Stone Algebras: 3-valued Logic and Rough Sets

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Stone Algebras: 3-valued Logic and Rough Sets

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Abstract In this article, we propose 3-valued semantics of the logics compatible with Stone and dual Stone algebras. We show that these logics can be considered as 3-valued by establishing soundness and completeness results. We also establish rough set semantics of these logics where the third value can be interpreted as not certain but possible.

Key words: Stone algebras, 3-valued logic, Rough sets.

1 Introduction

In 1940, Moisil introduced 3-valued Łukasiewicz algebras as the algebraic models of 3-valued Łukasiewicz logic.

Definition 1 ([1]) An abstract algebra $(A, \vee, \wedge, \sim, \triangledown, 0, 1)$ is a 3-valued Łukasiewicz algebra if for any $x, y \in A$:

1. $(A, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra, i.e.,
   (a) $\sim\sim x = x$,
   (b) $\sim (x \vee y) = \sim x \wedge \sim y$.
2. $\sim x \vee \triangledown x = 1$,
3. $x \wedge \sim x = \sim x \wedge \triangledown x$,
4. $\triangledown(x \wedge y) = \triangledown x \wedge \triangledown y$.

It is well-known that the various algebras appear as reduct algebras [1] of the 3-valued Łukasiewicz algebras. So, it is natural to ask the question.

– Can we provide 3-valued (n-valued) logics compatible with reduct algebras of 3-valued Łukasiewicz algebras?

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In [2], Kumar and Banerjee answered this question affirmatively in the case of Kleene algebras. They introduced a logic $\mathcal{L}_K$ for Kleene algebras, which is sound and complete with respect to a 3-valued consequence relation. In this paper, we show that the logic of Stone (dual Stone) algebras are sound and complete with respect to a 3-valued semantics defined via a 3-valued consequence relation.

In other aspects of this paper, we make explicit connections between logic of Stone (dual Stone) algebras and rough sets. Rough set theory, introduced by Pawlak [3,4] as a tool to deal with uncertainty in an information system. This deals with a domain U and an equivalence relation R on U. In Pawlakian rough sets theory, the equivalence relation R is interpreted as the *indiscernibility* relation on the domain U. xRy if and only if x is indiscernible to y with respect to attributes present in the information system. The pair (U, R) is called a (Pawlak) *approximation space*. For any $A \subseteq U$, one defines the *lower* and *upper approximations* of A in the approximation space (U, R), denoted $\mathsf{L}A$ and $\mathsf{U}A$ respectively, as follows. For $x \in U$, let $[x]$ denote the equivalence class of x modulo R,

$$\mathsf{L}A := \bigcup\{[x] : [x] \subseteq A\},$$
$$\mathsf{U}A := \bigcup\{[x] : [x] \cap A \neq \emptyset\}. \qquad (*)$$

Definition 2 Let (U, R) be an approximation space. For each $A \subseteq U$, the ordered pair $(\mathsf{L}A, \mathsf{U}A)$ is called a rough set in (U, R).

Notation 1 *$\mathcal{RS}$ denotes the collection of all rough sets for an approximation space (U, R).*

Notation 2 *Let U be a set. Then*

– *$\mathcal{P}(U)$ denote the power set of U.*
– *for any $A \subseteq U$, A^c denote the set theoretic complement of A in U.*

In rough set theory, the definition $(*)$ has been interpreted in the following manner.

1. x *certainly* belongs to A, if $x \in \mathsf{L}A$, i.e. all objects which are indiscernible to x are in A.
2. x *certainly does not* belong to A, if $x \notin \mathsf{U}A$, i.e. all objects which are indiscernible to x are not in A.
3. Membership of x in A is *not certain, but possible*, if $x \in \mathsf{U}A \setminus \mathsf{L}A$. This is the case when some objects indiscernible to x are in A, while some others, also indiscernible to x, are in A^c. In rough set terminology, sets of the form $\mathsf{U}A \setminus \mathsf{L}A$ are referred as *boundary* of A.

