A fuzzy jump-diffusion option pricing model based on Merton formula

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Abstract This paper proposes a fuzzy jump-diffusion option pricing model based on Merton's normal jump-diffusion price dynamics. The logarithm of the stock price is assumed to be a Gaussian fuzzy number and the diffusion and jump parameters of the Merton model are assumed to be triangular fuzzy numbers to model the impreciseness which occur due to the variation in financial markets. Using these assumptions, a fuzzy formula for a European call option has been proposed. Given any value of the option price, its belief degree is obtained by using the bisection search algorithm. The fuzzy call option prices have been defuzzified and it has been found that the fuzzy jump-diffusion model outperforms Wu's fuzzy Black-Scholes model. This is one of the first studies where the impreciseness of the stock price and input parameters has been modelled taking into account occasional large jumps in stock price trajectory and thereby proposing a fuzzy option pricing model.

Keywords Fuzzy · Jump-diffusion · Option · Bisection search · Black-Scholes

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1 Introduction and literature review

[Black and Scholes (1973)] proposed a closed-form solution for the price of a European option. Their model assumes that the stock prices follow a diffusion process. However, the changes in stock prices are observed to often consist of large jumps. In light of that, [Cox and Ross (1976)] discussed price dynamics if the stock prices follow a jump process. However, [Merton (1976)] proposed that the stock prices follow both diffusion and jump processes, taking into consideration two types of stock price behaviour over a period, the diffusion process which explains the small changes in stock prices in a small interval of time and the jump process which explains sudden large changes in stock prices in a small interval of time. [Heston (1993)] considered stochastic volatility instead of a constant one, that is the volatility changes with time. [Bates (1996)] and [Scott (1997)] used stochastic volatility in a jump-diffusion model to price options. Later, some authors discussed that the parameters of the option pricing models cannot always be expected in a precise sense because of variation in financial markets. [Muzzioli and Torricelli (2001)] and [Muzzioli and Torricelli (2004)] modelled the impreciseness in binomial option pricing models. [Yoshida (2003)], [Wu (2004)], and [Wu (2005)] modelled the impreciseness in the Black-Scholes option pricing model. To model impreciseness, they have used the fuzzy set theory of [Zadeh (1965)]. In the models of [Black and Scholes (1973)], [Cox and Ross (1976)], [Merton (1976)], [Heston (1993)], [Bates (1996)], and [Scott (1997)], the randomness of stock prices, volatility are modelled using random numbers from probability theory whereas, in the models of [Muzzioli and Torricelli (2001)], [Muzzioli and Torricelli (2004)], [Yoshida (2003)], [Wu (2004)], and [Wu (2005)], impreciseness of stock prices, volatility, interest rate are mo-

delled using fuzzy numbers from fuzzy set theory.
[Wu (2004)] justified that because different banks and financial institutions may have different risk-free interest rates, a financial analyst should assume the risk-free interest rate as a fuzzy number. He also justified that it is not reasonable to assume the volatility and stock prices as constants, price an option using those constant values, and then use that option price after a very short period (could be few seconds or minutes) for further decision making since by that time the volatility and stock prices have already changed and the respective option price should also be different. Further support to these arguments are given by [Wu (2005)], [Wu (2007)], [Thiagarajah et. al. (2007)], [Xu et. al. (2010)], [Guerra et. al. (2011)], and [Nowak and Romaniuk (2014)]. In this scenario, the variation of volatility and stock price in this very short period need not be measured with respect to stochastic volatility or stochastic stock price. The time index will remain the same (time usually measured in days) and the impreciseness should be measured by fuzzifying the parameters. Note that associating stochasticity to a parameter needs us to have its historical prices whereas associating fuzzy nature doesn't need so. This explains the difference between fuzzy volatility and stochastic volatility as well as between fuzzy stock price and stochastic stock price. The fuzzy interest rate is different from a stochastic interest rate since stochastic interest rate means interest rate varying with time and fuzziness of interest rate emerges because of different financial institutions and banks offering different interest rates ([Wu (2004)]). [Xu et. al. (2009)] extended the [Merton (1976)] normal jump-diffusion model by fuzzifying the jump parameters. The stochastic nature of the jump parameters addressed the uncertainty of randomness that is whether jumps occur or not whereas the fuzzy nature of the jump parameters addressed the uncertainty of impreciseness that is what is the exact number of jump times. This model however didn't assume the drift and volatility parameters as fuzzy numbers. [Zhang et. al. (2012)] proposed a fuzzy option pricing formula based on [Kou (2002)] double exponential jump-diffusion model where they assumed interest rate, drift rate, volatility, and the average jump intensity to be fuzzy numbers but haven't fuzzified the stock price. [Chen et. al. (2019)] assumed the stock return to be a Gaussian fuzzy number and priced European call option. Before them, all the authors working in the fuzzification of option pricing models were using triangular fuzzy numbers for input parameters. They also introduced the fuzzy Greeks for the sensitivity analysis of the fuzzy European option price with respect to the change in the pricing variables. In our paper, we will develop a fuzzy jump-diffusion option pricing model which will be an extension of normal jump-diffusion model of [Merton (1976)] and the fuzzy normal jump-diffusion model of [Xu et. al. (2009)]. We will assume all parameters, drift (and risk-free interest rate), volatility, jump parameters as well as the stock price to be fuzzy numbers. In the prior studies, all the input parameters of the Merton model were not assumed to be fuzzy numbers and we aim to close this research gap through our model. The structure of the paper is as follows. In section 2, we will give a review of Black-Scholes and Merton models and then formulate our fuzzy normal jump-diffusion model. In section 3, we will do an empirical analysis of our model using the stock index and option prices. We estimate the parameters of the Merton model. Then we find the alpha level sets of fuzzy option price for different values of alpha and belief degrees for different values of the option price. After that, we defuzzify the fuzzy option prices. In section 4, we discuss the results of our empirical analysis where we come up with a comparison of our model with [Wu (2004)] model using the defuzzified option prices. Finally, in section 5 we conclude that our fuzzy jump-diffusion model outperforms Wu model.

