

# Application of Extended Rational Trigonometric Techniques to Investigate Solitary Wave Solutions

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## Research Article

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# Application of Extended Rational Trigonometric Techniques to Investigate Solitary Wave Solutions

Nadia Mahak, Ghazala Akram\*

## Abstract

In this paper, a variety of novel exact traveling wave solutions are constructed for the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation *via* analytical techniques, namely, extended rational sine-cosine method and extended rational sinh-cosh method. The physical meaning of the geometrical structures for some of these solutions is discussed. Obtained solutions are expressed in terms of singular periodic wave, solitary waves, bright solitons, dark solitons, periodic wave and kink wave solutions with specific values of parameters. For the observation of physical activities of the problem, achieved exact solutions are vital. Moreover, to find analytical solutions of the proposed equation many methods have been used but given methodologies are effective, reliable and gave more and novel exact solutions.

**Keywords:** Nonlinear partial differential equations, Boiti-Leon-Manna-Pempinelli equation, Exact solutions, Extended rational sine-cosine method, Extended rational sinh-cosh method.

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## 1 Introduction

Investigating soliton wave solutions to nonlinear partial differential equations has long been a major impact in the field of mathematical physics. Traveling wave solutions of NPDEs in the type of soliton solutions have essential significance since they create a strong relation between mathematics and physics. Exact traveling wave solutions are considered best to understand the phenomena of natural sciences. A better deal of applications of NPDEs therefore appealed numerous researchers to look for their exact solutions. Many methods have been applied to find exact solutions of NPDEs such as, generalized exponential rational function [1], tanh method [2], the  $\exp(-\phi\xi)$ -expansion method [3], the extended rational sine-cosine approach and extended rational sinh-cosh approach [4, 5], F-expansion method [6], extended Fan sub-equation method [7], the  $(\frac{G'}{G})$ -expansion method [8], the first integral method [9, 10], the unified method [11], the extended  $(\frac{G'}{G^2})$ -expansion method [12], the generalised unified method [13], hyperbolic and exponential ansatz methods [14], the Hirota bilinear [15], the ansatz method [16], the modified Kudryashov and new auxiliary equation methods [17], the general bilinear techniques [18], Sine-Gordon expansion method [19], the Hirota method [20], and so on.

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However, the present work focus on the adoption of two novel approaches: the extended rational sine-cosine approach and extended rational sinh-cosh approach to seek exact traveling wave solutions of the (2+1)-dimensional Boiti-Leon-Manna-Pempinelli equation. The Asymmetric-Nizhnik-Novikov-Veselov equation (ANNV) which is actually a two-dimensional KdV equation depicted by the system of equations;

$$\begin{aligned}\phi_x - \psi_y &= 0 \\ \phi_t - 3(\phi\psi)_x + \phi_{xxx} &= 0,\end{aligned}$$

and Boiti *et al.* [21] initially derived the above system that is a model for an incompressible fluid where  $\phi$  and  $\psi$  are the segments of the dimensionless speed. The system has been generally investigated from different view points, for example, the investigation of its Painlevé property, Lie symmetries, its solutions using arbitrary exponential functions, its the conservation law forms, its exact solution using a separation of variable approach *etc.* By inserting the transformation:  $\psi = \Phi_x$  and  $\phi = \Phi_y$ , this system of equations yields

$$\Phi_{yt} + \Phi_{xxxy} - 3\Phi_{xx}\Phi_y - 3\Phi_{xy}\Phi_x = 0, \quad (1.1)$$

where  $\Phi = \Phi(x, y, t)$  and this equation (1.1) is called as Boiti-Leon-Manna-Pempinelli (BLMP) equation which was derived by Gilson *et al.* [22] during their researched a  $(2 + 1)$ -dimensional generalization of the AKNS shallow-water wave equation using the bilinear method. This equation was utilized to depict the  $(2 + 1)$ -dimensional interaction of the Riemann wave propagated along the  $y$ -axis with a long wave propagated along the  $x$ -axis. Many researchers are focusing to extract exact solutions of BLMP equation using various different methods such as, based on the binary Bell polynomials [23], Wronskian formalism and the Hirota method [24, 25], the extended homoclinic test approach [26], the rational sine-cosine method [27] and so on.

The strategy of the paper is summarized as follows: Demarcation of extended rational sine-cosine and extended rational sinh-cosh approaches are presented, in Section 2. In Section 3, application of these methods on the BLMP equation is investigated and graphs of some obtained solutions are drawn in Section 4. Conclusion is given in Section 5. Section 6 represents future recommendations.

## 2 Algorithms

Consider the nonlinear partial differential equation (NPDE):

$$F(\Phi, \Phi_x, \Phi_t, \Phi_{xx}, \Phi_{xt}, \dots) = 0, \quad (2.2)$$

where  $\Phi = \Phi(x, t)$  and inserting the following traveling wave transformation

$$\Phi(x, t) = \Phi(\psi), \quad \psi = x - ct,$$

where  $c$  refers the wave speed, which converts the NPD Eq.(2.2) into an ODE:

$$G(\Phi, -c\Phi', \Phi', c^2\Phi'', \Phi'', -c\Phi'', \dots) = 0, \quad (2.3)$$

where  $'$  denotes the derivative with respect to  $\psi$ .