These interpretations have led too much work in the study of connections between 3-valued algebras or logics and rough sets, see for instance [5–10, 2, 11, 12]. Its worth mention here some recent works.

- In [2], Kumar and Banerjee represented a given Kl--eene algebra in terms of Kleene algebra formed by rough sets for some appropriate approximation space. This imparted the 3-valued and rough set semantics of the logic $\mathcal{L}_K$ (of Kleene algebras). The interpretations 1, 2 and 3 have been explicitly captured in [13].
- In [11], Panicker and Banerjee adopted yet other definition of rough sets (first discussed by Pagliani [7]) to explore the C-algebraic structures of rough sets. As in [7], for an approximation space (U, R) and $A \subseteq U$, the pair $(\mathsf{L}A, (\mathsf{U}A)^c)$ is called a rough set. The collection of all the rough sets for an approximation space (U, R) forms a C-algebras. Further they have proved that a C-algebra is embeddable into C-algebra formed by rough sets for some appropriate approximation space. It is worth mention here that the C-algebras are the algebraic counterpart of McCarthy's three-valued logic (cf. [11]) and unlike our case where the set of truth values of proposed logic $\mathcal{L}_{\mathcal{S}}$ is a Stone algebra, a C-algebra may not form even a semi-lattice.

In the last part of this article, we capture interpretations 1, 2 and 3 via logics compatible with Stone and dual Stone algebras.

The rest of this paper is organized as follows. In Section 2, we present some basic results of Stone algebras that will be used in the sequel. In Section 3, we extend the distributive lattice logic [14] to obtain the logic $\mathcal{L}_{\mathcal{S}}$ ($\mathcal{L}_{\mathcal{DS}}$) of Stone (dual Stone) algebras. We further propose a 3-valued consequence relation $\vDash_1^S$ ($\vDash_0^{DS}$) and show that the logic $\mathcal{L}_{\mathcal{S}}$ ($\mathcal{L}_{\mathcal{DS}}$) is sound and complete with respect to the $\vDash_1^S$ ($\vDash_0^{DS}$). In Section 4, we provide the rough set semantics of the logic $\mathcal{L}_{\mathcal{S}}$ ($\mathcal{L}_{\mathcal{DS}}$) and capture the the interpretations 1, 2 and 3.

2 Stone algebras: Some known facts

Stone algebras (lattices) were introduced by Gratzer and Schmidt [15], and have been extensively studied in literature ([16–19], cf. [20]).

Definition 3 [15] An algebra $\mathcal{S} := (S, \vee, \wedge, \sim, 0, 1)$ is a Stone algebra if

1. $(S, \vee, \wedge, \sim, 0, 1)$ is a bounded distributive pseudo complemented lattice, i.e. $\forall a \in S$, $\sim a = max\{c \in S : a \wedge c = 0\}$ exists.
2. $\sim a \vee \sim\sim a = 1$, for all $a \in S$.

The dual notion of a given Stone algebra is known as dual Stone algebra. To make this article self-contained, we explicitly define the dual Stone algebra.

Definition 4 An algebra $\mathcal{DS} := (DS, \vee, \wedge, \neg, 0, 1)$ is a dual Stone algebra if

1. $(DS, \vee, \wedge, \neg, 0, 1)$ is a bounded distributive dual pseudo complemented lattice, i.e. $\forall a \in DS$, $\neg a = min\{c \in DS : a \vee c = 1\}$ exists.
2. $\neg a \wedge \neg\neg a = 0$, for all $a \in DS$ (dual Stone property).

Let $\mathcal{B} = (B, \vee, \wedge, 0, 1)$ be a Boolean algebra. Consider the set $B^{[2]} := \{(a, b) : a \leq b, a, b \in B\}$. It is well known that $\mathcal{B}^{[2]} = (B^{[2]}, \vee, \wedge, (0, 0), (1, 1))$ is a bounded distributive lattice, where $\vee$ and $\wedge$ are componentwise join and meet inherited from B.
Moreover, we have the following results.

