in [-\tau, 0] \\ S(t)\phi(0) + \int_0^t S(t-s)\big[(B_0u)(s) + (B_1u)(\alpha(s)) + f(s, x_{\alpha(s)}, u(\alpha(s)))\big]ds. \end{cases} \tag{2.1}$$

To express the dependence of the mild solution (2.1) on u and ϕ, we write $x(t) = x(t; u, \phi)$. The dependence shows the uniqueness property.

Proposition 2.3. *Under assumptions* $(A1) - (A3)$*, the integral equation* (2.1) *satisfies that the maps* $u \mapsto x(t; u, \phi)$ *and* $\phi \mapsto x(t; u, \phi)$ *are Lipschitz continuous for given* ϕ *and* u*, respectively.*

Proof. Let $x^1, x^2 \in \mathscr{C}_{T_{u,\phi}}$ be two mild solutions corresponding to controls $u_1, u_2 \in V$ for given $\phi \in \mathscr{C}_0$. Then

$$x^i(t) = x^i(t; u_i, \phi) = \begin{cases} \phi(t), \quad t \in [-\tau, 0] \\ S(t)\phi(0) + \int_0^t S(t-s)\big[(B_0u_i)(s) + (B_1u_i)(\alpha(s))\big]ds. \\ \qquad\qquad + \int_0^t S(t-s)f(s, x^i_{\alpha(s)}, u_i(\alpha(s)))ds, \ t \geq 0. \end{cases}$$

Since $u_i(t) = 0$ for $t \in [-\tau, 0]$, therefore we concern for $t \in (0, T_{u,\phi}]$. Let

$(x^i, u_i) \in B_R((\phi, 0); \mathscr{C}_0 \times U)$, $i = 1, 2$. Then

$$\begin{aligned} ||x^1(t) - x^2(t)|| &\leq M_0 \int_0^t \big(||B_0(u_1 - u_2)(t)|| + ||B_1(u_1 - u_2)(\alpha(t))||\big) ds \\ &+ M_0 N_R \int_0^t \big(||x^1_{\alpha(s)} - x^2_{\alpha(s)}||_0 + ||u_1(\alpha(s)) - u_2(\alpha(s))||_U\big) ds \\ &\leq M_0(2M_B + N_R)\sqrt{T_{u,\phi}}||u_1 - u_2||_V \\ &+ M_0 N_R \int_0^t ||x^1_{\alpha(s)} - x^2_{\alpha(s)}||_0 ds. \end{aligned} \tag{2.2}$$

For $s \in [0, T_{u,\phi}]$ and $\eta \in [-\tau, 0]$, we have $-\tau + \alpha(0) \leq \alpha(s) + \eta \leq s$. Thus

$$\begin{aligned} ||x^1_{\alpha(s)} - x^2_{\alpha(s)}||_0 &= \sup_{-\tau \leq \eta \leq 0} ||x^1(\alpha(s) + \eta) - x^2(\alpha(s) + \eta)|| \\ &\leq \sup_{-\tau \leq \xi \leq 0} ||x^1(\xi) - x^2(\xi)|| + \sup_{0 \leq \xi \leq s} ||x^1(\xi) - x^2(\xi)|| \\ &\leq \sup_{0 \leq \xi \leq s} ||x^1(\xi) - x^2(\xi)|| = ||x^1 - x^2||_s. \end{aligned}$$

Hence from (2.2), we get

$$||x^1 - x^2||_t \leq M_0(2M_B + N_R)\sqrt{T_{u,\phi}}||u_1 - u_2||_V + M_0 N_R \int_0^t ||x^1 - x^2||_s ds.$$

Now by Gronwall's inequality

$$||x^1 - x^2||_t \leq e^{M_0 N_R} M_0(2M_B + N_R)\sqrt{T_{u,\phi}}||u_1 - u_2||_V$$

which concludes the Lipschitz continuity of $u \mapsto x(t; u, \phi)$. Similarly, we can prove the other. □

Theorem 2.4. *The semilinear retarded control system* (1.3) *has a mild solution under assumptions* (A1) - (A3) *and is unique up to u and ϕ.*

