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Harmonic and non-periodic solutions of velocity-dependent conservative equations

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Abstract

We study in this paper a quadratic damping Helmholtz equation presumed to be velocity-dependent conservative nonlinear oscillator. We show that under the usual conditions of existence of particular and exact harmonic solutions, the equation can also exhibit exact and general non-periodic solutions. We show finally the existence of exact and explicit general harmonic and isochronous solutions without requiring that the system Hamiltonian must be identically zero.

Keywords: Quadratic Helmholtz oscillator equation, exact and general solution, quadratic damping, harmonic and isochronous solution.

Introduction

In many works in theory of nonlinear differential equations representing nonlinear oscillators, the purposes were often to calculate periodic solutions or to investigate their dynamical behaviors. To that end, as it is not in general possible to find exact and general periodic solutions, a great number of approximate methods is carried out in the literature to obtain approximations to solutions on the basis of predictions of qualitative theory of differential equations. Contrary to these predictions showing the existence of periodic solutions, some authors have been able to show explicitly and successfully that these equations can also have non-oscillatory solutions so that the application of approximate techniques to solve presumed nonlinear oscillator equations is not consistent. In other words, the use of approximate methods to investigate oscillations in nonlinear differential equations is valid when it is ensured that they have only periodic solutions for real time. As seen in the literature another way to investigate assumed oscillator equations is to use specific methods to secure exact periodic solutions. In general, these methods consist to suppose the analytical form of solution and to determinate the solution parameters by fitting techniques. However, these methods can suffer from the fact that they cannot in general ensure if the presumed conservative oscillator equation is effectively one, that is, if it has only periodic motions. In this paper we consider in this regard the mixed-parity velocity-dependent equation

$$\ddot{x} + \beta \dot{x}^2 + a_1 x + a_2 x^2 + a_3 = 0 \qquad (1)$$

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where overdot means derivative with respect to time, and a_1, a_2, a_3 and β are constants, presumed to be conservative nonlinear oscillator differential equation by the author of [1]. It is interesting to notice that when $a_3 = \beta = 0$, the equation (1) reduces to the presumed Helmholtz oscillator equation. However, in [2] Monsia and coworkers succeeded to show that this equation is a pseudo-oscillator since it can exhibit non-periodic solutions. The equation (1) belongs to the general class of quadratic Lienard type equations [3-5]. This class of equations has been deeply investigated in [4] from the Lie point symmetries point of view. In fact, this class of equations has gained a physical importance since the celebrated oscillator equation introduced in the literature by Mathews and Lakshmanan [6]. The authors [6] succeeded to prove explicitly for the first time the existence of a quadratic velocity-dependent nonlinear oscillator equation having general harmonic periodic solution but with amplitude-dependent frequency. It was an exceptional result obtained by these authors [6] since the properties of cosine function are well known in mathematics and very suitable for practical applications. Later Akande et al. [5] have successfully shown that the existence of a class of quadratic Lienard type equations that can exhibit harmonic periodic solutions. However, as it is known from the famous work of Akande et al. [3] conservative quadratic Lienard type equations can exhibit also non-oscillatory solutions. Indeed, the authors in [3] succeeded to show that the fascinating Mathews-Lakshmanan oscillator equation can admit real and complex-valued non-periodic solutions explicitly. Now, in [1] Gottlieb investigated the equation (1) viewed as a conservative oscillator equation using a specific method like those previously mentioned. Gottlieb claimed to exhibit conditions under which the equation (1) admits exact periodic solutions. The author [1] required that the initial conditions $x(0) = A$, and $\dot{x}(0) = 0$, and the Hamiltonian of the system must be identically zero for the existence of exact harmonic solutions. To that end, the author [1] exhibited exact harmonic solutions under the additional conditions $\operatorname{sgn} a_2 = \operatorname{sgn} \beta$, where $\operatorname{sgn} a_2$ designates the sign of non-zero a_2, in the form

$$x(t) = \alpha \cos wt + \lambda \tag{2}$$

where α and λ are constants, w is the angular frequency of oscillations under the condition that the first integral, that is the Hamiltonian of the system (1) must be identically zero. In this situation the author [1] claimed that the solutions to the equation (1) will not be harmonic but will remain periodic under other convenient initial conditions. To secure an exact harmonic solution of the form