Proposition 1 *[1, 21] Let $\mathcal{B} = (B, \vee, \wedge, ^c, 0, 1)$ be a Boolean algebra.*

1. *$\mathcal{B}^{[2]}_{\sim} := (B^{[2]}, \vee, \wedge, \sim, (0, 0), (1, 1))$ is a Stone algebra, where, for $(a, b) \in B^{[2]}$, $\sim (a, b) := (b^c, b^c)$.*
2. *$\mathcal{B}^{[2]}_{\neg} := (B^{[2]}, \vee, \wedge, \neg, (0, 0), (1, 1))$ is a dual Stone algebra, where, for $(a, b) \in B^{[2]}$, $\neg(a, b) := (a^c, a^c)$.*

Let $\mathbf{2} = (\{0, 1\}, \vee, \wedge, \sim, 0, 1)$ be the 2-element Boolean algebra and $\mathbf{3} = (\{0, a, 1\}, \vee, \wedge, 0, 1)$ be the 3-element lattice with $0 \leq a \leq 1$. Let $\mathbf{3}_{\sim}$ denote the Stone algebra $(\mathbf{3}, \sim)$ with $\sim 0 = 1 = \sim a$, $\sim 1 = 0$ and $\mathbf{3}_{\neg}$ denote the dual Stone algebra $(\mathbf{3}, \neg)$ with $\neg 0 = 1$, $\neg 1 = 0 = \neg a$.

Theorem 1 *[22, 21]*

1. *A Stone algebra $\mathcal{S} = (S, \vee, \wedge, \sim, 0, 1)$ is embedded into $\mathbf{2}^I \times \mathbf{3}^J_{\sim}$, for some index sets I and J.*
2. *A dual Stone algebra $\mathcal{DS} = (DS, \vee, \wedge, \neg, 0, 1)$ is embedded into $\mathbf{2}^I \times \mathbf{3}^J_{\neg}$, for some index sets I and J.*

Now, as $\mathbf{2}$ is embedded into algebras $\mathbf{3}_{\sim}$ and $\mathbf{3}_{\neg}$, hence the above theorem can be restated in terms of $\mathbf{3}_{\sim}$ and $\mathbf{3}_{\neg}$. So, in particular if B is a Boolean algebra, then the Stone algebra $\mathcal{B}^{[2]}_{\sim}$ and dual Stone algebra $\mathcal{B}^{[2]}_{\neg}$ can be embedded into $\mathbf{3}^I_{\sim}$ and $\mathbf{3}^J_{\neg}$ respectively, for appropriate index sets I and J.

Definition 5 [23] Let $\mathcal{L} := (L, \vee, \wedge, 0, 1)$ be a complete lattice.

(i) An element $a \in L$ is said to be completely join irreducible, if $a = \bigvee S$ implies that $a \in S$, for every subset S of L.

Notation 3 *Let $\mathcal{J}_L$ denote the set of all completely join irreducible elements of L, and $J(x) := \{a \in \mathcal{J}_L : a \leq x\}$, for any $x \in L$.*

(ii) A set S is said to be join dense in $\mathcal{L}$, provided for every element $a \in L$, there is a subset S' of S such that $a = \bigvee S'$.

The illustration of importance of completely join irreducible elements can be seen by a result of Birkhoff.

Lemma 1 *[24] Let L and K be two completely distributive lattices. Further, let $\mathcal{J}_L$ and $\mathcal{J}_K$ be join dense in L and K, respectively. Let $\phi : \mathcal{J}_L \to \mathcal{J}_K$ be an order isomorphism. Then the extension map $\Phi : L \to K$ given by $\Phi(x) := \bigvee(\phi(J(x)))$ (where $J(x) := \{a \in \mathcal{J}_L : a \leq x\}$), $x \in L$, is a lattice isomorphism.*

In [2] we characterized the completely join irreducible elements of lattices $\mathbf{3}^I$ and $B^{[2]}$, where B is a complete atomic Boolean algebra.

