

Fuzzy Rate Analysis of Operators and its Applications in Linear Spaces

Yijin Zhang

Chongqing University of Posts and Telecommunications

Honggang Li (✉ 382889721@qq.com)

Chongqing Technology and Business University

Maoming Jin

changjiang nanjing hangdao gongcheng ju: Nanjing Changjiang Waterway Engineering Bureau

Zongbing Lin

Panzhuhua University

Research Article

Keywords: Fuzzy rate, Operator, Membership function, Fixed Point Existence Theorem.

Posted Date: June 29th, 2021

DOI: <https://doi.org/10.21203/rs.3.rs-661983/v1>

License: © ⓘ This work is licensed under a Creative Commons Attribution 4.0 International License.

[Read Full License](#)

Fuzzy Rate Analysis of Operators and its Applications in Linear Spaces

Yijin Zhang¹, Honggang Li^{2*}, Maoming Jin³, Zongbing Lin⁴

1 Key Lab of Intelligent Analysis and Decision on Complex Systems,
Key Laboratory of Industrial Internet of Things and Networked Control,
Ministry of Education, Chongqing University of Posts and Telecommunications,
Chongqing 400065, P.R. China. E-mail: zhangyj@cqupt.edu.cn

2 Pass College of Chongqing Technology and Business University
Chongqing 401520, P.R. China. E-mail: 382889721@qq.com

3 Institute of Nonlinear Analysis Research, Changjiang Normal University
Chongqing 400803, P.R. China. E-mail: 19860025@yznu.cn

4 School of Mathematics and Computer Science, Panzhihua University,
Panzhuhua 617000, P.R. China. E-mail: zongbinglin@sohu.com

* The corresponding author: Honggang Li

Abstract In this paper, a new concept, the fuzzy rate of an operator in linear spaces is proposed for the very first time. Some properties and basic principles of it are studied. Fuzzy rate of an operator B which is specific in a plane is discussed. As its application, a new fixed point existence theorem is proved.

Key Words and Phrases Fuzzy rate; Operator; Membership function; Fixed Point Existence Theorem.

AMS Subject Classification 46S40; 03E99; 26E50.

1 Introduction

More and more classical analysis theory are being developed into fuzzy analysis theory. Fuzzy sets, fuzzy logic, fuzzy numbers, fuzzy topologies were introduced and studied[1-3]. Chang and Huang, Ding and Jong, Jin, Li and others studied several kinds of variational inequalities (inclusions) for fuzzy mappings[4-8].

Recently, Konwar and Nabanita introduce the notion of continuous linear operators and establish the uniform continuity theorem and Banach's contraction principle in an intuitionistic fuzzy n -normed linear space[9]. Wang investigates the concepts and some properties of interval-valued fuzzy ideals in B -algebras and the homomorphic inverse image of interval-valued intuitionistic fuzzy ideals[10]. Fixed Point Theorems in Partially Ordered Fuzzy Metric Spaces and Operator Theory and Fixed Points in Fuzzy Normed Algebras and Applications are studied in [11]. Fuzzy-wavelet-like operators via a real-valued scaling function are discussed in [12]. A linear fuzzy operator inequality approach is proposed for the first time in [13]. Fuzziness degree's quantity measure as to fuzzy operator is researched by means of fuzzy set theory in [14]. For more details, we reference to the readers [1-15].

In this work, we come up with the concept of fuzzy rate of an operator and consider its properties and applications. We also explore fuzzy rate which is produced by an operator effecting an element, as well as some properties and applications of it. These are new extension and applications of operator theory and fuzzy theory.

The remainder of this paper is organized as follows. In Section 2, we give an example which helps us introduce the concept of fuzzy rate of an operator. In Section 3, we propose the concept and prove some basic properties of it. In Section 4, a new Fixed Point Existence Theorem with the fuzzy rate of the operator B is obtained as its application.

2 An Example

Here, an example is given to introduce a new concept, the fuzzy rate of an operator in linear spaces.