Proof. Let us consider a function $\mathcal{F} : \mathscr{C}_{T_{u,\phi}} \to \mathscr{C}_{T_{u,\phi}}$ as

$$(\mathcal{F}x)(t) = \begin{cases} \phi(t), \ t \in [-\tau, 0] \\ S(t)\phi(0) + \int_0^t S(t-s)\big[(B_0 u)(s) + (B_1 u)(\alpha(s)) + f(s, x_{\alpha(s)}, u(\alpha(s)))ds\big] ds. \end{cases}$$

From the delay initial condition ϕ, let us define a priori function $\hat{\phi} : [-\tau, T_{u,\phi}] \to X$ as $\hat{\phi}(t) = \phi(t)$ if $t \in [-\tau, 0]$ and $\hat{\phi}(t) = \phi(0)$ if $t \in [0, T_{u,\phi}]$.

For given control $u \in L^2([0, T_{u,\phi}]; U)$ and $\phi \in \mathscr{C}_0$, let us take a ball of radius $r_{(u,\phi)} > 0$ and center at $\hat{\phi} \in \mathscr{C}_{T_{u,\phi}}$ as

$$B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_T) := \{x : ||x - \hat{\phi}||_T \leq r_{(u,\phi)}\},$$

where $r_{(u,\phi)}$ satisfies

$$\sup_{0 \leq t \leq T_{u,\phi}} ||(S(t) - I)\phi(0)|| \leq \frac{r_{(u,\phi)}}{2}$$

and $T_0 \in (0, T_{u,\phi}]$ be such that

$$M_0 T_0 N_R < \frac{1}{4}, \tag{2.3}$$

$$M_0 \sqrt{T_0}(2M_B + N_R)||u||_V \leq \frac{r_{(u,\phi)}}{4}. \tag{2.4}$$

We shall show that $\mathcal{F}$ maps $B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$ into itself.

Clearly $\phi \in B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$, therefore it is obvious for $t \in [-\tau, 0]$. Let $x \in B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$ and $t \in [0, T_0]$. Then

$$\begin{aligned}
||(\mathcal{F}x)(t) - \hat{\phi}(t)|| &\leq ||S(t)\phi(0) - \phi(0)|| + M_0 \int_0^t ||(B_0 u)(s) + (B_1 u)(\alpha(s))||ds \\
&+ M_0 \int_0^t ||f(s, x_{\alpha(s)}, u(\alpha(s)))||ds \\
&\leq ||(S(t) - I)\phi(0)|| + M_0 M_B \int_0^t \big(||u(s)||_U + ||u(\alpha(s))||_U\big)ds \\
&+ M_0 \int_0^t \big(||f(s, x_{\alpha(s)}, u(\alpha(s))) - f(s, \phi, 0)||ds \\
&\leq \frac{r_{(u,\phi)}}{2} + 2M_0 M_B \sqrt{T_0}||u||_V \\
&+ M_0 N_R \int_0^t \big(||x_{\alpha(s)} - \phi||_{T_0} + ||u(\alpha(s))||_U\big)ds \\
&\leq \frac{r_{(u,\phi)}}{2} + M_0 \sqrt{T_0}\big(2M_B + N_R\big)||u||_V + M_0 T_0 r_{(u,\phi)} N_R.
\end{aligned}$$

Since from the conditions (2.3) and (2.4), we get

$$M_0\big(\sqrt{T_0}(2M_B + N_R)||u||_V + T_0 r_{(u,\phi)} N_R\big) \leq \frac{r_{(u,\phi)}}{2},$$

therefore

$$||\mathcal{F}x - \hat{\phi}||_{T_0} \leq r_{(u,\phi)}.$$

Hence $\mathcal{F}$ maps $B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$ into itself. Further, suppose $x, y \in B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$ for a control $u \in V$. Then

$$\begin{aligned}
||\mathcal{F}x - \mathcal{F}y||_{T_0} &= \sup_{0<t\leq T_0} ||(\mathcal{F}x)(t) - (\mathcal{F}y)(t)|| \\
&= \sup_{0<t\leq T_0} M_0 \int_0^t \big|\big|f(s, x_{\alpha(s)}, u(\alpha(s))) - f(s, y_{\alpha(s)}, u(\alpha(s)))\big|\big|ds \\
&\leq M_0 N_R \sup_{0<t\leq T_0} \int_0^t ||x_{\alpha(s)} - y_{\alpha(s)}||_0 ds \\
&\leq T_0 M_0 N_R ||x - y||_{T_0}.
\end{aligned}$$

Since $T_0 M_0 N_R < \frac{1}{4}$, therefore $\mathcal{F}$ is a strict contraction. Hence $\mathcal{F}$ has unique fixed point $x \in B_{r_{(u,\phi)}}(\hat{\phi}; \mathscr{C}_{T_0})$ by the Banach Contraction Principle. This proves the existence of mild solution.