Let $i, k \in I$. Denote by f_i^x, $x \in \{a, 1\}$, the following element in $\mathbf{3}^I$.

$$f_i^x(k) := \begin{cases} x \text{ if } k = i \\ 0 \textit{ otherwise} \end{cases}$$

Proposition 2 *[2]*

1. *The set of completely join irreducible elements of $\mathbf{3}^I$ is given by:*
$$\mathcal{J}_{\mathbf{3}^I} = \{f_i^a, f_i^1 : i \in I\}.$$
Moreover, $\mathcal{J}_{\mathbf{3}^I}$ is join dense in $\mathbf{3}^I$.
2. *Let B be a complete atomic Boolean algebra. The set of completely join irreducible elements of $B^{[2]}$ is given by*
$$\mathcal{J}_{B^{[2]}} = \{(0, a), (a, a) : a \in \mathcal{J}_B\}.$$
Moreover, $\mathcal{J}_{B^{[2]}}$ is join dense in $B^{[2]}$.

Figure 1 shows the Hasse diagrams of $\mathcal{J}_{\mathbf{3}^I}$ and $\mathcal{J}_{B^{[2]}}$.

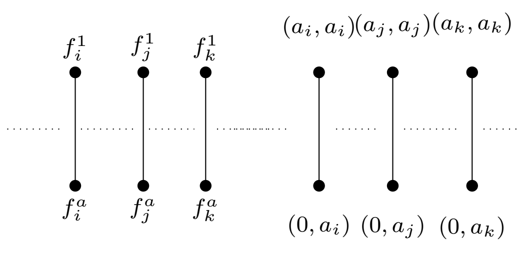

Fig. 1: Hasse diagram of $\mathcal{J}_{\mathbf{3}^I}$

We also established the following isomorphisms.

Theorem 2 *[2]*

1. *The sets of completely join irreducible elements of $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are order isomorphic.*
2. *The algebras $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are lattice isomorphic.*

Now, we know that the pseudo and dual pseudo negations (if exist) are defined via the order of the given partially ordered sets. Moreover, Stone and dual Stone algebras are equational algebras. Hence, using Lemma 1 and Theorem 2, we can re-write the Theorem 1 as

Theorem 3 *1. The Stone algebras $\mathbf{3}^I_\sim$ and $(\mathbf{2}^I)^{[2]}_\sim$ are isomorphic.*
2. *The dual Stone algebras $\mathbf{3}^I_\neg$ and $(\mathbf{2}^I)^{[2]}_\neg$ are isomorphic.*
3. *Let $\mathcal{S}$ be a Stone algebra. Then there is an (index) set I such that $\mathcal{S}$ can be embedded into Stone algebra $(\mathbf{2}^I)^{[2]}_\sim$.*
4. *Let $\mathcal{DS}$ be a dual Stone algebra. Then there is an (index) set I such that $\mathcal{DS}$ can be embedded into dual Stone algebra $(\mathbf{2}^I)^{[2]}_\neg$.*

3 3-valued semantics of logics for Stone and dual Stone algebras

In this section we focus on the study of the logics corresponding to the classes of Stone and dual Stone algebras and the structures $\mathcal{B}^{[2]}_\sim$ and $\mathcal{B}^{[2]}_\neg$. Our approach to the study is motivated by Dunn's 4-valued semantics of the De Morgan consequence system [25]: $\vDash_{0,1}$ (or $\vDash_0$ or $\vDash_1$), wherein valuations are defined in the 4-element De Morgan algebra. The 4-valued semantics arises from the fact that each element of a De Morgan algebra can be looked upon as a pair of sets.