Example 2.1 Let $U = R \times R$ be a real plane(Universe), c be a cycle or ellipse whose center is at $(0,0)$ in U . We suppose that the equation of c is, for a fixed $r > 0$ and an arbitrary $\lambda > 0$,

$$c_{(\lambda,r)} : x^2 + \lambda y^2 = r^2. \quad (2.1)$$

Assuming μ is a positive real number, when $\lambda = \mu$, we get a curve $c_{(\mu,r)}$, for fixed $\mu > 0, r > 0$,

$$c_{(\mu,r)} : x^2 + \mu y^2 = r^2. \quad (2.2)$$

Now we consider the membership degree for any point (x, y) in U belonging to the curve $c_{(\mu,r)}$.

If $(x, y) \in c_{(\mu,r)}$, the membership degree is 1.

If $(x, y) \notin c_{(\mu,r)}$, we compute the membership degree through the relationship between $c_{(\mu,r)}$ and $c_{(\lambda,r)}$ as following.

If there exists λ such that $(x, y) \in c_{(\lambda,r)}$, the membership degree is $e^{-(\lambda-\mu)^2}$.

If there doesn't exist λ such that $(x, y) \in c_{(\lambda,r)}$, that is, for every $\lambda > 0$, $(x, y) \notin c_{(\lambda,r)}$, the membership degree is 0.

To sum up, we define a membership function $F_{c_{(\mu,r)}}(x, y)$ for any point (x, y) in U belonging to the curve $c_{(\mu,r)}$, for every $\lambda > 0$ and certain $\mu > 0, r > 0$,

$$F_{c_{(\mu,r)}}(x, y) = \begin{cases} 0, & (x, y) \notin c_{(\mu,r)}, \notin c_{(\lambda,r)}; \\ e^{-(\lambda-\mu)^2}, & (x, y) \notin c_{(\mu,r)}, \in c_{(\lambda,r)}; \\ 1, & (x, y) \in c_{(\mu,r)}. \end{cases} \quad (2.3)$$

In (2.2), let $\mu = 1$, then the curve $c_{(1,r)}$ is a circle. We have $F_{c_{(1,r)}}(0, r) = 1$ for the point $(0, r) \in c_{(1,r)}$. $F_{c_{(1,r)}}(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{6}}) = e^{-(3-1)^2} = e^{-4}$ since point $(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{6}}) \notin c_{(1,r)}$, but $\in c_{(\lambda,r)}$ for $\lambda = 3$. However, $F_{c_{(1,r)}}(2r, r) = 0$ because point $(2r, r) \notin c_{(1,r)}$, $\notin c_{(\lambda,r)}$ for any $\lambda > 0$.

Set $B = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ be an operator, $b \neq 0$, (x, y) is a point or row vector in U , $B : (x, y) \mapsto (x, y)B$. Then, we get

$$F_{c_{(1,r)}}((0,r)B) = F_{c_{(1,r)}}(0, br) = e^{-(\frac{1}{b^2}-1)^2},$$

where the point $(0,r)B = (0, br) \in c_{(\frac{1}{b^2}, r)}$.

The value

$$\frac{F_{c_{(1,r)}}((0,r)B)}{F_{c_{(1,r)}}(0,r)} = \frac{e^{-(\frac{1}{b^2}-1)^2}}{1} = e^{-(\frac{1}{b^2}-1)^2}$$

expresses a fuzzy rate of operator B at the point $(0,r) \in c_{(1,r)}$ under $F_{c_{(1,r)}}$ for $\lambda = \frac{1}{b^2}$.

In addition, if $r = 1$, $b = \sqrt{2}$, then

$$\frac{F_{c_{(1,1)}}((0,1)B)}{F_{c_{(1,1)}}(0,1)} = \frac{e^{-(\frac{1}{b^2}-1)^2}}{1} = e^{-(\frac{1}{2}-1)^2} = e^{-\frac{1}{4}}.$$

Generally, suppose that \mathcal{X} is a linear space, $\emptyset \neq X \subseteq \mathcal{X}$, $x \in \mathcal{X}$, $F_X(x) : \mathcal{X} \rightarrow [0,1]$ is a membership function for x belonging to the set X , $A : \mathcal{X} \rightarrow \mathcal{X}$ is an operator. Then, $F_X(A(x))$ reflects the membership degree of the image of x belonging to the set X . It's clear that the value $\frac{F_X(A(x))}{F_X(x)}$, the ratio of $F_X(A(x))$ and $F_X(x)$, indicates the changing rate which is produced by the mapping $A : x \rightarrow A(x)$. Next we consider a special value $\sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)}$ to express a fuzzy rate of operator A at a point $y \in \mathcal{X}$ with $\mathcal{F}(\mathcal{X})$. It is very interesting to achieve the impact results and properties of the operator with respect to F_X .