2 Model formulation

Let us consider a European option $V = V(S(t), t)$ with exercise time T, strike (exercise) price K and payoff $p(S(T))$ where $S(t)$ is the stock price at time t. Now $p(S(T)) = \max(S(T) - K)$ for a call option and $p(S(T)) = \max(K - S(T))$ for a put option. We will assume the stock to be a non-dividend paying stock.

First, we will do a review of the Black-Scholes model and Merton's normal jump-diffusion model. Then we will propose our fuzzy normal jump- diffusion model.

2.1 Black-Scholes model

[Black and Scholes (1973)] proposed that the infinitesimal change in stock price follows the stochastic differential equation,

$$dS(t) = S(t)\big(\mu dt + \sigma dB(t)\big), \tag{1}$$

$$dB(t) \sim \mathcal{N}(0, dt) \, \text{for} \, t \in (0, T] \, , \text{and} \, S(0) > 0$$

where $S(t)$ denotes the stock price at time t, μ denotes the drift rate of the stock price, $\sigma > 0$ denotes the volatility of the stock price, and $B(t)$ is a standard Brownian motion. We assume that there are 252 trading days in a year and time t is measured in years.

Therefore, $\Delta t = \frac{1}{252}$.

Using Itô's lemma for diffusion process ([Itô (1944)]), it can be shown that

$$S(t) = S(0)\exp\Big\{\sigma B(t) + \Big(\mu - \frac{\sigma^2}{2}\Big)t\Big\} \quad (2)$$

is a solution to equation (1).

Now suppose we have $n+1$ historical observations, $S(t)$, $t \in \{0, \frac{1}{252}, ..., \frac{n}{252}\}$. Then the daily logarithmic stock return for $t \in \{\frac{1}{252}, \frac{2}{252}, ..., \frac{n}{252}\}$,

$$R(t) = \ln\frac{S(t)}{S(t-\Delta t)}$$

$$= \ln\left(\frac{S(0)\exp\Big\{\sigma B(t) + \Big(\mu - \frac{\sigma^2}{2}\Big)t\Big\}}{S(0)\exp\Big\{\sigma B(t-\Delta t) + \Big(\mu - \frac{\sigma^2}{2}\Big)(t-\Delta t)\Big\}}\right)$$

$$= \sigma\Big(B(t) - B(t-\Delta t)\Big) + \Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t$$

$$\stackrel{D}{=} \sigma B(\Delta t) + \Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t \sim \mathcal{N}\Big(\Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t, \sigma^2\Delta t\Big) \quad (3)$$

follows normal distribution with mean $\Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t$ and variance $\sigma^2\Delta t$. $\stackrel{D}{=}$ means equal in distribution. Throughout the paper, we will assume that the random variables in a sample are independent.

Now, the likelihood function of the sample $\{R(\frac{1}{252}), ..., R(\frac{n}{252})\}$ can be written as,

$$L_{BS}(\mu,\sigma) = \prod_{t=\frac{1}{252}}^{\frac{n}{252}} f_{R(t)}(\tilde{R}(t))$$

$$= \prod_{t=\frac{1}{252}}^{\frac{n}{252}} \frac{1}{\sigma\sqrt{2\pi\Delta t}}\exp\Big\{-\frac{(\tilde{R}(t) - (\mu - \frac{\sigma^2}{2})\Delta t)^2}{2\sigma^2\Delta t}\Big\} \quad (4)$$

where $f_{R(t)}(\tilde{R}(t))$ is the probability density function of $R(t)$, and the log-likelihood function of the sample can be written as,

$$logL_{BS}(\mu,\sigma)$$

$$= \sum_{t=\frac{1}{252}}^{\frac{n}{252}} \Big(-\ln\sigma\sqrt{2\pi\Delta t} - \frac{(\tilde{R}(t) - (\mu - \frac{\sigma^2}{2})\Delta t)^2}{2\sigma^2\Delta t}\Big) \quad (5)$$

The maximum likelihood estimates, $\hat{\mu}$ and $\hat{\sigma}$ can be obtained by taking the partial derivatives of $logL_{BS}(\mu,\sigma)$ with respect to μ and σ and then equating them to zero, which gives,

$$\frac{\partial}{\partial\mu}\Big(logL_{BS}(\mu,\sigma)\Big) = 0$$

$$\implies \Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t = \frac{1}{n}\sum_{t=\frac{1}{252}}^{\frac{n}{252}} \tilde{R}(t) = \mathrm{mean}(\tilde{R}(t))$$

Using this relation, we can rewrite equation (5) as,

$$logL_{BS}(\mu,\sigma) = \sum_{t=\frac{1}{252}}^{\frac{n}{252}} (-\ln\sigma\sqrt{2\pi}) - \frac{n}{2\sigma^2\Delta t}\mathrm{var}(\tilde{R}(t))$$

where

$$\mathrm{var}(\tilde{R}(t)) = \frac{1}{n}\sum_{t=\frac{1}{252}}^{\frac{n}{252}} (\tilde{R}(t) - \mathrm{mean}(\tilde{R}(t)))^2.$$

Therefore,

$$\frac{\partial}{\partial\sigma}\Big(logL_{BS}(\mu,\sigma)\Big) = 0 \implies \mathrm{var}(\tilde{R}(t)) = \sigma^2\Delta t$$

Hence, we obtain the parameter estimates as,

$$\hat{\sigma} = \sqrt{\frac{\mathrm{var}(\tilde{R}(t))}{\Delta t}},\ \hat{\mu} = \frac{\hat{\sigma}^2}{2} + \frac{\mathrm{mean}(\tilde{R}(t))}{\Delta t} \quad (6)$$