### 2.1 Extended rational sine-cosine method

**Step 1.** To obtain the solutions of Eq.(2.3), extended rational sine-cosine method asserts the general solution in the form

$$\Phi(\psi) = \frac{\xi_0 \sin(\mu\psi)}{\xi_2 + \xi_1 \cos(\mu\psi)}, \quad \cos(\mu\psi) \neq -\frac{\xi_2}{\xi_1}, \quad (2.4)$$

or,

$$\Phi(\psi) = \frac{\xi_0 \cos(\mu\psi)}{\xi_2 + \xi_1 \sin(\mu\psi)}, \quad \sin(\mu\psi) \neq -\frac{\xi_2}{\xi_1}, \quad (2.5)$$

where the unknown parameters  $\xi_0$ ,  $\xi_1$ ,  $\xi_2$  and  $\mu$  is the wave number can be determined later.

**Step 2.** By substituting Eq.(2.4) or Eq.(2.5) into Eq.(2.3), polynomials in  $\cos(\mu\psi)$  or  $\sin(\mu\psi)$  are obtained. Then collecting all coefficients with like powers of  $\cos(\mu\psi)^z$  or  $\sin(\mu\psi)^z$ , (where  $z$  is a positive integer) and equating them to zero. A set of algebraic equations can be obtained. The resulting equations are solved with the aid of Maple to get the values of unknown constants  $\xi_0$ ,  $\xi_1$ ,  $\xi_2$ ,  $c$  and  $\mu$ .

**Step 3.** Substituting the obtained unknown values from Step 2 into Eq.(2.4) or Eq.(2.5), the solution of Eq.(2.3) can be found.

## 2.2 Extended rational sinh-cosh method

**Step 1.** To obtain the solutions of Eq.(2.3), extended rational sinh-cosh method asserts the general solution in the form

$$\Phi(\psi) = \frac{\xi_0 \sinh(\mu\psi)}{\xi_2 + \xi_1 \cosh(\mu\psi)}, \quad \cosh(\mu\psi) \neq -\frac{\xi_2}{\xi_1}, \quad (2.6)$$

or,

$$\Phi(\psi) = \frac{\xi_0 \cosh(\mu\psi)}{\xi_2 + \xi_1 \sinh(\mu\psi)}, \quad \sinh(\mu\psi) \neq -\frac{\xi_2}{\xi_1}, \quad (2.7)$$

where the unknown parameters  $\xi_0$ ,  $\xi_1$ ,  $\xi_2$  and  $\mu$  refers the wave number can be determined later.

**Step 2.** By substituting Eq.(2.6) or Eq.(2.7) into Eq.(2.3), polynomials in  $\cosh(\mu\psi)$  or  $\sinh(\mu\psi)$  are obtained. Then collecting all coefficients with like powers of  $\cosh(\mu\psi)^z$  or  $\sinh(\mu\psi)^z$ , (where  $z$  is a positive integer) and equating them to zero. A set of equations can be obtained. The resulting equations are solved with the aid of Maple to get the values of unknown constants  $\xi_0$ ,  $\xi_1$ ,  $\xi_2$ ,  $c$  and  $\mu$ .

**Step 3.** Substituting the obtained unknown values from Step 2 into Eq.(2.6) or Eq.(2.7), the solution of Eq.(2.3) can be found.

## 3 Exact solutions of the Proposed PDE

The transformation:

$$\Phi(x, y, t) = U(\psi), \quad \psi = \lambda_1 x + \lambda_2 y - ct, \quad (3.8)$$

where  $\lambda_1$ ,  $\lambda_2$  and  $c$  are constants, is inserting into Eq.(1.1) and the resulting ODE can be written as

$$-c\lambda_2 U'' + \lambda_1^3 \lambda_2 U'''' - 6\lambda_1^2 \lambda_2 U' U'' = 0. \quad (3.9)$$

Integrating Eq.(3.9) and setting the constant of integration equals to zero which leads

$$-c\lambda_2 U' + \lambda_1^3 \lambda_2 U''' - 3\lambda_1^2 \lambda_2 (U')^2 = 0. \quad (3.10)$$

### 3.1 Exact solutions by extended rational sine-cosine method

Suppose that solution of Eq.(3.10) has the form

$$U(\psi) = \frac{\xi_0 \sin(\mu\psi)}{\xi_2 + \xi_1 \cos(\mu\psi)}. \quad (3.11)$$

Substituting Eq.(3.11) into Eq.(3.10), we get a polynomial in  $\cos(\mu\psi)$  and then collecting all coefficient of the like powers of  $\cos(\mu\psi)^z$  and setting them to zero. The following algebraic equations are obtained:

$$\begin{aligned}\cos(\mu\psi)^3 &: c\lambda_2\xi_2\xi_1^2 + \lambda_1^3\mu^2\xi_2\xi_1^2 = 0, \\ \cos(\mu\psi)^2 &: c\lambda_2\xi_1^3 + 2c\lambda_2\xi_2^2\xi_1 + 4\lambda_1^3\mu^2\xi_1^3 - 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 3\lambda_1^2\xi_0\mu a_2^2 = 0, \\ \cos(\mu\psi)^1 &: c\lambda_2\xi_2^3 + 2c\lambda_2\xi_2\xi_1^2 - 4\lambda_1^3\mu^2\xi_2a_1^2 + \lambda_1^3\mu^2\xi_2^3 + 6\lambda_1^2\xi_0\mu\xi_2\xi_1 = 0, \\ \cos(\mu\psi)^0 &: c\lambda_2\xi_2^2\xi_1 - 6\lambda_1^3\mu^2\xi_1^3 + 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 3\lambda_1^2\xi_0\mu\xi_1^2 = 0.\end{aligned}$$

The solutions of above equations are classified as

$$\textbf{Case 1. } \mu = \pm \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{-\frac{c\lambda_2}{\lambda_1}} \xi_1, \quad \xi_1 = \xi_1, \quad \xi_2 = 0.$$