$$x(t) = \pm\sqrt{\frac{-a_3}{a_2}}\cos\left(\sqrt{a_1}\, t\right) \tag{3}$$

where the bias λ is zero, Gottlieb [1] found that the conditions $\operatorname{sgn} a_2 = \operatorname{sgn}(-a_3)$, and $a_1 > 0$ such that $\beta = \frac{a_2}{a_1}$, are required. However, the author [1] was not able to find general harmonic periodic solutions. Therefore, the question is to ask whether the equation (1) can admit general non-periodic solutions, and general harmonic solutions under the Gottlieb conditions where the Hamiltonian of the system is identically different from zero, on the other hand. In this context, the intention in this paper is to show clearly that special initial conditions $x(0) = A$, and $\dot{x}(0) = 0$, where the first integral of the equation (1), that is the system Hamiltonian is identically zero, are not necessary for the existence of explicit and exact harmonic solutions, contrary to the theory by Gottlieb. In other words we show the existence of exact and general harmonic solutions, and prove that under the condition $\operatorname{sgn} a_2 = \operatorname{sgn} \beta$, required in [1] to find exact harmonic solutions, the equation (1) can have exact and general non-periodic solutions. On the other hand, we show under the Gottlieb conditions [1] that $-\frac{a_3}{a_2} > 0$, and $a_1 > 0$, exact and explicit general harmonic solutions that is to say, exact harmonic solutions for arbitrary initial conditions, without the necessity to take the Hamiltonian of the system (1) equal to zero identically, can be easily ensured. In this perspective, we carry out the required theory (section 2) and exhibit (section 3) the non-periodic solutions of the equation (1), and secondly (section 4) calculate its exact and explicit general harmonic solutions. We address finally a conclusion for the work.

2-The general theory

To state the quadratic velocity-dependent Helmholtz equation (1), consider the general class of quadratic Lienard type equations [3]

$$\ddot{x} + \frac{1}{2}\frac{g'(x)}{g(x)}\dot{x}^2 + \frac{a}{2}\frac{f'(x)}{g(x)}x^{\ell} + \frac{a\ell}{2}\frac{f(x)}{g(x)}x^{\ell-1} - \frac{1}{2}\frac{b'(x)}{g(x)} = 0 \tag{4}$$

where prime means differentiation with respect to the argument, $b(x)$, $g(x) \neq 0$ and $f(x)$ are functions of x, a and ℓ are arbitrary constants. The equation

$$b(x) = g(x)\dot{x}^2 + a\, f(x)x^{\ell} \tag{5}$$

denotes, as can be verified, a first integral of the equation (4). Setting $g(x) = e^{2\beta x}$, and $f(x) = \left(c_1 x + c_2 x^2 + c_3 x^3 + c_4\right)e^{2\beta x}$, reduces the equation (4) to

$$\begin{aligned} &\ddot{x} + \beta\dot{x}^2 + a\left[\left(c_2 + \beta c_1\right)x + \left(\beta c_2 + \frac{3c_3}{2}\right)x^2 + \beta c_3 x^3 + \left(\beta c_4 + \frac{c_1}{2}\right)\right]x^{\ell} + \\ &\frac{a\ell}{2}\left(c_1 x + c_2 x^2 + c_3 x^3 + c_4\right)x^{\ell-1} - \frac{b'(x)}{2}e^{-2\beta x} = 0 \end{aligned} \tag{6}$$

Applying $\ell = 0$, yields the equation

$$\ddot{x} + \beta\dot{x}^2 + a\left(c_2 + \beta c_1\right)x + a\left(\beta c_2 + \frac{3c_3}{2}\right)x^2 + \beta a c_3 x^3 + a\left(\beta c_4 + \frac{c_1}{2}\right) - \frac{b'(x)}{2}e^{-2\beta x} = 0 \tag{7}$$