2.2 Merton model

A jump-diffusion model has been first proposed by [Merton (1976)] which states that the infinitesimal change in stock price follows the stochastic differential equation,

$$dS(t) = S(t)\big((\mu - \lambda k)dt + \sigma dB(t)\big) + S(t-)dC(t), \quad (7)$$

$$dB(t) \sim \mathcal{N}(0,dt),\ C(t) = \sum_{i=1}^{\eta(t)}(D_i - 1),\ D_i = \frac{S(\tau_i)}{S(\tau_i -)} > 0,$$

$$k = E[D_i - 1],\ F_i = \ln D_i \sim \mathcal{N}(m,\delta^2)\ \text{for}\ t \in (0,T],\ i \in \mathbb{N}$$

and $S(0) > 0$, where $S(t)$ denotes the stock price at time t and $B(t)$ is a standard Brownian motion. This model additionally assumes that there are jumps in the path of the stock price. The jumps occur at random times $\{\tau_1, \tau_2, ...\}$, $\tau_i > 0\ \forall i$. $\eta(t) = \{\#i : \tau_i \in (0,t]\}$ is a counting process with the further assumption that it is a Poisson process with intensity $\lambda > 0$ which denotes the number of times jumps occur. Hence, $\eta(t) \sim Po(\lambda t)$ follows Poisson distribution with parameter $\lambda t > 0$. The waiting times of the jump events are denoted by $T_i = \tau_i - \tau_{i-1}$ and the expected value of T_i is $\frac{1}{\lambda}$. The jumps occur from $S(\tau_i -)$ to $S(\tau_i)$ where $D_i = \frac{S(\tau_i)}{S(\tau_i -)}$ is the price ratio and $D_i - 1$ is the relative change in stock price if jump occurs. Hence, the sum of all such relative changes in stock prices, also called the jump sizes, which we denote by $C(t)$, is a compound Poisson process. $\{D_i\}$ are assumed to be identical and independent log-normal random variables with parameters m and $\delta > 0$, and k is the expected value of $D_i - 1$, i.e.

$k = \exp\{m + \frac{\delta^2}{2}\} - 1$. μ denotes the drift rate of the stock price and $\sigma > 0$ denotes the volatility of the stock price, conditional on no jumps in the path of stock prices. We assume that there are 252 trading days in a year and time t is measured in years. Therefore, $\Delta t = \frac{1}{252}$.

The right-hand side of equation (7) has two parts, the diffusion part $S(t)\big((\mu - \lambda k)dt + \sigma dB(t)\big)$ and the jump part $S(t-)dC(t)$. The diffusion part represents the "normal vibrations in price", and the jump part represents "abnormal vibrations in price". These abnormal vibrations are because of some incoming "new information about the stock" which is modelled by a Poisson process and is assumed to have a good impact on price, thereby explaining the large jumps occurring in the real scenario in the markets. Without the jump part, Merton's jump-diffusion model becomes the Black Scholes model. The model also assumes that $B(t)$, $\eta(t)$, and D_i are independent of each other.

Using Itô's lemma for the jump-diffusion process ([Cont and Tankov (2004)]), it can be shown that,

$$S(t) = S(0) \exp\Big\{\sigma B(t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)t\Big\} \prod_{i=1}^{\eta(t)} D_i \quad (8)$$

is a solution to equation (7).

Now suppose we have $n+1$ historical observations, $S(t)$, $t \in \{0, \frac{1}{252}, ..., \frac{n}{252}\}$. Then the daily logarithmic stock return for $t \in \{\frac{1}{252}, \frac{2}{252}, ..., \frac{n}{252}\}$,

$$R(t) = \ln \frac{S(t)}{S(t-\Delta t)}$$

$$= \ln \left(\frac{S(0) \exp\Big\{\sigma B(t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)t\Big\} \prod_{i=1}^{\eta(t)} D_i}{S(0) \exp\Big\{\sigma B(t-\Delta t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)(t-\Delta t)\Big\} \prod_{i=1}^{\eta(t-\Delta t)} D_i} \right)$$

$$= \sigma(B(t) - B(t-\Delta t)) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)\Delta t + \sum_{i=\eta(t-\Delta t)+1}^{\eta(t)} F_i$$

$$\stackrel{D}{=} \sigma(B(t) - B(t-\Delta t)) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)\Delta t + \sum_{i=1}^{\eta(t)-\eta(t-\Delta t)} F_i$$

$$\stackrel{D}{=} \sigma B(\Delta t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)\Delta t + \sum_{i=1}^{\eta(\Delta t)} F_i. \quad (9)$$

We have used the fact that the increments of $B(t)$ and $\eta(t)$ are stationary.

Next, the cumulative distribution function of $R(t)$ can be expressed as (similar to [Hanson and Westman (2002)] and [Synowiec (2008)]),

$$P(R(t) < \tilde{R}(t))$$

$$= \sum_j P(R(t) < \tilde{R}(t) | \eta(\Delta t) = j) P(\eta(\Delta t) = j)$$

$$= \sum_j P\Big(\sigma B(\Delta t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)\Delta t + \sum_{i=1}^{j} F_i < \tilde{R}(t)\Big) P(\eta(\Delta t) = j)$$

$$= \sum_j P(H(t) < \tilde{R}(t)) P(\eta(\Delta t) = j), \quad (10)$$

where $H(t) = \sigma B(\Delta t) + \Big(\mu - \lambda k - \frac{\sigma^2}{2}\Big)\Delta t + \sum_{i=1}^{j} F_i \sim \mathcal{N}\Big((\mu - \lambda k - \frac{\sigma^2}{2})\Delta t + jm,\, \sigma^2 \Delta t + j\delta^2\Big)$.