For the case 1 the solutions of Eq.(1.1):

$$\Phi_{1_1}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \tan \left[ \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]. \quad (3.12)$$

$$\Phi_{1_2}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \tan \left[ \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right].$$

$$\textbf{Case 2. } \mu = \pm \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{-\frac{c\lambda_2}{\lambda_1}} \xi_1, \quad \xi_1 = \pm \xi_2, \quad \xi_2 = \xi_2.$$

For the case 2 the solutions of Eq.(1.1):

$$\Phi_{2_1}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\sin \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \cos \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{2_2}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\sin \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \cos \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{2_3}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\sin \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \cos \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{2_4}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\sin \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \cos \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

**OR**

Suppose that Eq.(3.10) has solution in the form, as

$$U(\psi) = \frac{\xi_0 \cos(\mu\psi)}{\xi_2 + \xi_1 \sin(\mu\psi)}.$$

Substituting Eq.(3.13) into Eq.(3.10), we get a polynomial in  $\sin(\mu\psi)$  and collecting all coefficients of the like powers of  $\sin(\mu\psi)$  and setting them to zero. The following algebraic equations are obtained:

$$\begin{aligned}\sin(\mu\psi)^3 &: -c\lambda_2\xi_1^2\xi_2 + \lambda_1^3\mu^2\xi_2\xi_1^2 = 0, \\ \sin(\mu\psi)^2 &: -2c\lambda_2\xi_1\xi_2^2 - c\lambda_2\xi_1^3 + 4\lambda_1^3\mu^2\xi_1^3 - 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 3\lambda_1^2\xi_0\mu d_2^2 = 0, \\ \sin(\mu\psi)^1 &: -c\lambda_2\xi_2^3 - 2c\lambda_2\xi_1^2\xi_2 + \lambda_1^3\mu^2\xi_2^3 - 4\lambda_1^3\mu^2\xi_2\xi_1^2 + 6\lambda_1^2\xi_0\mu\xi_2\xi_1 = 0, \\ \sin(\mu\psi)^0 &: -c\lambda_2\xi_1\xi_2^2 - 6\lambda_1^3\mu^2\xi_1^3 + 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 3\lambda_1^2\xi_0\mu\xi_1^2 = 0.\end{aligned}$$

The solutions of above equations are classified as

$$\textbf{Case 3. } \mu = \pm \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \xi_0, \quad \xi_1 = \pm \sqrt{\frac{\lambda_1}{c\lambda_2}} \xi_0, \quad \xi_2 = 0.$$

For the case 3 the solutions of Eq.(3.10):

$$\begin{aligned}\Phi_{3_1}(x, y, t) &= \sqrt{\frac{c\lambda_2}{\lambda_1}} \cot \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right], \\ \Phi_{3_2}(x, y, t) &= -\sqrt{\frac{c\lambda_2}{\lambda_1}} \cot \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right].\end{aligned}$$

$$\textbf{Case 4. } \mu = \pm \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{\frac{c\lambda_2}{\lambda_1}} \xi_2, \quad \xi_1 = \pm \xi_2, \quad \xi_2 = \xi_2.$$

For the case 4 the solutions of Eq.(3.10):

$$\begin{aligned}\Phi_{4_1}(x, y, t) &= \sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\cos \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \sin \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}, \\ \Phi_{4_2}(x, y, t) &= \sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\cos \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \sin \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}, \\ \Phi_{4_3}(x, y, t) &= -\sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\cos \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \sin \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}, \\ \Phi_{4_4}(x, y, t) &= -\sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\cos \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \sin \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.\end{aligned}$$

### 3.2 Exact solutions by extended rational sinh-cosh method

Suppose that the traveling wave solution of Eq.(3.10) has the form,

$$U(\psi) = \frac{\xi_0 \sinh(\mu\psi)}{\xi_2 + \xi_1 \cosh(\mu\psi)}.$$

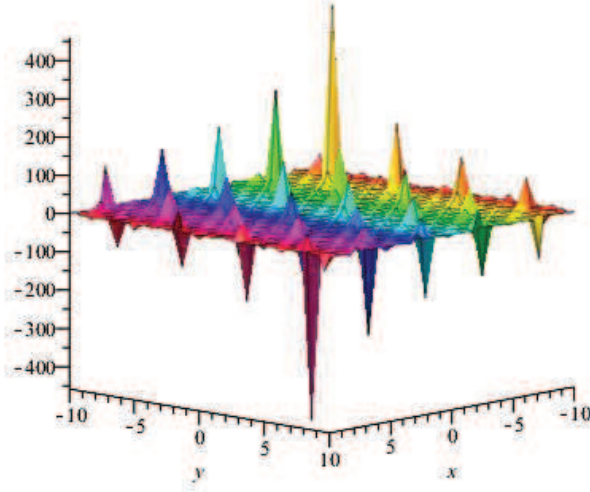


Figure 1: 3D graphics of  $\Phi_{11}$  with  $\lambda_1 = 0.09$ ,  $\lambda_2 = -1.78$  and  $c = 0.078$ .