The choice of $b(x) = c\,e^{2\beta x}$, leads to the equation

$$\ddot{x} + \beta\dot{x}^2 + a(c_2 + \beta c_1)x + a\left(\beta c_2 + \frac{3c_3}{2}\right)x^2 + \beta a c_3 x^3 + a\left(\beta c_4 + \frac{c_1}{2}\right) - \beta c = 0 \tag{8}$$

Now, putting $c_3 = 0$, allows one to get the desired equation

$$\ddot{x} + \beta\dot{x}^2 + a(c_2 + \beta c_1)x + a\beta c_2 x^2 + \frac{ac_1}{2} + \beta(ac_4 - c) = 0 \tag{9}$$

where β, c, c_1, c_2, c_3, and c_4, are arbitrary constants. As observed the equation (9) is identical with the equation (1) when $a_1 = a(c_2 + \beta c_1)$, $a_2 = \beta a c_2$, and $a_3 = \frac{ac_1}{2} + \beta(ac_4 - c)$. The first integral (5), taking into account the expressions of $b(x)$, $g(x)$ and $f(x)$, becomes

$$ce^{2\beta x} = e^{2\beta x}\dot{x}^2 + a(c_1 x + c_2 x^2 + c_4)e^{2\beta x} \tag{10}$$

that is, reduces to the first integral interpreted as the system Hamiltonian

$$c = \dot{x}^2 + a(c_1 x + c_2 x^2 + c_4) \tag{11}$$

which can give

$$\frac{dx}{\sqrt{c - ac_4 - a(c_1 x + c_2 x^2)}} = \pm dt \tag{12}$$

from which the general solution of the equation (9) can be obtained by integration. In this way the equation (12) can be rewritten

$$\frac{dx}{\sqrt{\left(x + \frac{c_1}{2c_2}\right)^2 - \frac{c_1^2}{4c_2^2} - \left(\frac{c - ac_4}{ac_2}\right)}} = \pm\sqrt{-ac_2}\,dt \tag{13}$$

The change of variable $X = x + \frac{c_1}{2c_2}$, where $dx = dX$, and $q = -\frac{c_1^2}{4c_2^2} - \left(\frac{c - ac_4}{ac_2}\right)$ allows one to obtain

$$\frac{dX}{\sqrt{X^2 + q}} = \pm\sqrt{-ac_2}\,dt \tag{14}$$

so that one can secure

$$\frac{dz}{\sqrt{z^2 + 1}} = \pm\sqrt{-ac_2}\,dt \tag{15}$$

where $z = \frac{X}{\sqrt{q}}$. From the equation (15) one can write

$$\operatorname{arcsin} h(z) = \pm\sqrt{-ac_2}\,(t+K) \tag{16}$$

where K is a constant of integration. Using the equation (16), the general solution of the equation (9) can take the form

$$x(t) = \frac{-c_1}{2c_2} + \sqrt{-\frac{c_1^2}{4c_2^2} - \left(\frac{c - ac_4}{ac_2}\right)} \sinh\left[\pm\sqrt{-ac_2}\,(t+K)\right] \tag{17}$$

The general solution (17) allows, following the sign of parameters, to get periodic or non-periodic solutions.

3-The exact and explicit general non-periodic solutions

This part is devoted to calculate non-periodic solutions to equation (9).

3.1 Case $ac_2 > 0$.

In this case $\sinh\left[\sqrt{-ac_2}\,(t+K)\right] = i\sin\left[\sqrt{ac_2}\,(t+K)\right]$ such that the general solution (17) becomes

$$x(t) = \frac{-c_1}{2c_2} + i\sqrt{\frac{c_4}{c_2} - \frac{c}{ac_2} - \frac{c_1^2}{4c_2^2}} \sin\left[\sqrt{ac_2}\,(t+K)\right] \tag{18}$$

where $\frac{c_4}{c_2} - \frac{c}{ac_2} - \frac{c_1^2}{4c_2^2} > 0$. In this situation the general solution (18) is a complex-valued formula. This solution (18) satisfies the Gottlieb conditions [1] to obtain the exact harmonic solution (2), that is $\operatorname{sgn} a_2 = \operatorname{sgn} \beta$ or $ac_2 > 0$, and $\frac{1}{2}\frac{c_1}{\beta_0 c_2} + \frac{ac_4 - c}{ac_2} < \frac{2c_1}{\beta_0 c_2} + \frac{c_1^2}{c_2^2}$, when $\frac{ac_4 - c}{ac_2} > \frac{c_1^2}{4c_2^2}$, and $ac_4 - c > 0$, for some values of c_1, c_2, c_4, and a. For $\frac{ac_4 - c}{ac_2} = \frac{c_1^2}{4c_2^2}$, that is $q = 0$, the equation (14) leads to the general solution of the equation (9) in the form