Now, since time is discrete in our case, we have,

$$\eta(\Delta t) = \eta\Big(\frac{1}{252}\Big) = \Big\{\#i : \tau_i \in \Big(0, \frac{1}{252}\Big]\Big\}$$

$$= \Big\{\#i : \tau_i = \frac{1}{252}\Big\} = 1 \text{ or } 0.$$

So, we can express equation (10) as a finite sum,

$$P(R(t) < \tilde{R}(t)) = \sum_{j=0}^{1} P(H(t) < \tilde{R}(t)) P(\eta(\Delta t) = j) \quad (11)$$

Clearly, the cumulative distribution function of R(t) is differentiable, which implies the probability density function of $R(t)$ can be expressed as,

$$f_{R(t)}(\tilde{R}(t)) = \sum_{j=0}^{1} f_{H(t)}(\tilde{R}(t)) P(\eta(\Delta t) = j)$$

$$= \sum_{j=0}^{1} \frac{1}{h_2\sqrt{2\pi}} e^{-\frac{(\tilde{R}(t)-h_1)^2}{2{h_2}^2}} e^{-\lambda \Delta t} \frac{(\lambda \Delta t)^j}{j!}$$

$$= \sum_{j=0}^{1} \frac{(\lambda \Delta t)^j}{j! h_2 \sqrt{2\pi}} \exp\Big\{-\frac{(\tilde{R}(t)-h_1)^2}{2{h_2}^2} - \lambda \Delta t\Big\} \quad (12)$$

where $h_1 = (\mu - \lambda k - \frac{\sigma^2}{2})\Delta t + jm$ and $h_2 = \sqrt{\sigma^2 \Delta t + j\delta^2}$.

Now, the likelihood function of the sample $\{R(\frac{1}{252}), ..., R(\frac{n}{252})\}$ can be written as,

$$L_M(\mu, \sigma, m, \delta, \lambda) = \prod_{t=\frac{1}{252}}^{\frac{n}{252}} f_{R(t)}(\tilde{R}(t))$$

$$= \prod_{t=\frac{1}{252}}^{\frac{n}{252}} \sum_{j=0}^{1} \frac{(\lambda \Delta t)^j}{j! h_2 \sqrt{2\pi}} \exp\Big\{-\frac{(\tilde{R}(t)-h_1)^2}{2{h_2}^2} - \lambda \Delta t\Big\} \quad (13)$$

and the log-likelihood function of the sample can be written as,

$$logL_M(\mu, \sigma, m, \delta, \lambda)$$

$$= \sum_{t=\frac{1}{252}}^{\frac{n}{252}} \ln\left(\sum_{j=0}^{1} \frac{(\lambda \Delta t)^j}{j! h_2 \sqrt{2\pi}} \exp\Big\{-\frac{(\tilde{R}(t)-h_1)^2}{2{h_2}^2} - \lambda \Delta t\Big\}\right) \quad (14)$$

2.3 A fuzzy jump-diffusion (FJD) model

We will be using Merton's jump-diffusion model in a fuzzy environment assuming the stock price and the input parameters to be fuzzy numbers. As pointed out in literature review that it is necessary to consider two types of stock price trajectory - the diffusion as well as jump processes and to model the impreciseness of the stock price and input parameters, we are proposing a fuzzy jump-diffusion model. The following mathematical derivations follow from the equations in the previous section on the Merton model.

From equation (8), we get

$$l(t) = \ln S(t) = \ln S(0) + \sigma B(t) + (\mu - \lambda k - \frac{\sigma^2}{2})t + \sum_{i=1}^{\eta(t)} F_i \tag{15}$$

We assume the logarithm of stock price to be a Gaussian fuzzy number (gfn). This is the best assumption we can take since we cannot derive a membership function based on its exact distribution. Instead, we consider a similar random variable,

$$\psi(t) = \ln S(0) + \sigma B(t) + (\mu - \lambda k - \frac{\sigma^2}{2})t + \sum_{i=1}^{E[\eta(t)]} F_i$$

where $E[\eta(t)] = \lambda t$ and based on its normal distribution, we assume

$$\tilde{l}(t) = gfn\Big(\bar{\mu}, \bar{\sigma}, \bar{\sigma}\Big) \tag{16}$$

and its membership function is

$$f_{\tilde{l}(t)}(x) = \exp\Big\{ - \frac{(x-\bar{\mu})^2}{2\bar{\sigma}^2} \Big\}, \ \ x \in \mathbb{R}$$

where $\bar{\mu} = \ln S(0) + (\mu - \lambda k - \frac{\sigma^2}{2})t + \lambda t m$ and $\bar{\sigma} = \sqrt{\sigma^2 t + \lambda t \delta^2}$.

The α-level set

$$\tilde{l}(t)_\alpha = \{x : f_{\tilde{l}(t)}(x) \geq \alpha\}$$
$$= \{x : \bar{\mu} - \bar{\sigma}\sqrt{-2\ln\alpha} \leq x \leq \bar{\mu} + \bar{\sigma}\sqrt{-2\ln\alpha}\}$$

which implies

$$\tilde{l}(t)^L_\alpha = \bar{\mu} - \bar{\sigma}\sqrt{-2\ln\alpha} \quad \text{and} \quad \tilde{l}(t)^R_\alpha = \bar{\mu} + \bar{\sigma}\sqrt{-2\ln\alpha} \tag{17}$$

are the left and right end points of $\tilde{l}(t)_\alpha$ respectively.

Since the logarithm of stock price is a fuzzy number, the stock price is also a fuzzy number, $\tilde{S}(t)$.

$$\tilde{S}(t)^L_\alpha = \exp\{\bar{\mu} - \bar{\sigma}\sqrt{-2\ln\alpha}\} \ \text{ and}$$
$$\tilde{S}(t)^R_\alpha = \exp\{\bar{\mu} + \bar{\sigma}\sqrt{-2\ln\alpha}\} \tag{18}$$

are the left and right end points of $\tilde{S}(t)_\alpha$ respectively.