Substituting Eq.(3.13) into Eq.(3.10), we get a polynomial in  $\cosh(\mu\psi)$  and collecting all terms with the like powers of  $\cosh(\mu\psi)^z$  and setting them to zero. The following algebraic equations are obtained:

$$\begin{aligned} \cosh(\mu\psi)^3 &: c\lambda_2\xi_2\xi_1^2 - \lambda_1^3\mu^2\xi_2\xi_1^2 = 0, \\ \cosh(\mu\psi)^2 &: 2c\lambda_2\xi_2^2\xi_1 + c\lambda_2\xi_1^3 - 4\lambda_1^3\mu^2\xi_1^3 + 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 3\lambda_1^2\xi_0\mu a_2^2 = 0, \\ \cosh(\mu\psi)^1 &: c\lambda_2\xi_2^3 + 2c\lambda_2\xi_2\xi_1^2 - \lambda_1^3\mu^2\xi_2^3 + 4\lambda_1^3\mu^2\xi_2\xi_1^2 + 6\lambda_1^2\xi_0\mu\xi_2\xi_1 = 0, \\ \cosh(\mu\psi)^0 &: c\lambda_2\xi_2^2\xi_1 - 4\lambda_1^3\mu^2\xi_2^2\xi_1 + 6\lambda_1^3\mu^2\xi_1^3 + 3\lambda_1^2\xi_0\mu\xi_1^2 = 0. \end{aligned}$$

The solutions of above equations are classified as

$$\textbf{Case 5. } \mu = \pm \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{\frac{c\lambda_2}{\lambda_1}} \xi_1, \quad \xi_1 = \xi_1, \quad \xi_2 = 0.$$

For the case 5 the solutions of Eq.(3.10):

$$\begin{aligned} \Phi_{5_1}(x, y, t) &= \sqrt{\frac{c\lambda_2}{\lambda_1}} \tanh \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right], \\ \Phi_{5_2}(x, y, t) &= -\sqrt{\frac{c\lambda_2}{\lambda_1}} \tanh \left[ \frac{1}{2\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]. \end{aligned} \tag{3.13}$$

$$\textbf{Case 6. } \mu = \pm \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{\frac{c\lambda_2}{\lambda_1}} \xi_2, \quad \xi_1 = \pm \xi_2, \quad \xi_2 = \xi_2.$$

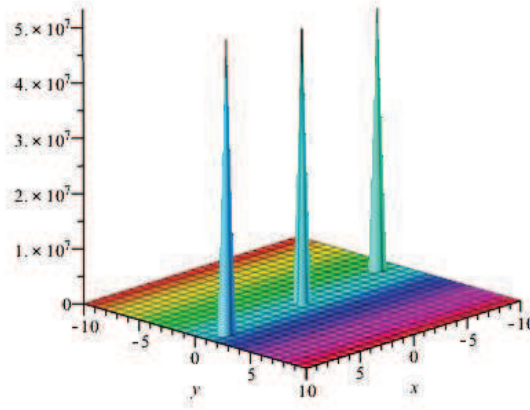


Figure 2: 3D graphics of  $\Phi_{24}$  with  $\lambda_1 = 0.09$ ,  $\lambda_2 = -0.6$  and  $c = 0.78$ .

For the case 6 the solutions of Eq.(3.10):

$$\Phi_{61}(x, y, t) = \sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\sinh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \cosh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{62}(x, y, t) = -\sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\sinh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \cosh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{63}(x, y, t) = \sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\sinh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \cosh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{64}(x, y, t) = -\sqrt{\frac{c\lambda_2}{\lambda_1}} \frac{\sinh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \cosh \left[ \frac{1}{\lambda_1} \sqrt{\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

**OR**

Suppose that Eq.(3.10) has solution in the form, as

$$U(\psi) = \frac{\xi_0 \cosh(\mu \psi)}{\xi_2 + \xi_1 \sinh(\mu \psi)}.$$

Substituting Eq.(3.14) into Eq.(3.10), we get a polynomial in  $\sinh(\mu \psi)$  and collecting all terms of the like powers of  $\sinh(\mu \psi)^z$  and setting them to zero. The following algebraic equations are obtained:

$$\begin{aligned} \sinh(\mu \psi)^3 &: c\lambda_2 \xi_1^2 \xi_2 + \lambda_1^3 \mu^2 \xi_1^2 \xi_2 = 0, \\ \sinh(\mu \psi)^2 &: 2c\lambda_2 \xi_2^2 \xi_1 - c\lambda_2 \xi_1^3 - 4\lambda_1^3 \mu^2 \xi_1^3 - 4\lambda_1^3 \mu^2 \xi_2^2 \xi_1 - 3\lambda_1^2 \xi_0 \mu a_2^2 = 0, \\ \sinh(\mu \psi)^1 &: c\lambda_2 \xi_2^3 - 2c\lambda_2 \xi_1^2 \xi_2 + \lambda_1^3 \mu^2 \xi_2^3 + 4\lambda_1^3 \mu^2 \xi_1^2 \xi_2 + 6\lambda_1^2 \xi_0 \mu \xi_2 \xi_1 = 0, \\ \sinh(\mu \psi)^0 &: -c\lambda_2 \xi_2^2 \xi_1 - 6\lambda_1^3 \mu^2 \xi_1^3 - 4\lambda_1^3 \mu^2 \xi_2^2 \xi_1 - 3\lambda_1^2 \xi_0 \mu \xi_1^2 = 0. \end{aligned}$$



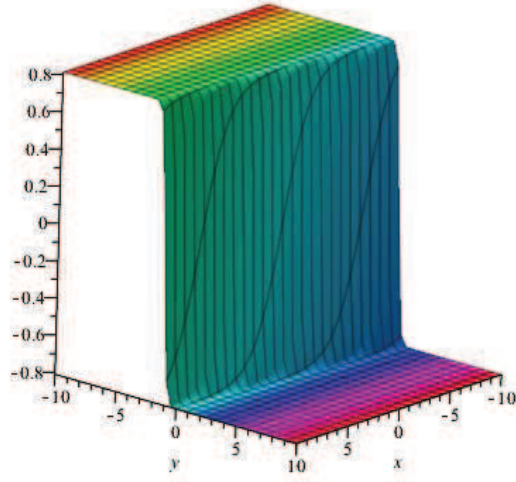


Figure 3: 3D graphics of  $\Phi_{5_2}$  with  $\lambda_1 = 0.09$ ,  $\lambda_2 = 0.76$  and  $c = 0.078$ .