Now to prove the uniqueness up to u and ϕ, we have to show the following:

(i) $u \mapsto x(t; u, \phi)$ is Lipschitz map for given ϕ.

(ii) $\phi \mapsto x(t; u, \phi)$ is Lipschitz map for given u.
(iii) $(u, \phi) \mapsto x(t; u, \phi)$ is Lipschitz map.

The results (i) and (ii) are obtained in Proposition 2.3. So we prove (iii). Let $x^i(\cdot, u_i, \phi_i) \in \mathscr{C}_{T_0}$ be a mild solution corresponding to $(u_i, \phi_i) \in V \times \mathscr{C}_0$, $i = 1, 2$. Then

$$x^i(t) = x^i(t; u_i, \phi_i) = \begin{cases} \phi_i(t), \quad t \in [-\tau, 0] \\ S(t)\phi_i(0) + \int_0^t S(t-s)\big[(B_0 u_i)(s) + (B_1 u_i)(\alpha(s))\big]ds \\ \qquad\qquad + \int_0^t S(t-s) f(s, x^i_{\alpha(s)}, u_i(\alpha(s)))ds, \; t > 0. \end{cases}$$

The Lipschitz condition trivially holds for $t \in [-\tau, 0]$ with constant $1 + \delta$, $\delta > 0$. So, consider $t \in [0, T_0]$ for which we get

$$||x^1 - x^2||_t \le M_0\big(||\phi_1 - \phi_2||_0 + (2M_B + N_R)\sqrt{T_0}||u_1 - u_2||_V\big) + M_0 N_R \int_0^t ||x^1 - x^2||_s ds.$$

By Gronwall's inequality, we obtain

$$||x^1 - x^2||_t \le e^{M_0 N_R} M_0\big(||\phi_1 - \phi_2||_0 + (2M_B + N_R)\sqrt{T_0}||u_1 - u_2||_V\big).$$

This concludes the Lipschitz continuity of $(u, \phi) \mapsto x(t; u, \phi)$.

Next, we extend the mild solution $x \in \mathscr{C}_{T_0}$ of (1.3) to the interval $[-\tau, 2T_0]$ by defining $x(t) = h(t)$ on $[T_0, 2T_0]$ as follows:
if $0 < \tau < T_0$, then

$$h(t) = S(t - T_0)x(T_0) + \int_{T_0}^t S(t-s)\big[(B_0 u)(s) + (B_1 u)(\alpha(s)) + f(s, h_{\alpha(s)}, u(\alpha(s)))\big]ds$$

and if $0 < T_0 < \tau$, then

$$h(t) = \begin{cases} \psi(t), \quad t \in [T - \tau, 0] \\ S(t)\psi(0) + \int_0^t S(t-s)\big[(B_0 u)(s) + (B_1 u)(\alpha(s))\big]ds \\ \qquad\qquad + \int_0^t S(t-s) f(s, h_{\alpha(s)}, u(\alpha(s)))ds, \; t \in [0, T_0] \\ S(t - T_0)x(T_0) + \int_{T_0}^t S(t-s)\big[(B_0 u)(s) + (B_1 u)(\alpha(s))\big]ds, \\ \qquad\qquad + \int_{T_0}^t S(t-s) f(s, h_{\alpha(s)}, u(\alpha(s)))ds, \; t \in [T_0, 2T_0], \end{cases}$$

where ψ is restriction of ϕ on $[T - \tau, 0]$. Proceeding this way, the interval of existence and uniqueness of the mild solution can be extended to $[-\tau, nT_0]$, where $n \in \mathbb{N}$ is such that $T_{u,\phi} \le nT_0$. This completes the proof of theorem. □