In a similar way, we exploit Theorem 3 to provide a 3-valued semantics of the logic for Stone algebras. However, by an easy consequence of Stone's representation theorem and Theorem 3, we have:

Theorem 4 *1. Let $\mathcal{S} = (S, \vee, \wedge, \sim, 0, 1)$ be a Stone algebra. Then there is a set U such that $\mathcal{S}$ can be embedded into Stone algebra formed by $(\mathcal{P}(U))^{[2]}_\sim$.*
2. *Let $\mathcal{DS} = (DS, \vee, \wedge, \neg, 0, 1)$ be a dual Stone algebra. Then there is a set U such that $\mathcal{DS}$ can be embedded into dual Stone algebra formed by $(\mathcal{P}(U))^{[2]}_\neg$.*

3.1 Bounded Distributive Lattice Logic with Negation

Bounded distributive lattices are algebraic models of the bounded distributive lattice logic ($BDLL$), an extension of distributive lattice logic introduced by Dunn

[14]. The study of logics in this section is based on $BDLL$. Let us present the logic. The language consists of

- the set $\mathcal{P}$ of propositional variables, whose elements are denoted by $p, q, r, \ldots$.
- propositional constants $\top$ and $\bot$,
- logical connectives $\vee$ and $\wedge$.

The set $\mathcal{F}$ of well-formed formulas of the logic is then given by the scheme:

$$\alpha := p \mid \top \mid \bot \mid \alpha \vee \beta \mid \alpha \wedge \beta \ ,$$

where p is a propositional variable.

Definition 6 [25] The bounded distributive lattice logic ($BDLL$) is a binary consequence system $\vdash \subseteq \mathcal{F} \times \mathcal{F}$ with the following postulates and rules:

1. $\alpha \vdash \alpha$ (Reflexivity),
2. $\alpha \vdash \beta, \beta \vdash \gamma / \alpha \vdash \gamma$ (Transitivity),
3. $\alpha \wedge \beta \vdash \alpha$, $\alpha \wedge \beta \vdash \beta$ (Conjunction Elimination),
4. $\alpha \vdash \beta, \alpha \vdash \gamma / \alpha \vdash \beta \wedge \gamma$ (Conjunction Introduction),
5. $\alpha \vdash \alpha \vee \beta$, $\beta \vdash \alpha \vee \beta$ (Disjunction Introduction),
6. $\alpha \vdash \gamma, \beta \vdash \gamma / \alpha \vee \beta \vdash \gamma$ (Disjunction Elimination),
7. $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ (Distributivity),
8. $\alpha \vdash \top$ (Top).
9. $\bot \vdash \alpha$ (Bottom).

The postulates and rules from 1 to 7 precisely define the distributive lattice logic. The term $\alpha \vdash \beta$ in the above representation of logic is called a *consequent.* Intuitively, $\alpha \vdash \beta$ reflects that β is a consequence of α.

Let us add a unary connective $-$ to the language of $BDLL$. Let $\mathcal{F}_-$ be the set of formulas defined using the following rule:

$$\alpha := p \mid \top \mid \bot \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid -\alpha,$$

By an extension $\mathcal{L}$ of $BDLL$, we mean a binary consequence system $\vdash \subseteq \mathcal{F}_- \times \mathcal{F}_-$ which contains all the postulates and rules of the logic $BDLL$. By $\alpha \vdash_{\mathcal{L}} \beta$, we shall mean that the consequent $\alpha \vdash \beta$ is derivable in the logical system $\mathcal{L}$ (where the notion of derivability is defined in the classical manner).

In this paper, the various semantics of a logic $\mathcal{L}$ are defined using valuations.

Definition 7 Let $\mathcal{A} = (A, \vee, \wedge, -, 0, 1)$ be a lattice-based algebra, where $-$ is a unary operation on A. A map $v : \mathcal{F}_- \to A$ is called a valuation on A if $\forall \alpha, \beta \in \mathcal{F}_-$

1. $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$,
2. $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$,
3. $v(-\alpha) = -v(\alpha)$,
4. $v(\bot) = 0$, $v(\top) = 1$.

The notion of local (global) validity is defined in the following manner:

Definition 8 Let $(A, \vee, \wedge, -, 0, 1)$ be a lattice-based algebra.

- A consequent $\alpha \vdash \beta$ is *valid in A under the valuation* v, if $v(\alpha) \leq v(\beta)$. If the consequent is valid under all valuations on A, then it is *valid in* A, and denote it as $\alpha \vDash_A \beta$.