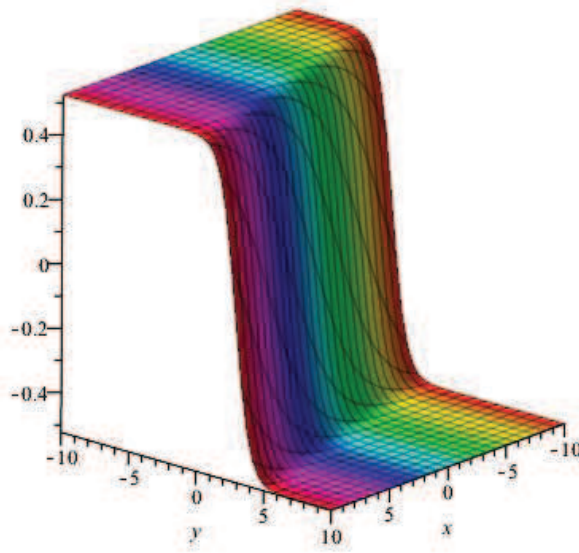


Figure 4: 3D graphics of  $\Phi_{6_1}$  with  $\lambda_1 = 0.5$ ,  $\lambda_2 = -1.76$  and  $c = 0.078$ .

The solutions of above equations are classified as

$$\textbf{Case 7. } \mu = \pm \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \xi_0, \quad \xi_1 = \pm \iota \sqrt{\frac{\lambda_1}{c\lambda_2}} \xi_0, \quad \xi_2 = 0.$$

For the case 7 the solutions of Eq.(3.10):

$$\Phi_{7_1}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \coth \left[ \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right].$$

$$\Phi_{7_2}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \coth \left[ \frac{1}{2\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right].$$

$$\textbf{Case 8. } \mu = \pm \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}}, \quad \xi_0 = \pm \sqrt{\frac{c\lambda_2}{\lambda_1}} \xi_2, \quad \xi_1 = \pm \iota \xi_2, \quad \xi_2 = \xi_2.$$

For the case 8 the solutions of Eq.(3.10):

$$\Phi_{8_1}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\cosh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \iota \sinh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

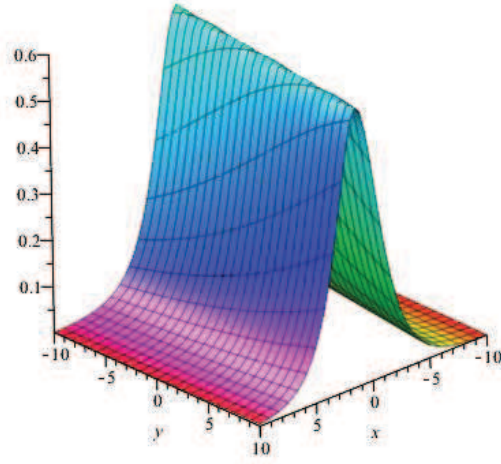
$$\Phi_{8_2}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\cosh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 + \iota \sinh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

$$\Phi_{8_3}(x, y, t) = \sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\cosh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \iota \sinh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

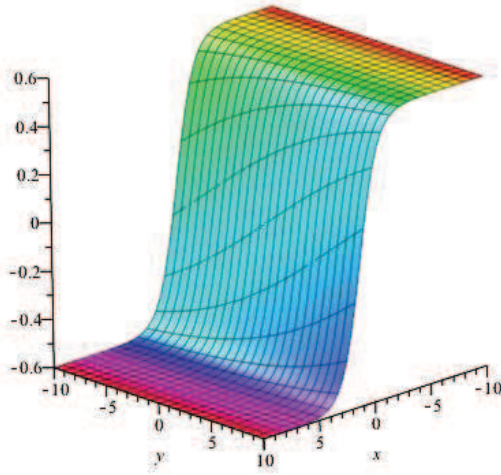
$$\Phi_{8_4}(x, y, t) = -\sqrt{-\frac{c\lambda_2}{\lambda_1}} \frac{\cosh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}{1 - \iota \sinh \left[ \frac{1}{\lambda_1} \sqrt{-\frac{c\lambda_2}{\lambda_1}} (\lambda_1 x + \lambda_2 y - ct) \right]}.$$