3. Approximate Controllability

The problem of controlling chaos was explored by E. Ott, C. Grebogi and J.A. Yorke [21]. They explained that the chaotic trajectory may become periodic by the application of suitable control. It was also demonstrated by Controlled Closing Lemma [1]. This lemma presents the condition for the existence of control by which a chaotic motion is turned into periodic in finite time. This notion of closed loop trajectory is extended to establish sufficient conditions for the approximate controllability of continuous-time systems with deterministic chaos. Deterministic chaos is assumed to be noise-free unlike the stochastic chaos. Stochastic chaos is noise-induced and is dealt with stochastic differential equations. The control of deterministic chaos prevails two peculiar features: sensitivity to the initial state and dependency on the desired state. Accordingly we have the next definition.

Definition 3.1. *A chaotic control system is approximately controllable from an initial state* $\phi(0)$ *to any desired state* $\hat{x} \in X$ *if for every* $\varepsilon > 0$ *there exists a control* $u \in V$ *and finite time* T *such that the mild solution* $x \in \mathscr{C}_T$ *satisfies* $||x(T) - \hat{x}|| < \varepsilon$.

Remark 3.2. The approximate controllability of a chaotic control system is analogous to steering the trajectory from an initial state to the vicinity of the desired state in finite time by the action of control. The final time T directly depends upon the given initial and desired states, however inversely on ε and control u.

Let us write the linear control delay system as

$$\dot{y}(t) = Ay(t) + (B_0u)(t) + (B_1u)(\alpha(t)), \quad t \geq 0 \tag{3.1a}$$
$$y_0 = \phi \quad \text{on} \quad [-\tau, 0], \tag{3.1b}$$

and the associated linear control system without delay as

$$\dot{z}(t) = Az(t) + (B_0u)(t), \quad t \geq 0 \tag{3.2a}$$
$$z(0) = \phi(0). \tag{3.2b}$$

The mild solution of (3.1) is

$$y(t) = \begin{cases} \phi(t), & t \in [-\tau, 0] \\ S(t)\phi(0) + \int_0^t S(t-s)\big[B_0u(s) + B_1u(\alpha(s))\big]ds, & t \geq 0 \end{cases} \tag{3.3}$$

and the mild solution of (3.2) is

$$z(t) = S(t)\phi(0) + \int_0^t S(t-s)B_0u(s)ds. \tag{3.4}$$

The approximate controllability of linear control systems (3.1) and (3.2) is defined same as the Definition 3.1. The solution and controllability properties of (3.2) are well-explained in the books by Curtain and Zwart [22], and Zabczyk [23]. So, we assume the following:

(A4) The linear control system (3.2) is approximately controllable.

Theorem 3.3. *Under assumptions* (A1), (A2) *and* (A4)*, the linear control delay system* (3.1) *is approximately controllable.*

PROOF. Let $\hat{y}$ be the desired state. We have to show that for all $\varepsilon > 0$ there exists a control $u \in V$ and T such that $||y(T; u) - \hat{y}|| < \varepsilon$.

From equations (3.3) and (3.4), we have $y(t; u) = z(t; u) + \xi_t$, where

$$\xi_t = \int_0^t S(t-s)(B_1 u)(\alpha(s))ds,\ t \geq 0.$$

Let $t_1 > 0$ be such that $\alpha(s) \leq 0$ for $s \in (0, t_1]$. Take a sequence $0 = t_0 < t_1 < t_2 < ... < t_n < t_{n+1} = T$ such that $\alpha(s) \leq t_i$ for $s \in (0, t_{i+1}]$ and $i = 0, 1, ..., n$. Let $\hat{y}_1, ..., \hat{y}_{n+1} = \hat{y} \in X$ be given and take $\hat{z}_1 = \hat{y}_1$. Then by approximate controllability of (3.2), we get a control $u_1 \in L^2([0, t_1]; U)$ such that $||z(t_1, u_1) - \hat{z}_1|| < \varepsilon$. Let us take $w_1(t) = u_1(t),\ t \in [0, t_1]$. Since $y(t) \equiv z(t)$ as $\xi_t \equiv 0$ for $t \in (0, t_1]$, therefore