3 Fuzzy Rate of Operators

In this section, we first give the concept of fuzzy rate of an operator, then we show some basic properties of it.

Definition 3.1 Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0,1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0,1]\}$ be a collection of all membership functions over \mathcal{X} . For any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, if

$$\|A\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)} \quad (3.1)$$

exists, then $\|A\|_y$ is called a fuzzy rate of the operator A at the point $y \in \mathcal{X}$ on $\mathcal{F}(\mathcal{X})$.

In Example 2.1, letting $\mathcal{F}(U) = \{F_{c_{(\mu,r)}} | F_{c_{(\mu,r)}} : \mu, r > 0\}$, by (2.3) we can achieve, obviously

$$\begin{aligned}
\|B\|_{(0,r)} &= \sup_{F_{c(\mu,r)} \in \mathcal{F}(U)} \frac{F_{c(\mu,r)}((0,r)B)}{F_{c(\mu,r)}(0,r)} \\
&= \sup_{F_{c(\mu,r)} \in \mathcal{F}(U)} \frac{F_{c(\mu,r)}(0,br)}{F_{c(\mu,r)}(0,r)} \\
&= \sup_{F_{c(\mu,r)} \in \mathcal{F}(U)} \frac{e^{-(\frac{1}{b^2}-\mu)^2}}{e^{-(1-\mu)^2}} \\
&= \sup_{\mu > 0} e^{(1-\frac{1}{b^4})} e^{2\mu(\frac{1}{b^2}-1)} \\
&= \begin{cases} +\infty, & |b| < 1, \\ e^{(1-\frac{1}{b^4})}, & |b| \geq 1. \end{cases}
\end{aligned}$$

At the same time, we have the following theorem about the relationship between a fuzzy rate of the operator and fuzzy sets.

Theorem 3.2 *Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} and $\|A\|_y$ be the fuzzy rate of the operator A at the point $y \in \mathcal{X}$ on $\mathcal{F}(\mathcal{X})$. Then for any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, there exist two membership functions $F, G \in \mathcal{F}(\mathcal{X})$ such that, for each $y \in \mathcal{X}$,*

$$\|A\|_y F(y) = G(A(y)). \quad (3.2)$$

Proof. For any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, since

$$\|A\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)} < +\infty,$$

then $F(y) \neq 0$. $\forall n, \exists G_n \in \mathcal{F}(\mathcal{X})$, we arrive at

$$\|A\|_y \geq \frac{G_n(A(y))}{G_n(y)} > \|A\|_y - \frac{1}{n},$$

and

$$\|A\|_y = \lim_{n \rightarrow \infty} \frac{G_n(A(y))}{G_n(y)} < +\infty$$

for each $y \in \mathcal{X}$.

Since $0 < G_n(y), G_n(A(y)) \leq 1$, there exist $G_{n_k}(y) \rightarrow F(y)$ as $k \rightarrow +\infty$, and $G_{n_{k_m}}(A(y)) \rightarrow G(A(y))$ as $m \rightarrow +\infty$ for any $y \in \mathcal{X}$.

If $0 < F(y) \leq 1$, it follows that

$$\|A\|_y = \lim_{n \rightarrow \infty} \frac{G_n(A(y))}{G_n(y)} = \lim_{m \rightarrow \infty} \frac{G_{n_{k_m}}(A(y))}{G_{n_{k_m}}(y)} = \frac{\lim_{m \rightarrow \infty} G_{n_{k_m}}(A(y))}{\lim_{m \rightarrow \infty} G_{n_{k_m}}(y)} = \frac{G(A(y))}{F(y)},$$

and $\|A\|_y F(y) = G(A(y))$ for every $y \in \mathcal{X}$.