$$x(t) = \frac{-c_1}{2c_2} + K_1 e^{\pm\sqrt{-ac_2}\,t} \tag{19}$$

where K_1 is an arbitrary constant. Such a non-periodic solution satisfies also the preceding Gottlieb conditions to obtain exact harmonic solutions of the type (2). Thus, the previous Gottlieb conditions [1] are not sufficient to ensure only exact harmonic solutions to the equation (1).

3.2 Case $ac_2 < 0$.

3.2.1 The real-valued non-periodic solution.

In this case, $\frac{c_4}{c_2}-\frac{c}{ac_2}-\frac{c_1^2}{4c_2^2}>0$, and the general solution (17) is a real-valued non-periodic solution.

3.2.2 The complex-valued solution

Setting $\frac{c_4}{c_2}-\frac{c}{ac_2}-\frac{c_1^2}{4c_2^2}<0$, leads the general solution (17) to become a complex-valued formula. Now the existence of exact and explicit general harmonic solutions can be discussed.

4-The harmonic and isochronous solutions

To calculate the exact and general harmonic solutions to the equation (9), consider the general solution (17). Then, taking $ac_2>0$, and $\frac{c_4}{c_2}-\frac{c}{ac_2}-\frac{c_1^2}{4c_2^2}<0$, reduces the solution (17) to the general harmonic solutions

$$x(t)=\frac{-c_1}{2c_2}\pm\sqrt{\frac{c}{ac_2}+\frac{c_1^2}{4c_2^2}-\frac{c_4}{c_2}}\sin\left[\sqrt{ac_2}\left(t+K_2\right)\right] \tag{20}$$

where K_2 is an arbitrary constant. The equation (20) is amplitude-dependent frequency due to the term ac_2. Thus, for $ac_2=1$, the formula (20) becomes isochronous solutions. In this context we have successfully proved that exact harmonic solutions according to the form (2) can exist under arbitrary initial conditions where the first integral of the system (1) is not identically zero, contrary to the results of Gottlieb [1]. Let us consider now $c_1=0$. Then the general equation (9) becomes

$$\ddot{x}+\beta\dot{x}^2+ac_2x+a\beta c_2x^2+\beta\left(ac_4-c\right)=0 \tag{21}$$

and the solution (20) gives the general harmonic solutions of the equation (21) as

$$x(t)=\pm\sqrt{\frac{c-ac_4}{ac_2}}\sin\left[\sqrt{ac_2}\left(t+K_3\right)\right] \tag{22}$$

where K_3 is an arbitrary constant, and $c-ac_4>0$. The equation (22) shows also the existence of exact harmonic solutions according to the form (3) under arbitrary initial conditions without taking the first integral of the equation (21) equal identically to zero, contrary to the existence conditions determined by Gottlieb [1]. On the other hand, putting $ac_2=1$, leads to the equation

$$\ddot{x}+\beta\dot{x}^2+x+\beta x^2+\beta\left(ac_4-c\right)=0 \tag{23}$$

which admits the general harmonic and isochronous periodic solution

$$x(t) = \pm\sqrt{c - ac_4}\sin(t + K_4) \tag{24}$$

where K_4 is an arbitrary constant, and $c - ac_4 > 0$. In this perspective, the period of the solution (24) is equal to $T = 2\pi$, and the equation (23) is equivalent to the linear harmonic oscillator

$$\ddot{x} + x = 0 \tag{25}$$

where the amplitude of oscillations is taken as $\sqrt{c - ac_4}$, without requiring $\beta = 0$.

Now we can consider numerical applications to illustrate the usefullness of the theory.