A triangular fuzzy number (tfn), $\tilde{n} = tfn(n_1, n_2, n_3)$ is defined by its membership function,

$$f_{\tilde{n}}(x) = \begin{cases} \frac{x-n_1}{n_2-n_1} & \text{if } n_1 \leq x \leq n_2 \\ \frac{n_3-x}{n_3-n_2} & \text{if } n_2 < x \leq n_3 \\ 0 & \text{otherwise} \end{cases}$$

and its α-level set is $[(1-\alpha)n_1 + \alpha n_2, (1-\alpha)n_3 + \alpha n_2]$.

We assume the volatility σ, interest rate r, and the jump parameters m, δ and λ to be triangular fuzzy numbers. The stock is assumed to be a non-dividend-paying stock, i.e. $\mu = r$.

The European call option price according to the Black-Scholes model is,

$$C_{BS}(S, t, T, K, r, \sigma) = S\Phi(d_1) - K\exp\{-r(T-t)\}\Phi(d_2) \tag{19}$$

where $d_1 = \frac{1}{\sigma\sqrt{T-t}}\Big(\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)\Big)$, $d_2 = d_1 - \sigma\sqrt{T-t}$, and Φ is the cumulative distribution function of a standard normal random variable.

The European call option price according to the Merton model is ([Cont and Tankov (2004)]),

$$C_M(S, t, T, K, r, \sigma, m, \delta, \lambda)$$
$$= \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} C_{BS}(S_n, t, K, r, \sigma_n) \tag{20}$$

where $S_n = S\exp\{nm + \frac{n\delta^2}{2} - \lambda(T-t)e^{m+\frac{\delta^2}{2}} + \lambda(T-t)\}$ and $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{T-t}$.

From equation (19), we can write the fuzzy call option price based on Black-Scholes model as ([Wu (2004)]),

$$\tilde{C}_{BS}(\tilde{S}, t, T, K, \tilde{r}, \tilde{\sigma}) = \Big(\tilde{S} \otimes \tilde{N}(\tilde{d}_1)\Big) \ominus \Big(\tilde{1}_{\{K\}} \otimes e^{-\tilde{r}\otimes\tilde{1}_{\{T-t\}}} \otimes \tilde{N}(\tilde{d}_2)\Big) \tag{21}$$

where $\tilde{d}_1 = \Bigg(\ln(\tilde{S} \oslash \tilde{1}_{\{K\}}) \oplus \Big(\tilde{r} \oplus (\tilde{\sigma} \otimes \tilde{\sigma} \oslash \tilde{1}_{\{2\}})\Big) \otimes \tilde{1}_{\{T-t\}}\Bigg) \oslash \Bigg(\tilde{\sigma} \otimes \sqrt{\tilde{1}_{\{T-t\}}}\Bigg)$ and $\tilde{d}_2 = \tilde{d}_1 \ominus \Bigg(\tilde{\sigma} \otimes \sqrt{\tilde{1}_{\{T-t\}}}\Bigg)$.

In a similar way, we propose a fuzzy call option pricing formula based on Merton's jump-diffusion model using equation (20),

$$\tilde{C}_M(\tilde{S},t,T,K,\tilde{r},\tilde{\sigma},\tilde{m},\tilde{\delta},\tilde{\lambda}) = \oplus\sum_{n=0}^{\infty}\Big(e^{-\tilde{\lambda}\otimes\tilde{1}_{\{T-t\}}}$$

$$\otimes(\tilde{\lambda}\otimes\tilde{\lambda}\otimes ...\otimes\tilde{\lambda})\otimes(\tilde{1}_{\{T-t\}}\otimes\tilde{1}_{\{T-t\}}\otimes ...\otimes\tilde{1}_{\{T-t\}})$$

$$\oslash\tilde{1}_{\{n!\}}\otimes\tilde{C}_{BS}(\tilde{S}_n,t,K,\tilde{r},\tilde{\sigma}_n)\Big) \tag{22}$$

where

$$\tilde{\sigma}_n = \sqrt{(\tilde{\sigma}\otimes\tilde{\sigma})\oplus\Big(\tilde{1}_{\{n\}}\otimes\tilde{\delta}\otimes\tilde{\delta}\oslash\tilde{1}_{\{T-t\}}\Big)} \quad \text{and}$$

$$\tilde{S}_n = \tilde{S}\otimes\exp\Bigg\{\Big(\tilde{1}_{\{n\}}\otimes\tilde{m}\Big)\oplus\Big(\tilde{1}_{\{n\}}\otimes\tilde{\delta}\otimes\tilde{\delta}\oslash\tilde{1}_{\{2\}}\Big)$$

$$\ominus\Bigg(\tilde{\lambda}\otimes e^{\tilde{m}\oplus\Big(\tilde{\delta}\otimes\tilde{\delta}\oslash\tilde{1}_{\{2\}}\Big)}\Bigg)\oplus\Big(\tilde{\lambda}\otimes\tilde{1}_{\{T-t\}}\Big)\Bigg\},$$

$\oplus\sum$ denotes summation with respect to fuzzy addition.

The α-level set of $\tilde{\sigma_n}$ is $[\tilde{\sigma}^L_{n,\alpha},\tilde{\sigma}^U_{n,\alpha}]$ where

$$\tilde{\sigma}^L_{n,\alpha} = \sqrt{(\tilde{\sigma}^L_\alpha)^2+\frac{n(\tilde{\delta}^L_\alpha)^2}{T-t}} \text{ and } \tilde{\sigma}^U_{n,\alpha} = \sqrt{(\tilde{\sigma}^U_\alpha)^2+\frac{n(\tilde{\delta}^U_\alpha)^2}{T-t}}. \tag{23}$$