$$||y(t_1; w_1) - \hat{y}_1|| = ||z(t_1; u_1) - \hat{z}_1|| < \varepsilon.$$

Now, we set $\hat{z}_i = \hat{y}_i - \xi_{t_i}$ for $i = 2, ..., n+1$; where $\xi_{t_i} = \int_0^{t_i} S(t_i - s) w_{i-1}(\alpha(s))ds$ and define $w_i \in L^2([0, t_i]; U)$ as

$$w_i(t) = \begin{cases} w_{i-1}(t), & t \in [0, t_{i-1}] \\ u_i(t), & t \in (t_{i-1}, t_i], \end{cases} \qquad \text{for} \quad u_i \in L^2([t_{i-1}, t_i]; U).$$

For $i = 2$, by approximate controllability of linear control system, there exists a control $u_2 \in L^2([t_1, t_2]; U)$ such that (3.4) satisfies $||z(t_2; u_2) - \hat{z}_2|| < \varepsilon$. Further, the mild solution (3.3) of linear control delay system gives $y(t; w_2) = z(t; u_2) + \xi_t$ for $t \in (t_1, t_2]$. Thus

$$||y(t_2; w_2) - \hat{y}_2|| = ||z(t_2; u_2) - (\hat{y}_2 - \xi_{t_2})|| < \varepsilon.$$

Proceeding similarly, finally at $i = n + 1$, there exists $u_{n+1} \in L^2([t_n, t_{n+1}]; U)$ such that $||z(t_{n+1}; u_{n+1}) - \hat{z}_{n+1}|| < \varepsilon$. Then from (3.3), we get

$$||y(t_{n+1}; w_{n+1}) - \hat{y}_{n+1}|| = ||z(t_{n+1}; u_{n+1}) - (\hat{y}_{n+1} - \xi_{t_{n+1}})|| < \varepsilon.$$

Thus we get $w_{n+1} \in L^2([0, T]; U)$ satisfying $||y(T; w_{n+1}) - \hat{y}|| < \varepsilon$. This completes the proof of theorem. □

We shall denote the mild solution as $x(t) = x(0, t; u) \in X$ for the upcoming discussion to express the trajectory from starting time to current time under some control $u \in V$.

If we write $\mathfrak{B}(t, u(t)) = (B_0 u)(t) + (B_1 u)(\alpha(t))$, then we can define a Nemytskii operator $\mathcal{B} : V \to Y$ as $\mathcal{B}u(\cdot) = \mathfrak{B}(\cdot, u(\cdot)) = (B_0 u)(\cdot) + (B_1 u)(\alpha(\cdot))$. Similarly, define a Nemytskii operator $F : \mathscr{C}_0 \times V \to Y$ by $F(x, u)(\cdot) = f(\cdot, x_{\alpha(\cdot)}, u(\alpha(\cdot)))$. Then the mild solution (2.1) of semilinear retarded control system is written as

$$x(0,t;u) = S(t)\phi(0) + \int_0^t S(t-s)\big[\mathcal{B}u(s) + F(x,u)(s)\big]ds$$

and the mild solution (3.3) of linear control delay system is written as

$$y(0,t;u) = S(t)\phi(0) + \int_0^t S(t-s)\mathcal{B}u(s)ds.$$

Let $t_1 > 0$ be such that $\alpha(t) \le 0$ for $t \in [0,t_1]$. Then $u(\alpha(t)) = 0$ and $F \equiv 0$ on $[0,t_1]$. The chaotic behavior of the Nemytskii operator F initially dominates the control function, but after the application of control for sufficient time T, it is overlooked by the control operator. With this concern, the following assumption has practical significance:

(A5) $\mathcal{R}(F) \subset \overline{\mathcal{R}(\mathcal{B})}$ on $[t_1,T]$.