5. Numerical applications

To compare analytical results with the solutions obtained by numerical integration, it is convenient to write the analytical solutions using the general initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$.

5.1. Expression of the solution (20) in terms of x_0 and v_0

Under the conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$, one can obtain from the solution (20) written as

$$x(t) = \frac{-c_1}{2c_2} + \sqrt{\frac{c}{ac_2} + \frac{c_1^2}{4c_2^2} - \frac{c_4}{c_2}}\sin\left[\sqrt{ac_2}\,(t + K_2)\right] \tag{26}$$

the system of algebraic equations

$$\begin{cases} x_0 = -\dfrac{c_1}{2c_2} + \sqrt{\dfrac{c}{ac_2} + \dfrac{c_1^2}{4c_2^2} - \dfrac{c_4}{c_2}}\sin\left(K_2\sqrt{ac_2}\right) \\ v_0 = \sqrt{ac_2\left(\dfrac{c}{ac_2} + \dfrac{c_1^2}{4c_2^2} - \dfrac{c_4}{c_2}\right)}\cos\left(K_2\sqrt{ac_2}\right) \end{cases} \tag{27}$$

From this system of algebraic equations, one can get

$$K_2 = \frac{\sqrt{ac_2}}{ac_2} arc\cot an\left[\frac{v_0\sqrt{ac_2}}{ac_2\left(x_0 + \dfrac{c_1}{2c_2}\right)}\right] \tag{28}$$

so that the general solution (26) takes the form

$$x(t) = \frac{-c_1}{2c_2} + \sqrt{\frac{c}{ac_2} + \frac{c_1^2}{4c_2^2} - \frac{c_4}{c_2}} \sin\left[\sqrt{ac_2}\, t + arc\cot an\left[\frac{v_0\sqrt{ac_2}}{ac_2\left(x_0 + \frac{c_1}{2c_2}\right)}\right]\right] \tag{29}$$

The comparison of the analytical result (29) in circles line with the numerical solution in solid line of equation (9) is shown in the Figure 1 under the conditions that $x_0 = 2;\ v_0 = 0{,}2;\ \beta = 0{,}2;\ a = -1;\ c = 1;\ c_1 = 1;\ c_2 = -1;$ and $c_4 = 1$.