The α-level set of $\tilde{S}_n$ is $[\tilde{S}^L_{n,\alpha},\tilde{S}^U_{n,\alpha}]$ where

$$\tilde{S}^L_{n,\alpha} = \tilde{S}^L_\alpha\exp\Bigg\{n\tilde{m}^L_\alpha+\frac{n(\tilde{\delta}^L_\alpha)^2}{2}-\tilde{\lambda}^U_\alpha\exp\Big\{\tilde{m}^U_\alpha+\frac{(\tilde{\delta}^U_\alpha)^2}{2}\Big\}$$

$$+\tilde{\lambda}^L_\alpha(T-t)\Bigg\},$$

$$\tilde{S}^U_{n,\alpha} = \tilde{S}^U_\alpha\exp\Bigg\{n\tilde{m}^U_\alpha+\frac{n(\tilde{\delta}^U_\alpha)^2}{2}-\tilde{\lambda}^L_\alpha\exp\Big\{\tilde{m}^L_\alpha+\frac{(\tilde{\delta}^L_\alpha)^2}{2}\Big\}$$

$$+\tilde{\lambda}^U_\alpha(T-t)\Bigg\}. \tag{24}$$

Hence, the α-level set of fuzzy call option price, $\tilde{C}_M(\tilde{S},t,T,K,\tilde{r},\tilde{\sigma},\tilde{m},\tilde{\delta},\tilde{\lambda})$ is $[\tilde{C}^L_{M,\alpha}(t),\tilde{C}^U_{M,\alpha}(t)]$ where

$$\tilde{C}^L_{M,\alpha}(t) = \sum_{n=0}^{\infty}\frac{\exp\Big\{-\tilde{\lambda}^U_\alpha(T-t)\Big\}(\tilde{\lambda}^L_\alpha)^n(T-t)^n}{n!}$$

$$\times\tilde{C}^L_{BS,\alpha}(S_n,t,T,K,r,\sigma_n),$$

$$\tilde{C}^U_{M,\alpha}(t) = \sum_{n=0}^{\infty}\frac{\exp\Big\{-\tilde{\lambda}^L_\alpha(T-t)\Big\}(\tilde{\lambda}^U_\alpha)^n(T-t)^n}{n!}$$

$$\times\tilde{C}^U_{BS,\alpha}(S_n,t,T,K,r,\sigma_n) \tag{25}$$

For solving numerically the belief degree using a computer, we approximate the left and right end points of $\tilde{C}_{M,\alpha}$ with finite sums,

$$\tilde{C}^L_{M,\alpha} = \sum_{n=0}^{100}\frac{\exp\Big\{-\tilde{\lambda}^U_\alpha(T-t)\Big\}(\tilde{\lambda}^L_\alpha)^n(T-t)^n}{n!}$$

$$\times\tilde{C}^L_{BS,\alpha}(S_n,t,T,K,r,\sigma_n),$$

$$\tilde{C}^U_{M,\alpha} = \sum_{n=0}^{100}\frac{\exp\Big\{-\tilde{\lambda}^L_\alpha(T-t)\Big\}(\tilde{\lambda}^U_\alpha)^n(T-t)^n}{n!}$$

$$\times\tilde{C}^U_{BS,\alpha}(S_n,t,T,K,r,\sigma_n) \tag{26}$$

Wu's technique ([Wu (2004)]) of finding the belief degree for fuzzy Black-Scholes model is being applied to our fuzzy jump-diffusion model. Given any European call option price c of the fuzzy price $\tilde{C}_M$ at time t, we can get its belief degree $f_{\tilde{C}_M}(c)$ by solving the optimisation problem (OP),

$$\text{OP 1:}\begin{cases} \max & \alpha \\ \text{subject to} & (\tilde{C}^L_{M,\alpha})\le c\le(\tilde{C}^L_{M,\alpha}) \\ & 0\le\alpha\le 1. \end{cases}$$

where $f_{\tilde{C}_M}(c)=\max\alpha$. OP 1 is further relaxed into

$$\text{OP 2:}\begin{cases} \max & \alpha \\ \text{subject to} & g(\alpha)=(\tilde{C}^L_{M,\alpha})\le c \\ & 0\le\alpha\le 1. \end{cases}$$

and

$$\text{OP 3:}\begin{cases} \max & \alpha \\ \text{subject to} & c\le(\tilde{C}^L_{M,\alpha})=h(\alpha) \\ & 0\le\alpha\le 1. \end{cases}$$

We solve OP 2 and OP 3 by bisection search algorithm.

Bisection search algorithm for OP 2:
1st step - Tolerance level, $\epsilon = 10^{-7}$. Set $\alpha_0 = 0.9$, $\beta_1 = 0$, $\beta_2 = 1$. Pick a value for c.
2nd step - Given values of t, T, K, and $S(0)$, calculate $\tilde{C}^L_{M,\alpha}$. Set $g(\alpha)=\tilde{C}^L_{M,\alpha}$.
3rd step - If $g(\alpha)\le c$ go to 4th step, else go to 5th step.
4th step - If $c-g(\alpha)<\epsilon$, then exit the loop and put max $=\alpha$, else set $\beta_1=\alpha$, $\alpha=\frac{\beta_1+\beta_2}{2}$, and go to 2nd step.
5th step - $\beta_2=\alpha$, $\alpha=\frac{\beta_1+\beta_2}{2}$, and go to 2nd step.

Bisection search algorithm for OP 3:
1st step - Tolerance level, $\epsilon = 10^{-7}$. Set $\alpha_0 = 0.9$, $\beta_1 = 0$, $\beta_2 = 1$. Pick a value for c.
2nd step - Given values of t, T, K, and $S(0)$, calculate $\tilde{C}^U_{M,\alpha}$. Set $h(\alpha)=\tilde{C}^U_{M,\alpha}$.

3rd step - If $-h(\alpha) \leq -c$ go to 4th step, else go to 5th step.
4th step - If $-c + h(\alpha) < \epsilon$, then exit the loop and put $\max = \alpha$, else set $\beta_1 = \alpha$, $\alpha = \frac{\beta_1+\beta_2}{2}$, and go to 2nd step.
5th step - $\beta_2 = \alpha$, $\alpha = \frac{\beta_1+\beta_2}{2}$, and go to 2nd step.