If $F(y) = 0$, it implies that

$$\|A\|_y = \lim_{n \rightarrow \infty} \frac{G_n(A(y))}{G_n(y)} = \lim_{m \rightarrow \infty} \frac{G_{n_{k_m}}(A(y))}{G_{n_{k_m}}(y)} = \frac{\lim_{m \rightarrow \infty} G_{n_{k_m}}(A(y))}{\lim_{m \rightarrow \infty} G_{n_{k_m}}(y)} < +\infty,$$

and $\lim_{m \rightarrow \infty} G_{n_{k_m}}(A(y)) = 0$, $G(A(y)) = \lim_{m \rightarrow \infty} G_{n_{k_m}}(A(y)) = 0$ [15].

Therefore, $\|A\|_y F(y) = G(A(y))$ holds for any $y \in \mathcal{X}$. The proof is complete.

It is easy to verify that the converse proposition of Theorem 3.2 holds. We reach

Theorem 3.3 *Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . For any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, if there exist $F, G \in \mathcal{F}(\mathcal{X})$ and $F(y) \neq 0$ such that*

$$\|A\|_y F(y) = G(A(y)),$$

then for any $y \in \mathcal{X}$,

$$\|A\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)} = \frac{G(A(y))}{F(y)} < +\infty,$$

that is, the fuzzy rate of the operator A exists.

Now, we state some basic properties of the fuzzy rate of the operator A as the next theorem. These properties are very useful for further applications.

Theorem 3.4 *Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . For any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, $A_1, A_2 \in \mathcal{A}(\mathcal{X})$ and the identity operator $I \in \mathcal{A}(\mathcal{X})$, then*

- (1) $\|A\|_y > 0$ for any $A \in \mathcal{A}(\mathcal{X})$;
- (2) $\|I\|_y = 1$ for any $y \in \mathcal{X}$;
- (3) If A_1 is a linear operator and $a > 0$ is a real number, then $\|aA_1\|_y \leq \|A_1\|_{(ay)} \|aI\|_y$;
- (4) If $F(A_1(y)) \geq F(A_2(y))$ for any $y \in \mathcal{X}$, then

$$\|A_1\|_y \geq \|A_2\|_y;$$

- (5) If $F(A_1(y) + A_2(y)) = F(A_1(y)) + F(A_2(y))$ for any $y \in \mathcal{X}$, then

$$\|A_1 + A_2\|_y \leq \|A_1\|_y + \|A_2\|_y;$$

- (6) If $F(A_1(y) - A_2(y)) = F(A_1(y)) - F(A_2(y)) \geq 0$ for any $y \in \mathcal{X}$, then

$$0 \leq \|A_1\|_y - \|A_2\|_y \leq \|A_1 - A_2\|_y;$$

- (7) If $(A_1 A_2)(y) = A_1(A_2(y))$ for any $y \in \mathcal{X}$, and there exist $\|A_1\|_{A_2(y)}$ and $\|A_2\|_y$, then

$$\|A_1 A_2\|_y \leq \|A_1\|_{A_2(y)} \|A_2\|_y;$$

(8) If $\mathcal{F}_1(\mathcal{X}) \subseteq \mathcal{F}_2(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$,

$$\|A\|_{y, \mathcal{F}_1(\mathcal{X})} = \sup_{F \in \mathcal{F}_1(\mathcal{X})} \frac{F(A(y))}{F(y)}$$

and

$$\|A\|_{y, \mathcal{F}_2(\mathcal{X})} = \sup_{F \in \mathcal{F}_2(\mathcal{X})} \frac{F(A(y))}{F(y)},$$

where $\|A\|_{y, \mathcal{F}_1(\mathcal{X})}$ represents the fuzzy rate of the operator A at the point $y \in \mathcal{X}$ on $\mathcal{F}_1(\mathcal{X})$ and $\|A\|_{y, \mathcal{F}_2(\mathcal{X})}$ on $\mathcal{F}_2(\mathcal{X})$, then for any $y \in \mathcal{X}$,

$$\|A\|_{y, \mathcal{F}_1(\mathcal{X})} \leq \|A\|_{y, \mathcal{F}_2(\mathcal{X})}.$$

Proof. (1) On the one hand, it follows that $\|A\|_y \geq 0$ for any $A \in \mathcal{A}(\mathcal{X})$ from Definition 3.1. On the other hand, if $\|A\|_y = 0$, then for any $F \in \mathcal{F}(\mathcal{X})$, $F(A(y)) = 0$. It is false because there exists a membership function $F \in \mathcal{F}(\mathcal{X})$ such that $F(A(y)) > 0$ from Example 2.1.