### 3.3 Results and discussion

Some obtained exact rational trigonometric solutions of the (2 + 1)-dimensional Boiti-Leon-Manna-Pempinelli equation are shown by graphs along with the physical explanations which are plotted only for  $-10 \leq x \leq 10$ ,  $-10 \leq y \leq 10$  and  $t = 0$ . Eq.(3.12) represents the periodic wave solutions, as tangent is a periodic function with period  $2\pi$ . Fig.(1) illustrates the evolution of singular periodic wave solutions for  $\Phi_{1_1}(x, y, t)$  for  $\lambda_1 = 0.09$ ,  $\lambda_2 = -1.78$  and  $c = 0.078$ . Taking  $\lambda_1 = 0.09$ ,  $\lambda_2 = -0.6$  and  $c = 0.78$ , Fig.(2) shows periodic solutions for  $\Phi_{2_4}(x, y, t)$ . Eq.(3.13) shows dark soliton solutions. Further, with  $\lambda_1 = 0.09$ ,  $\lambda_2 = 0.76$  and  $c = 0.078$ , Fig.(3) of  $\Phi_{5_2}(x, y, t)$  describes kink wave solutions as Kink waves are solitons that rise or descend from one asymptotic state to another. With  $\lambda_1 = 0.5$ ,  $\lambda_2 = -1.76$  and  $c = 0.078$ , Fig.(4) also represents kink type wave solutions for  $\Phi_{6_1}(x, y, t)$ . With the supposition that  $\lambda_1 = -0.5$ ,  $\lambda_2 = 0.06$  and  $c = 3$ , part (a) of Fig.(5) illustrates the graph of the real part of  $\Phi_{8_1}(x, y, t)$  which represents bright soliton wave solutions, while part (b) of Fig.(5) shows the graph of the imaginary part of  $\Phi_{8_1}(x, y, t)$  which describes kink wave solutions.



(a)



(b)

Figure 5: 3D graphics of real and imaginary parts of  $\Phi_{8_1}$  with  $\lambda_1 = -0.5$ ,  $\lambda_2 = 0.06$  and  $c = 3$ .

## 4 Conclusion

Exact rational trigonometric solutions of the  $(2 + 1)$ -dimensional Boiti-Leon-Manna-Pempinelli equation have been retrieved by using extended rational sine-cosine and extended rational sinh-cosh methods. Obtained solitary wave solutions are extremely useful in the study of NPDEs in the context of shallow water waves. Some new graphical representations are obtained with the help of these methods. It is found that some of the obtained exact solutions have likely comparable with [27]. As solutions of Eq.(47) and Eq.(59) in [27] are likely similar with our solutions Eq.(3.13) and Eq.(3.12) respectively. To our knowledge, remaining obtained solutions such as solitary waves, singular periodic wave, bright soliton, dark soliton, periodic wave and kink wave solutions are new and novel.

## 5 Future recommendations

This paper obtained solitary wave solutions to a nonlinear evolution equation that appears in mathematical physics. These solutions are going to be indeed valuable for conducting future research in this field. One novel future aspect is to consider high dimensional equations for examples with perturbation term(s), fractional temporal evolutions will lead to additional interesting results that will be further closer to realistic situations. Seeking analytical solutions for such equations will be a daunting task. It is believed that the proposed methods are powerful, effective, and may play an important role describing the physical features of various nonlinear complex models.

## 6 Declarations

**Funding** Not applicable.

**Conflicts of interest** The Authors have no conflict of interest.

**Availability of data and material** Not applicable.

**Code availability** The computations involved in the work are done with the help of Maple and Mathematica.

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Figures

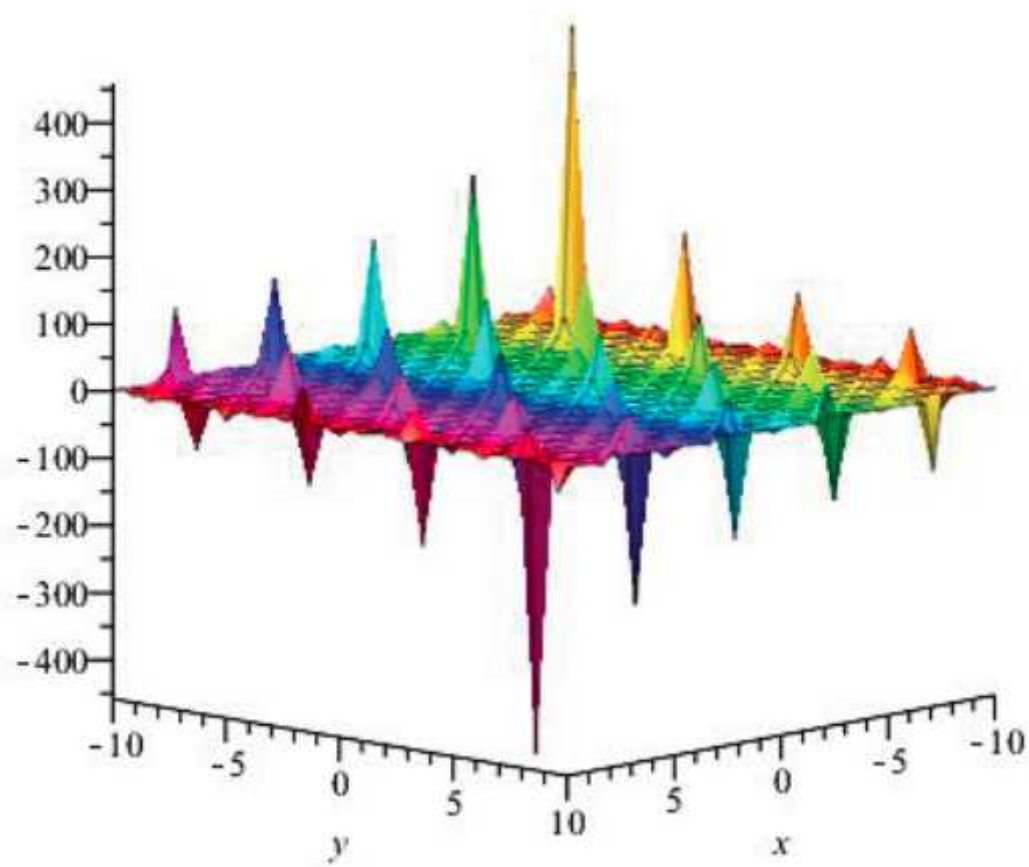
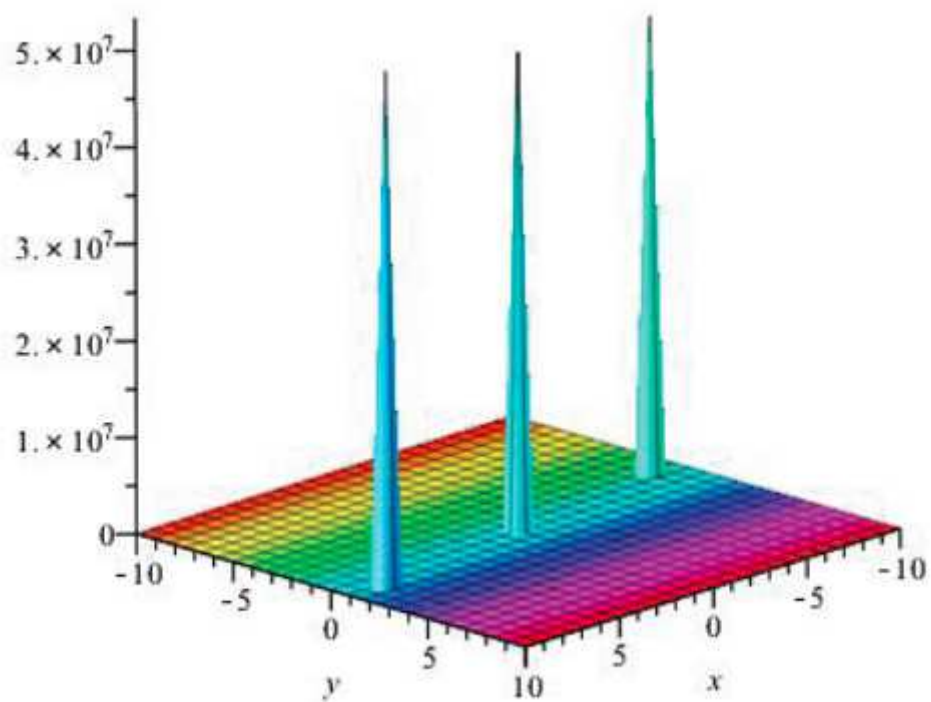