3 Empirical analysis of the FJD model

First, we estimate the Merton model parameters. We take the daily closing prices of S&P 500, the benchmark stock index of the New York Stock Exchange, from January 4, 2010, to April 27, 2020, which is 2596 trading days. The data set is taken from the website of Yahoo Finance, https://in.finance.yahoo.com/quote/%5EGSPC/history?p=%5EGSPC.

The initial closing price of S&P 500, $S(0)$ is taken to be it's closing price on January 4, 2010. Time, t (measured in years) varies from $t = 0$ to $t = \frac{n}{252}$, where $n = 2596$. $\Delta t = \frac{1}{252}$ denotes the difference between two consecutive trading days. We use R for doing the numerical analysis.

We consider the jump events in our study as the outlier events. Hence, we calculate the historical drift rate of the stock price conditional on no jumps in the path of stock prices, $\tilde{\mu}_M$ as the arithmetic mean of those observed daily logarithmic stock returns which are not outliers and the historical volatility of the stock price conditional on no jumps in the path of stock prices, $\tilde{\sigma}_M$ as the sample standard deviation of those observed daily logarithmic stock returns which are not outliers. Next, we calculate historical m as the arithmetic mean of those observed daily logarithmic stock returns which are outliers and historical δ as the sample standard deviation of those observed daily logarithmic stock returns which are outliers. Finally, the historical λ is calculated as 252/(arithmetic mean of historical waiting times of the jump events). The parameter estimates for Merton's jump-diffusion model are obtained using maximum likelihood estimation and Lange Algorithm using *stats4* package and *constrOptim* function in R. In the *constrOptim* function, we take the negative of log-likelihood function of the sample in equation (16) as the objective function and the constraints, $\sigma > 0$, $\delta > 0$, and $\lambda > 0$ as the linear inequality constraints, and obtain the parameter estimates as $\hat{\mu}_M = 0.105895904$, $\hat{\sigma}_M = 0.106873983$, $\hat{m} = -0.005354184$, $\hat{\delta} = 0.025212291$, and $\hat{\lambda} = 28.598633803$.

Now, let us consider a S&P 500 index European option, SPX200619C02575000 which is being traded on April 27, 2020. Here initial closing price, $S(0) = 2878.48$ USD is the closing price on April 27, 2020. This option has a strike price of 2575 USD and matures on June 19, 2020. There are 39 trading days starting from April 27, 2020 till June 19, 2020. Hence, exercise time, $T = \frac{38}{252}$. We take
$\tilde{l}(t) = gfn\Big(\ln S(0), 0, 0\Big)$, $\tilde{r} = tfn(0.09, \hat{\mu}_M, 0.11)$,
$\tilde{\sigma} = tfn(0.09, \hat{\sigma}_M, 0.11)$, $\tilde{m} = tfn(-0.0054, \hat{m}, -0.0052)$,
$\tilde{\delta} = tfn(0.024, \hat{\delta}, 0.026)$, and $\tilde{\lambda} = tfn(27, \hat{\lambda}, 29)$.

First, we compute the alpha level sets of fuzzy call option price at time, $t = 0$ for different values of α (see table 1). Next, we compute the belief degrees, $f_{\tilde{C}_M}(c)$ for different call option prices, c (see table 2). Finally, we defuzzify the fuzzy call option prices using the centre of gravity defuzzifier ([Wang (1996)]) method. The centre of gravity defuzzifier for each α is obtained as,

$$\text{Defuzz}[\tilde{C}_M](\alpha) = \frac{\int_{\tilde{C}_{M,\alpha}} x f_{\tilde{C}_M}(x)dx}{\int_{\tilde{C}_{M,\alpha}} f_{\tilde{C}_M}(x)dx}$$

Now by resolution identity ([Zadeh (1971)]; [Wu (2004)]), $f_{\tilde{C}_M}(x) = 1$ for $x \in \tilde{C}_{M,\alpha}$ which implies

$$\text{Defuzz}[\tilde{C}_M](\alpha) = \frac{\int_{\tilde{C}_{M,\alpha}} xdx}{\int_{\tilde{C}_{M,\alpha}} dx} = \frac{(\tilde{C}^U_{M,\alpha})^2 - (\tilde{C}^L_{M,\alpha})^2}{2(\tilde{C}^U_{M,\alpha} - \tilde{C}^L_{M,\alpha})} \tag{27}$$

The defuzzified call option prices derived from our FJD model as well as Wu model are given in table 3. We use the same defuzzifier for Wu model as well. Using the defuzzified prices, we compare our model with Wu model.

4 Results and discussion

From table 1, we can say that with an α value of 0.99, one can select any value between 321.436 and 373.0986 for the option price at $t = 0$ if he follows FJD model and any value between 351.6046 and 352.9012 if he follows Wu model. The width of the α-level set of the fuzzy call option price is the difference between its left and right endpoints. As α increases, the width decreases. We also notice that the width is greater for FJD model than Wu model. $c = 348.05$ is the observed value of the call option. We see that the belief degrees for the observed value of the option as well as for its nearby values according to FJD model are greater than those according to Wu model. Using equation (19), the call option price according to Black-Scholes model is 344.3056 and using equation (20) (approximating the infinite sum with a finite sum taking $n = 0$ to 100), the call option price according to Merton model is 347.1855. Clearly, the Merton model estimated option price is nearer to the observed option price than the Black-Scholes model estimated option price is, which implies the Merton model is superior to the Black-Scholes model. From table 4, we find that for $\alpha \geq 0.92$, the absolute difference between defuzzified option price according to FJD model and the Merton model estimated option price is lesser than the absolute difference between defuzzified option price according to Wu model and the Black-Scholes model estimated option price. From table 5 we find that for $\alpha \geq 0.94$, the absolute difference between the defuzzified option price according to FJD model and the observed option price is lesser than the absolute difference between the defuzzified option price according to Wu model and the observed option price. This implies that with very high belief degrees, our FJD model performs much better than Wu model. This means that if one wants to be very appropriate (appropriateness measured in terms of belief degree), say 95% or more, one should pick an option price estimate based on FJD model instead of Wu model. This is because Merton model incorporates jumps in its stock price dynamics whereas Black-Scholes model fails to do it and the FJD model which is a fuzzification of Merton's normal jump-diffusion price dynamics gives better option price estimates than Wu's fuzzification of the Black-Scholes price dynamics. This is one of the first studies where impreciseness of all the input parameters and stock price are modelled for a jump-diffusion stock price dynamics.