(2) Because I is an identity operator, we obtain

$$\|I\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(I(y))}{F(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(y)}{F(y)} = 1$$

for all $y \in \mathcal{X}$.

(3) If A_1 is a linear operator and a is a positive real number, then

$$\begin{aligned} \|aA_1\|_y &= \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(aA_1(y))}{F(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \left(\frac{F(A_1(ay))}{F(ay)} \frac{F(ay)}{F(y)} \right) \\ &\leq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(ay))}{F(ay)} \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(ay)}{F(y)} = \|A_1\|_{(ay)} \|aI\|_y. \end{aligned}$$

(4) If $F(A_1(y)) \geq F(A_2(y))$ for any $y \in \mathcal{X}$, then

$$\|A_1\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(y))}{F(y)} \geq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_2(y))}{F(y)} = \|A_2\|_y.$$

(5) If $F(A_1(y) + A_2(y)) = F(A_1(y)) + F(A_2(y))$ for any $y \in \mathcal{X}$, we get

$$\begin{aligned} \|A_1 + A_2\|_y &= \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F((A_1 + A_2)(y))}{F(y)} \\ &\leq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(y))}{F(y)} + \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_2(y))}{F(y)} = \|A_1\|_y + \|A_2\|_y. \end{aligned}$$

(6) Let $F(A_1(y) - A_2(y)) = F(A_1(y)) - F(A_2(y)) \geq 0$ for any $y \in \mathcal{X}$, which means

$$\frac{F(A_1(y)) - F(A_2(y))}{F(y)} \geq 0$$

and

$$\begin{aligned}\|A_1\|_y &= \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(y))}{F(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{[F(A_1(y)) - F(A_2(y))] + F(A_2(y))}{F(y)} \\ &\leq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(y)) - F(A_2(y))}{F(y)} + \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_2(y))}{F(y)} = \|A_1 - A_2\|_y + \|A_2\|_y.\end{aligned}$$

Therefore, $0 \leq \|A_1\|_y - \|A_2\|_y \leq \|A_1 - A_2\|_y$ holds.

(7) Set $(A_1 A_2)(y) = A_1(A_2(y))$ for any $y \in \mathcal{X}$. Then there exist $\|A_1\|_{A_2(y)}$ and $\|A_2\|_y$ such that

$$\begin{aligned}\|A_1 A_2\|_y &= \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(A_2(y)))}{F(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(A_2(y)))}{F(A_2(y))} \frac{F(A_2(y))}{F(y)} \\ &\leq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_1(A_2(y)))}{F(A_2(y))} \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A_2(y))}{F(y)} = \|A_1\|_{A_2(y)} \|A_2\|_y.\end{aligned}$$

(8) It is clear that the result holds by $\mathcal{F}_1(\mathcal{X}) \subseteq \mathcal{F}_2(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$ and Definition 3.1. The proof is completed.

The following is also a property of the fuzzy rate of the operator A based on its basic properties.

Corollary 3.5 *Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . For any $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, if $A^n(y) = A^{n-1}(A(y))$ for $n = 1, 2, \dots$, there exist $\|A\|_{A^{k-1}(y)}$ for $k = 1, 2, \dots, n$, such that*

$$\frac{1}{\prod_{1 \leq k \leq n} \|A\|_{A^{k-1}(y)}} \leq \frac{1}{\|A^n\|_y}, \quad (3.3)$$

where $A^0 = I$ is an identity operator.

Proof. We have

$$\begin{aligned}\|A^n\|_y &= \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(A^{n-1}(y)))}{F(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F((A(A^{n-1}(y))))}{F(A^{n-1}(y))} \frac{F(A^{n-1}(y))}{F(y)} \\ &\leq \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F((A(A^{n-1}(y))))}{F(A^{n-1}(y))} \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A^{n-1}(y))}{F(y)} \\ &= \|A\|_{A^{n-1}(y)} \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A^{n-1}(y))}{F(y)} \leq \dots \leq \prod_{1 \leq k \leq n} \|A\|_{A^{k-1}(y)}.\end{aligned}$$

It follows that the result (3.3) holds. This completes the proof.

In what follows, we will apply the above properties to prove a new Fixed Point Existence Theorem.