Theorem 3.4. *Under assumptions* $(A1)-(A5)$*, the semilinear retarded control system* (1.3) *is approximately controllable.*

Proof. Let $\hat{x} \in X$ be the desired state and $\varepsilon > 0$ be given. Let $T > 0$ be such that assumptions $(A4)$ and $(A5)$ hold in $[0,T]$ and $[t_1,T]$, respectively. Then by assumption $(A4)$, there exists a control $u_0 \in V$ such that $||\hat{x} - z(0,T;u_0)|| < \varepsilon$. We have $u_0(\alpha(t)) = 0$ and $F \equiv 0$ on $[0,t_1]$, so $x(0,t_1;u_0) = y(0,t_1;u_0) = z(0,t_1;u_0)$. Since assumption $(A4)$ and Theorem 3.3 imply that the linear control delay system is approximately controllable, there exists $u_1 \in L^2([t_1,T];U)$ such that

$$||\hat{x} - y(0,T;w_1)|| < \frac{\varepsilon}{2},$$

where $w_1(t) = \begin{cases} u_0(t), & t < t_1 \\ u_1(t), & t \ge t_1 \end{cases}$, and $y(0,T;w_1) = y(t_1,T;u_1)$.

By assumption $(A5)$, for every $w \in L^2([t_1,T];U)$ there exists $v \in L^2([t_1,T];U)$ such that

$$||\mathcal{B}v - F(x,w)||_Y < \frac{\varepsilon}{2M_0\sqrt{T-t_1}}. \tag{3.5}$$

Define $w_2(t) := \begin{cases} u_0(t), & t < t_1 \\ w(t), & t \ge t_1 \end{cases}$. Then

$$\begin{aligned} x(0,T;w_2) &= S(T)\phi(0) + \int_0^T S(T-s)\big[\mathcal{B}w_2(s) + F(x,w_2)(s)\big]ds \\ &= y(0,T;w_2) + \int_{t_1}^T S(T-s)F(x,w_2)(s)ds. \end{aligned}$$

Now

$$\begin{aligned}||\hat{x} - x(0,T;w_2)|| &\leq ||\hat{x} - y(0,T;w_1)|| + ||y(0,T;w_1) - x(0,T;w_2)|| \\ &< \varepsilon/2 + ||y(0,T;w_1) - x(0,T;w_2)|| \qquad (3.6)\end{aligned}$$

and

$$\begin{aligned}&||y(0,T;w_1) - x(0,T;w_2)|| \\ &= ||y(0,T;w_1) - y(0,T;w_2) - \int_{t_1}^{T} S(T-s)F(x,w_2)(s)ds|| \\ &= ||\int_{t_1}^{T} S(T-s)\mathcal{B}(u_1 - w)(s)ds - \int_{t_1}^{T} S(T-s)F(x,w_2)(s)ds|| \\ &\leq M_0 \int_{t_1}^{T} ||\mathcal{B}(u_1 - w)(s) - F(x,w)(s)||ds \\ &\leq M_0\sqrt{T-t_1}||\mathcal{B}(u_1 - w) - F(x,w)||_Y.\end{aligned}$$

Let us choose $v = u_1 - w$. Then using (3.5), we obtain

$$||y(0,T;w_1) - x(0,T;w_2)|| < \frac{\varepsilon}{2}.$$

Thus from (3.6), we get $||\hat{x} - x(0,T;w_2)|| < \varepsilon$. This completes the proof. □

4. Application

Example 4.1. Consider the following parabolic partial differential equation representing the diffusion:

$$\begin{aligned}\frac{\partial y}{\partial t}(t,x) &= \frac{\partial^2 y}{\partial x^2}(t,x) + b(x)\int_0^t u(s,x)ds + u(\alpha(t),x) \\ &\quad + f(t, y_{\alpha(t)}(x), u(\alpha(t),x)), \quad t \in [0,T],\ x \in [0,\pi] \qquad (4.1a)\end{aligned}$$

$$\frac{\partial y}{\partial x}(t,0) = \frac{\partial y}{\partial x}(t,\pi) = 0, \quad t \in [0,T] \qquad (4.1b)$$

$$y_0(\theta,x) = \phi(\theta,x), \quad \theta \in [-\frac{1}{T},0],\ x \in [0,\pi], \qquad (4.1c)$$

where $y(t,x)$ is the density at time t and at point x, $b \in L^\infty[0,\pi]$ is weight for the linear action of control u, and $\alpha(t) = \dfrac{t^2-1}{T}$. Clearly α satisfies the delay property $\alpha(t) \leq t$ and $\mathcal{R}(\alpha) = [-\frac{1}{T}, T - \frac{1}{T}]$.