Table 2: Belief degrees for different call option prices, c.

c	$f_{\tilde{C}_M}(c)$	$f_{\tilde{C}_{BS}}(c)$ (Wu model)
344	0.9987673	0.8780257
345	0.9991544	0.8927706
346	0.9995414	0.9075092
347	0.9999283	0.9222416
348.05	0.999665	0.937704
349	0.999297	0.951688
350	0.9989097	0.9664021
351	0.9985225	0.9811104
352	0.9981354	0.995813
353	0.9977485	0.9883962

Table 3: Defuzzified call option prices at $t = 0$ for different values of α.

α	Defuzz$[\tilde{C}_M(0)]$	Defuzz$[\tilde{C}_{BS}(0)]$ (Wu model)
0.90	358.0762	351.9654
0.91	355.93	351.9972
0.92	354.0337	352.0292
0.93	352.3802	352.0611
0.94	350.9633	352.0931
0.95	349.7776	352.125
0.96	348.8185	352.1569
0.97	348.0823	352.1889
0.98	347.5658	352.2209
0.99	347.2673	352.2529

Table 1: α-level sets of fuzzy call option price at $t = 0$ for different values of α.

α	$\tilde{C}_{M,\alpha}(0)$	$\tilde{C}_{BS,\alpha}(0)$ (Wu model)
0.90	[89.12601,627.0263]	[345.4905,358.4402]
0.91	[115.7527,596.1074]	[346.169,357.8255]
0.92	[142.0361,566.0313]	[346.8478,357.2106]
0.93	[168.0327,536.7276]	[347.5268,356.5955]
0.94	[193.8016,508.125]	[348.206,355.9802]
0.95	[219.4039,480.1514]	[348.8853,355.3647]
0.96	[244.9027,452.7343]	[349.5648,354.7491]
0.97	[270.3631,425.8014]	[350.2446,354.1333]
0.98	[295.8515,399.2801]	[350.9245,353.5173]
0.99	[321.436,373.0986]	[351.6046,352.9012]

Table 4: Absolute differences between defuzzified option prices according to FJD model and Merton model estimated option prices and absolute differences between defuzzified option prices according to Wu model and Black-Scholes model estimated option prices at $t = 0$ for different values of α.

α	$\lvert$Defuzz$[\tilde{C}_M(0)] - C_M(0)\rvert$	$\lvert$Defuzz$[\tilde{C}_{BS}(0)] - C_{BS}(0)\rvert$
0.90	10.8907	7.6598
0.91	8.7445	7.6916
0.92	6.8482	7.7236
0.93	5.1947	7.7555
0.94	3.7778	7.7875
0.95	2.5921	7.8194
0.96	1.633	7.8513
0.97	0.8968	7.8833
0.98	0.3803	7.9153
0.99	0.0818	7.9473

Table 5: Absolute differences between the defuzzified option prices according to FJD model and the observed option price and the absolute differences between the defuzzified option price according to Wu model and the observed option price at $t = 0$ for different values of α. $c = 348.05$ is the observed option price.

α	$\lvert \mathrm{Defuzz}[\tilde{C}_M(0)] - 348.05 \rvert$	$\lvert \mathrm{Defuzz}[\tilde{C}_{BS}(0)] - 348.05 \rvert$
0.90	10.0262	3.9154
0.91	7.88	3.9472
0.92	5.9837	3.9792
0.93	4.3302	4.0111
0.94	2.9133	4.0431
0.95	1.7276	4.075
0.96	0.7685	4.1069
0.97	0.0323	4.1389
0.98	0.4842	4.1709
0.99	0.7827	4.2029

5 Conclusion

In this paper, we proposed a fuzzy jump-diffusion option pricing model based on Merton's normal jump-diffusion price dynamics. We have considered the Merton model for its parameter fuzzification since it also takes into consideration the occasional large jumps which occur in the stock price dynamics in the real scenario because of certain factors such as economic and financial crises, geographical incidents such as environmental disasters, political incidents and military conflicts. The stock price, diffusion parameters and jump parameters of the Merton model are assumed to be fuzzy numbers to model the impreciseness. We have taken an appropriate consideration of Gaussian fuzzification of the logarithm of stock price and the diffusion and jump parameters are assumed to be triangular fuzzy numbers. A fuzzy call option formula is hence obtained which is the main contribution of this paper. We have also empirically verified the performance of our FJD model. Given any option price, its belief degree is obtained by solving a maximisation problem using bisection search algorithm which is thoroughly discussed. Given any value of α, we can tell in which interval the option price will lie. We have also deduced the crisp versions of the fuzzy option prices by defuzzifying using the centre of gravity defuzzifier. Comparing the defuzzified option prices with the observed option price, we have shown that our FJD model outperforms the Wu model. This is the first time a normal fuzzy jump-diffusion model has been proposed as well as empirically tested taking into consideration the impreciseness occurring from all input parameters of the jump-diffusion price dynamics and stock price. Our proposed methodology can also be applied to the stochastic volatility jump-diffusion option pricing models by [Bates (1996)] and [Scott (1997)], which is the scope for future research.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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