4 Application—a new Fixed Point Existence Theorem

Fixed Point theory is very important and most generally useful one in classical function analysis. In this section, we prove a new Fixed Point Existence Theorem with the fuzzy rate of the operator A as its application. First, we have

Lemma 4.1 *Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . For $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$, suppose*

$$\|A\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)} < +\infty.$$

If for $\delta \in (0, 1]$, there exists a natural number N such that $\|A^n\|_y \geq \delta$ as $n \geq N$, then there exists a $F_0 \in \mathcal{F}(\mathcal{X})$ such that

$$F_0(A^n(y)) = F_0(A^{n-1}(y)), \quad (4.1)$$

or

$$F_0(A(A^{n-1}(y))) = F_0(A^{n-1}(y)), \quad (4.2)$$

for $n \geq N$.

Proof. Note that $A^n(y) = A^{n-1}(A(y))$ for $n = 1, 2, \dots$.

If $\delta \in (0, 1]$, there exists a natural number N such that $\|A^n\|_y \geq \delta$ as $n \geq N$. Then we know

$$1 \leq \frac{1}{\prod_{1 \leq k \leq n} \|A\|_{A^{k-1}(y)}} \leq \frac{1}{\|A^n\|_y} \leq \frac{1}{\delta} < +\infty$$

for $n \geq N$, and

$$0 \leq \sum_{k=1}^n -\ln \|A\|_{A^{k-1}(y)} \leq -\ln \|A^n\|_y \leq -\ln \delta < +\infty,$$

hence $\lim_{k \rightarrow +\infty} \ln \|A\|_{A^{k-1}(y)} = 0$ and $\lim_{k \rightarrow +\infty} \|A\|_{A^{k-1}(y)} = 1$ [15].

It follows that for any natural number m there exists a natural number M , as $k > \max\{M, N\}$ such that

$$1 - \frac{1}{m} < \|A\|_{A^{k-1}(y)} < 1 + \frac{1}{m}.$$

Since

$$\|A\|_{A^{k-1}(y)} = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(A^{k-1}(y)))}{F(A^{k-1}(y))} < +\infty,$$

there exists a membership function $F_0 \in \mathcal{F}(\mathcal{X})$ such that

$$1 - \frac{1}{m} - \frac{1}{m} < \|A\|_{A^{k-1}(y)} - \frac{1}{m} < \frac{F_0(A(A^{k-1}(y)))}{F_0(A^{k-1}(y))} < \|A\|_{A^{k-1}(y)} + \frac{1}{m} < 1 + \frac{1}{m} + \frac{1}{m}.$$

Letting $m \rightarrow +\infty$, we obtain

$$\frac{F_0(A(A^{k-1}(y)))}{F_0(A^{k-1}(y))} = 1,$$

that's to say, $F_0(A(A^{k-1}(y))) = F_0(A^{k-1}(y))$. The proof is completed.

Then, we give the definition of a quasi-fixed point of the operator A with respect to the membership function F .

Definition 4.2 Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . For $y \in \mathcal{X}$, if there exists $F \in \mathcal{P}(\mathcal{X})$ such that $F(A(y)) = F(y)$, then y is called a quasi-fixed point of the operator A with respect to F .

By the proof of Lemma 4.1, $A^{k-1}(y)$ is a quasi-fixed point of the operator A with respect to F_0 .

Now, Fixed Point Existence Theorem with respect to F is presented.

Theorem 4.3 (new Fixed Point Existence Theorem) Let \mathcal{X} be a linear space, $A : \mathcal{X} \rightarrow \mathcal{X}$ be an operator, $\mathcal{A}(\mathcal{X}) = \{A | A : \mathcal{X} \rightarrow \mathcal{X}\}$, $F : \mathcal{X} \rightarrow [0, 1]$ be a membership function over \mathcal{X} , $\mathcal{P}(\mathcal{X}) = \{F | F : \mathcal{X} \rightarrow [0, 1]\}$ be a collection of all membership functions over \mathcal{X} . Presume

$$\|A\|_y = \sup_{F \in \mathcal{F}(\mathcal{X})} \frac{F(A(y))}{F(y)} < +\infty$$

for $\emptyset \neq \mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$. If $\delta \in (0, 1]$, there exists a natural number N such that $\|A^n\|_y \geq \delta$ as $n \geq N$. Then there exists an injection functional $F_0 \in \mathcal{F}(\mathcal{X})$ such that

$$F_0(A^n(y)) = F_0(A^{n-1}(y)), \tag{4.3}$$

$A^{n-1}(y)$ is a fixed point of the operator A with respect to F_0 , and y is a fixed point of the operator A^n with respect to F_0 for $n \geq N$.