We transform the heat equation (4.1) into the abstract differential form by constructing suitable Hilbert spaces.

Let $X = L^2[0,\pi]$ be the state space, $y(t,\cdot)$ be the state and $Y = L^2([0,T];X)$. Define $Ay = \frac{\partial^2 y}{\partial x^2}$ with $D(A) = H^2[0,\pi] \cap H_0^1[0,\pi]$. Then A generates a

C_0-semigroup $S(t), t \geq 0$. Further, $\{\psi_n(x) = \sqrt{\frac{2}{\pi}}\cos(nx) : 0 \leq x \leq \pi\}$ forms an orthonormal basis for X associated to the eigenspectrum $\{\lambda_n = -n^2\}$, $n \in \{0\} \cup \mathbb{N}$, of operator A. Then $S(t)y = \sum_{n=0}^{\infty} e^{-n^2 t}\langle y, \psi_n\rangle\psi_n$.

Let us take $U = L^2[0, \pi]$ and $V = L^2([0, T]; U)$ as the control space. Define control operators B_0 and B_1 as follows:

$$(B_0 u)(t, x) = b(x)\int_0^t u(s, x)ds \quad \text{and} \quad (B_1 u)(\alpha(t), x) = u(\alpha(t), x).$$

Thus B_0 and $B_1 = I$ are bounded linear operators. Further, the nonlinear function f is given as

$$f(t, y_{\alpha(t)}(\cdot, x), u(\alpha(t)), x) = t||y_{\alpha(t)}(\cdot, x)||_0\psi_n(x) + t^2||u(\alpha(t), x)||_V\psi_{n+k}(x).$$

Then the parabolic control system (4.1) resembles the abstract form (1.3). Now, we need to verify that the appropriate operators satisfy the assumptions in the previous sections. For $y^1, y^2 \in \mathscr{C}_0$ corresponding to controls $u_1, u_2 \in V$, we have

$$\begin{aligned}
&||f(t, y^1_{\alpha(t)}(\cdot, x), u_1(\alpha(t), x)) - f(t, y^2_{\alpha(t)}(\cdot, x), u_2(\alpha(t), x))||_X \\
&\leq t||y^1_{\alpha(t)}(\cdot, x) - y^2_{\alpha(t)}(\cdot, x)||_0 + t^2||u_1(\alpha(t), x) - u_2(\alpha(t), x)||_U \\
&\leq \sup_{t\in[0,T]}\{t, t^2\}\big(||y^1_{\alpha(t)}(\cdot, x) - y^2_{\alpha(t)}(\cdot, x)||_0 + ||u_1(\alpha(t), x) - u_2(\alpha(t), x)||_U\big).
\end{aligned}$$

The Nemytskii operator $\mathcal{B}$ is given by $\mathcal{B}u = B_0 u(\cdot, x) + B_1 u(\alpha(\cdot), x)$ and the Nemytskii operator of f is $F(y, u) = f(\cdot, y_{\alpha(\cdot)}(\cdot, x), u(\alpha(\cdot), x))$. Clearly F and $\mathcal{B}$ satisfy the range condition $\mathcal{R}(F) \subset \overline{\mathcal{R}(\mathcal{B})}$. Thus, all the assumptions $(A1)-(A5)$ are satisfied. Hence, (4.1) is approximately controllable.

5. Conclusion

A class of retarded nonlinear deterministic chaotic abstract control systems is studied. The notion of control-induced delay is introduced, and its linear and nonlinear effects are broadly explored. The nonlinearity caused by the delay function is emphasized and its practical significance is demonstrated. It is sometimes difficult to control the chaotic behavior of the system. In such case, suitable choice of control and its sufficient action of time play key role to control the chaos. So, deterministic chaos is analysed and its controllability is established under natural assumptions. To study the control of stochastic chaos and the optimal control are future concern.

Acknowledgement

The author is thankful to IGNTU Amarkantak for support to accomplish this research work.

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Declaration

Funding: The work is supported by IGNTU Amarkantak, India.
Conflicts of interest/Competing interests: Not applicable.
Availability of data and material: Not applicable.
Code availability: Not applicable.