Figure 1

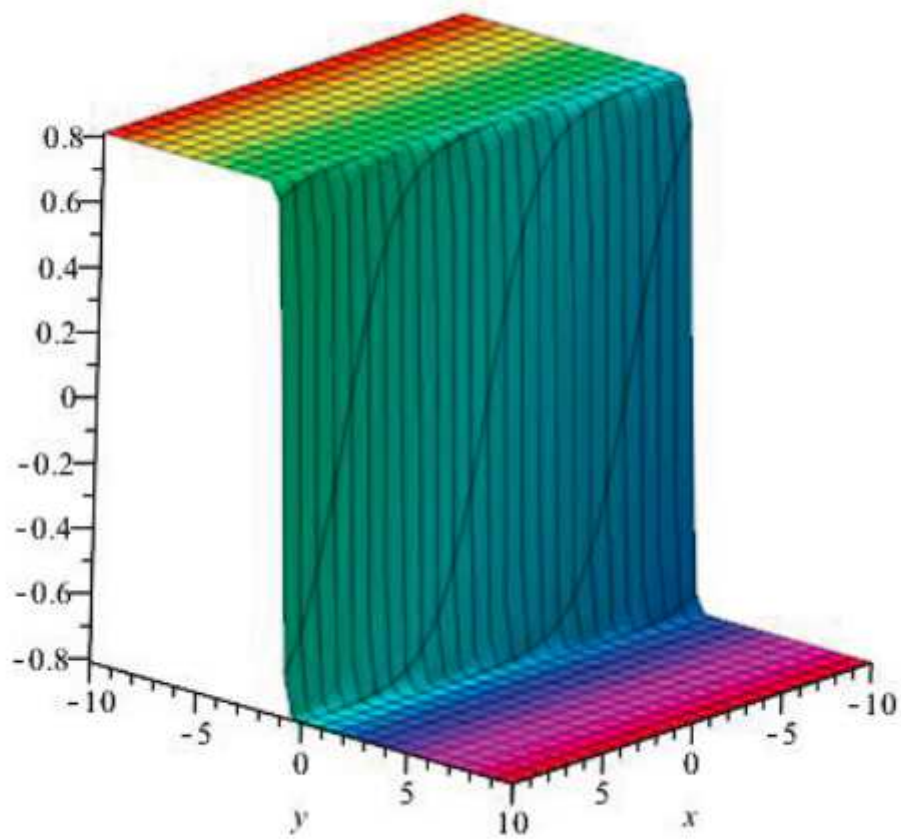
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**Figure 2**