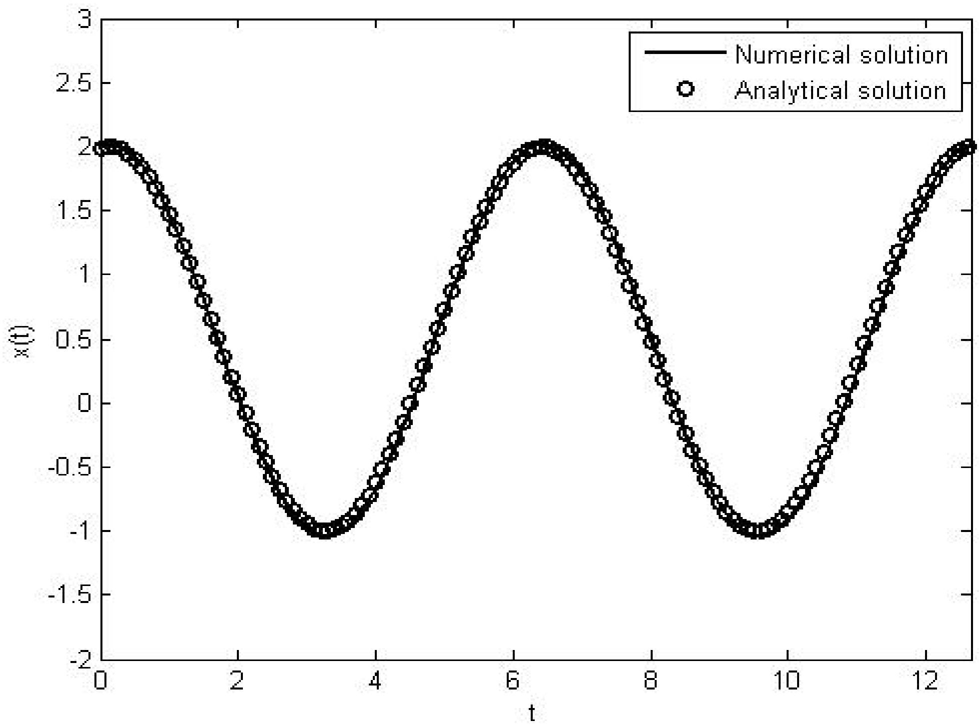

Figure 1: Comparison of solution (29) with numerical solution of equation (9). Typical values are: $x_0 = 2;\ v_0 = 0{,}2;\ \beta = 0{,}2;\ a = -1;\ c = 1;\ c_1 = 1;\ c_2 = -1;$ and $c_4 = 1$.

5.2. Expression of the solution (22) in terms of x_0 and v_0

As can be verified, in this case, the solution (22) written as

$$x(t) = \sqrt{\frac{c - ac_4}{ac_2}} \sin\left[\sqrt{ac_2}\,(t + K_3)\right] \tag{30}$$

takes the form

$$x(t) = \sqrt{\frac{c - ac_4}{ac_2}} \sin\left[\sqrt{ac_2}\, t + arc\cot an\left[\frac{v_0\sqrt{ac_2}}{x_0 ac_2}\right]\right] \tag{31}$$

where $K_3 = \frac{\sqrt{ac_2}}{ac_2} arc\cot an\left[\frac{v_0\sqrt{ac_2}}{x_0 ac_2}\right]$. The Figure 2 shows the comparison of the formula (31) in circles line with the numerical solution in solid line of equation (21) under the conditions that $x_0 = \sqrt{2}; v_0 = 0{,}2;\ \beta = 0{,}2;\ a = -1; c = 1;\ c_2 = -1;$ and $c_4 = 1$.

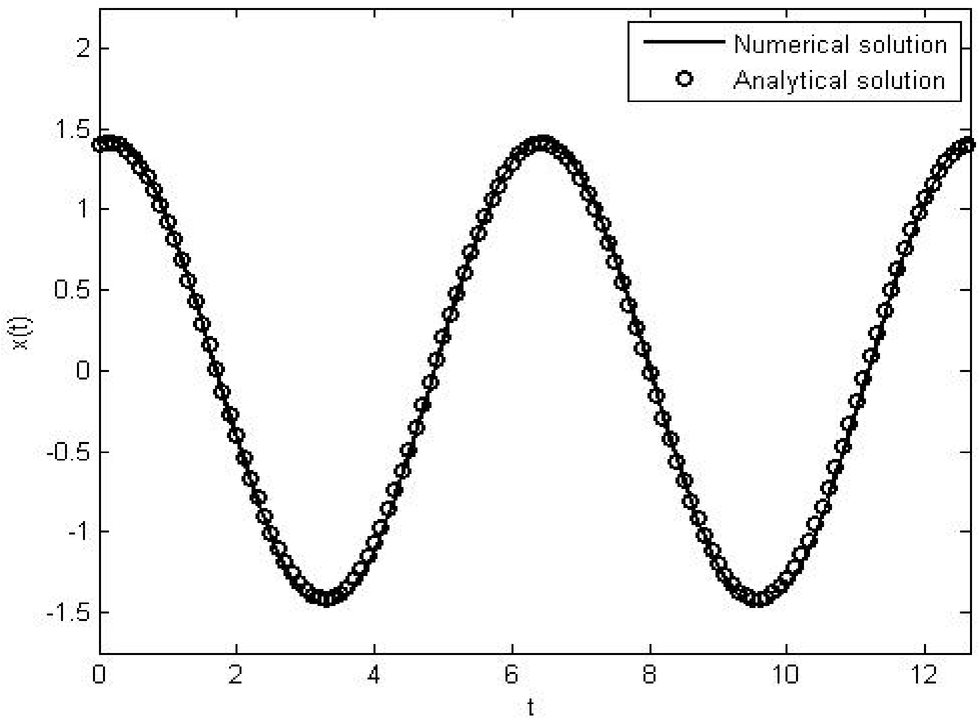

Figure 2: Comparison of solution (31) with numerical solution of equation (21). Typical values are: $x_0 = \sqrt{2}; v_0 = 0{,}2;\ \beta = 0{,}2;\ a = -1; c = 1;\ c_2 = -1;$ and $c_4 = 1$.