Proof. It follows directly that there exists an injection functional $F_0 \in \mathcal{F}(\mathcal{X})$ such that $F_0(A^n(y)) = F_0(A^{n-1}(y))$ from Lemma 4.1. This completes the proof.

Like the classical fixed point theory applied to differential equations, we believe that the Fixed Point Existence Theorem with respect to fuzzy theory might be applied to fuzzy equations or fuzzy differential equations. They are worth further studying in the future.

5 Conclusions

In this work, we have obtained the following results:

- The fuzzy rate of an operator in linear spaces is introduced and some properties and basic principles of it are studied.
- The fuzzy rate of an diagonal matrix B in a plane is discussed.

- A new Fixed Point Existence Theorem is obtained.

Acknowledgments

This work was supported by Natural Science Foundation Project of Chongqing (Grant No.cstc2019jcyj-msxmX0716, cstc2020jcyj-msxm1397) and Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No.KJQN202000601).

Competing interests

The authors declare that they have no competing interests regarding the publication of this article.

References

- [1] M. Burgin, A. Sostak, Towards the theory of continuity defect and continuity measure for mappings of metric spaces[J]. *Matematika*, 1992: 45-62.
- [2] M. Burgin, A. Sostak, Fuzzification of the theory of continuous functions[J]. *Fuzzy Sets and Systems*, 1994, 62(1): 71-81.
- [3] J. J. Buckley, Y. Qu, Fuzzy complex analysis I: differentiation[J]. *Fuzzy Sets and Systems*, 1991, 41(3): 269-284.
- [4] S. Chang, N. Huang, Generalized complementarity problems for fuzzy mappings[J]. *Fuzzy sets and systems*, 1993, 55(2): 227-234.
- [5] X. P. Ding, J. Y. Park, A new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings[J]. *Journal of Computational and Applied Mathematics*, 2002, 138(2): 243-257.
- [6] Y. P. Fang, N. J. Huang, H-monotone operator and resolvent operator technique for variational inclusions[J]. *Applied Mathematics and Computation*, 2003, 145(2-3): 795-803.
- [7] M. M. Jin, Perturbed Proximal point algorithm for general quasi-variational inclusions with fuzzy set-valued mappings, *OR Transactions*, 2005, 9(3):31-38.
- [8] H. G. Li, Iterative algorithm for a new class of generalized nonlinear fuzzy set-variational inclusions involving (H, eta)-monotone mappings[J]. *Advances in Nonlinear Variational Inequalities*, 2007, 10(1): 89-100.
- [9] N. Konwar, P. Debnath, Continuity and Banach contraction principle in intuitionistic fuzzy n-normed linear spaces[J]. *Journal of Intelligent and Fuzzy Systems*, 2017, 33(4):1-11.
- [10] F. Wang, Interval-valued intuitionistic fuzzy ideals of B-algebras[J]. *Journal of Intelligent and Fuzzy Systems*, 2017, 33(4): 2609-2615.
- [11] Y. J. Cho, T. M. Rassias, R. Saadati, *Fuzzy Operator Theory in Mathematical Analysis*[M]. Springer, 2018.

- [12] R. Ezzati, F. Mokhtarnejad, N. Hassasi, Some Fuzzy-Wavelet-Like Operators and Their Convergence[J]. *Mathematical Problems in Engineering*, 2013, 2013:1-10.
- [13] Z. Tai, Exponential stability of non-linear hyperbolic distributed complex-valued parameter systems: The linear fuzzy operator inequality approach[J]. *Applied Mathematics Letters*, 2012, 25(10): 1404-1409.
- [14] Z. J. Liu, General Expression Forms on the Fuzziness Degree of Fuzzy Operator[C]. *Applied Mechanics and Materials*. Trans Tech Publications, 2011, 40: 835-838.
- [15] V. A. Zorich, O. Paniagua, *Mathematical analysis II*[M]. Berlin: Springer, 2016.