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**Figure 3**

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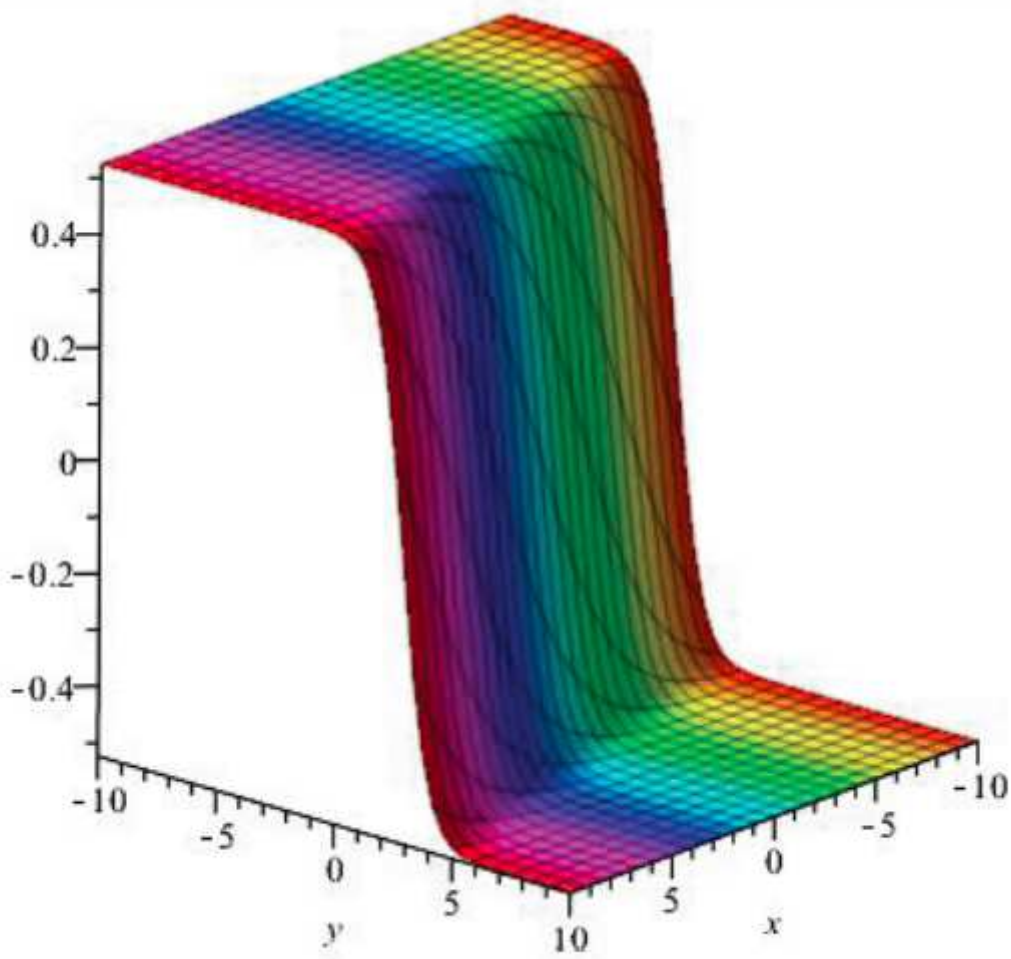
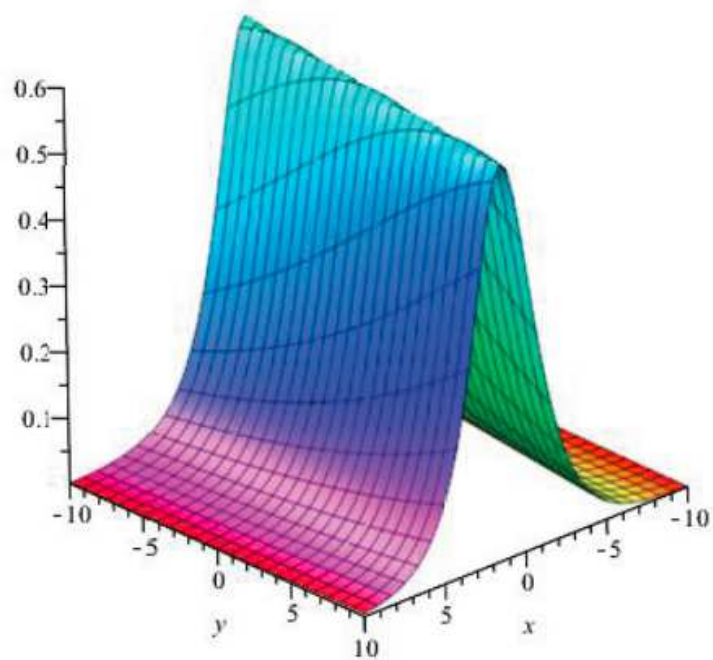
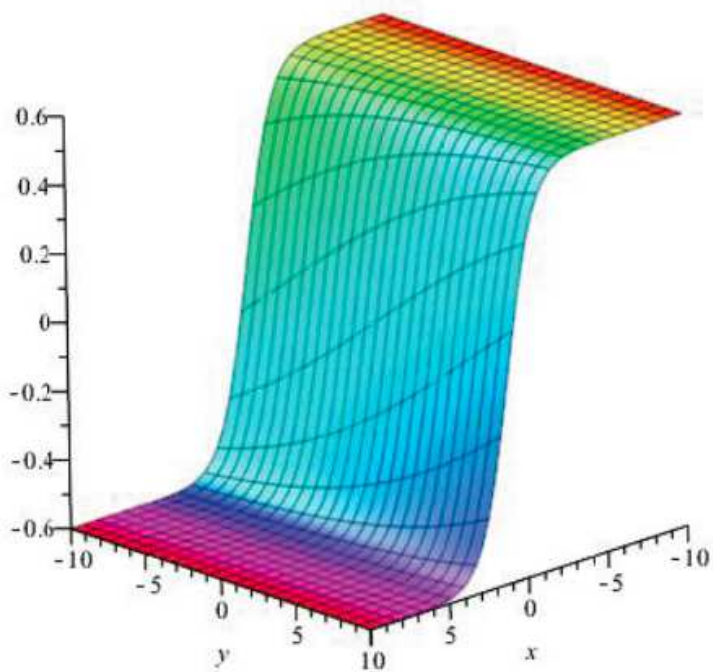


Figure 4

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(a)



(b)

**Figure 5**

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