5.3. Expression of the solution (24) in terms of x_0 and v_0

This case reduces, using the initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$, the general solution

$$x(t) = \sqrt{c - ac_4}\sin(t + K_4) \qquad (32)$$

to the expression

$$x(t) = \pm\sqrt{c - ac_4}\sin\left(t + arc\cot an\left(\frac{v_0}{x_0}\right)\right) \qquad (33)$$

where $K_4 = arc\cot an\left(\frac{v_0}{x_0}\right)$. The comparison of the analytical solution (33) in circles line with the result obtained by numerical integration in solid line of the equation (23) under the conditions that $x_0 = \sqrt{2}; v_0 = 0,2; \beta = 0,2; a = -1; c = 1;$ and $c_4 = 1$ is graphically represented in the Figure 3.

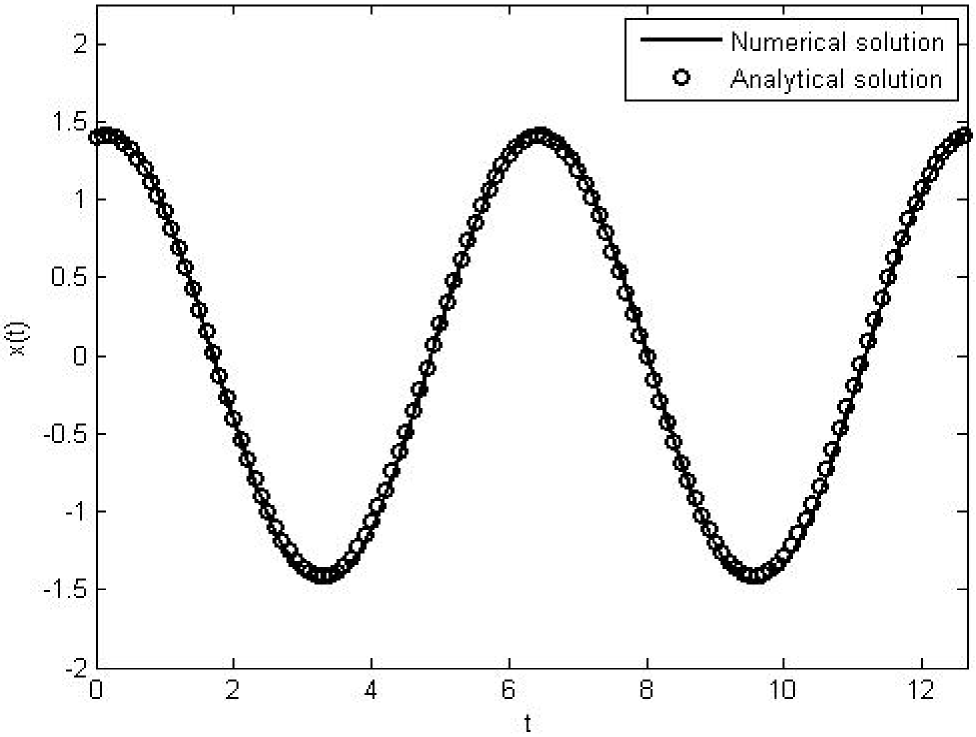

Figure 3: Comparison of solution (33) with numerical solution of equation (23). Typical values are: $x_0 = \sqrt{2}; v_0 = 0,2; \beta = 0,2; a = -1; c = 1;$ and $c_4 = 1$.

Conclusion

This paper has been devoted to study a well-known velocity-dependent conservative equation with mixed-parity presumed to be a conservative nonlinear oscillator. We have successfully shown that under the usual conditions of existence of exact harmonic solutions, this equation can have exact and explicit general non-periodic solutions so that it cannot be considered as a conservative nonlinear oscillator. By contrast, we have succeeded to show the existence of exact and explicit general harmonic and isochronous solutions for this equation without requiring that the Hamiltonian of the system becomes identically zero. Results obtained by numerical integration show an excellent agreement with the analytical solutions by comparison.

Ethics

Authors declare no conflict of interest

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