Let $\mathcal{A}$ be a class of algebras of the type $(A, \vee, \wedge, -, 0, 1)$.

- If the consequent $\alpha \vdash \beta$ is valid in each algebra of $\mathcal{A}$, then we say $\alpha \vdash \beta$ *is valid in* $\mathcal{A}$, and denote it as $\alpha \vDash_{\mathcal{A}} \beta$.

3.2 The logics $\mathcal{L_S}$, $\mathcal{L_{DS}}$ and their 3-valued Semantics

Let U be a set and $A \subseteq U$. Then for any $x \in U$, either $x \in A$ or $x \in A^c$. This distinguished property of '$\in$' leads to the **T**rue-**F**alse semantics of classical propositional logic. Now, if v is a valuation from classical propositional sentences to $\mathcal{P}(U)$, then v determines a family of 2-valued valuations $\{v_x : x \in U\}$ on classical propositional sentences, where $v_x(\gamma) = 1$ if $x \in v(\gamma)$ and $v_x(\gamma) = 0$ if $x \notin v(\gamma)$. Utilizing this fact along with Stone's representation theorem, one establishes the equivalency between **T**rue-**F**alse semantics, set theoretic semantics and algebraic semantics of classical propositional logic.

In this section, we follow the same approach to establish the completeness results for $\mathcal{L_S}$ and $\mathcal{L_{DS}}$ (defined below).

Definition 9 Let $\sim$ be a unary connective added to the language of $BDLL$. Then, for $\alpha, \beta \in \mathcal{F}_\sim$, $\mathcal{L_S}$ denotes the logic $BDLL$ along with following rules and postulates.

1. $\alpha \vdash \beta / \sim \beta \vdash \sim \alpha$(Contraposition)
2. $\sim \alpha \wedge \sim \beta \vdash \sim (\alpha \vee \beta)$ ($\vee$-linearity).
3. $\top \vdash \sim \bot$ (Nor).
4. $\alpha \wedge \beta \vdash \gamma / \alpha \wedge \sim \gamma \vdash \sim \beta$,
5. $\alpha \wedge \sim \alpha \vdash \bot$,
6. $\top \vdash \sim \alpha \vee \sim\sim \alpha$,

Definition 10 Let $\neg$ be a unary connective added to the language of $BDLL$. Then, for $\alpha, \beta \in \mathcal{F}_\neg$, $\mathcal{L_{DS}}$ denotes the logic $BDLL$ along with following rules and postulates.

1. $\alpha \vdash \beta / \neg \beta \vdash \neg \alpha$ (Contraposition)

2. $\neg(\alpha \wedge \beta) \vdash \neg\alpha \vee \neg\beta$ ($\wedge$-linearity).
3. $\neg\top \vdash \bot$
4. $\gamma \vdash \alpha \vee \beta / \neg\beta \vdash \alpha \vee \neg\gamma$.
5. $\top \vdash \alpha \vee \neg\alpha$.
6. $\neg\alpha \wedge \neg\neg\alpha \vdash \bot$.

Now, we introduce the following classes of algebras.

$\mathcal{A}_S$:= class of all Stone algebras, $\mathcal{A}_{DS}$:= class of all dual Stone algebras, $\mathcal{S}B^{[2]}$: = class of all $\mathcal{B}^{[2]}_{\sim}$, $\mathcal{DS}B^{[2]}$:= class of all $\mathcal{B}^{[2]}_{\neg}$, $\mathcal{S}(\mathcal{P}(U)^{[2]}$: class of all Stone algebras formed by the collection $\mathcal{P}(U)^{[2]}$ for all sets U, $\mathcal{DS}(\mathcal{P}(U)^{[2]}$:= class of all dual Stone algebras formed by the collection $\mathcal{P}(U)^{[2]}$ for all sets U.

Now, utilizing Theorem 3 and Theorem 4, in the classical manner we get the results.

Theorem 5 *1. For $\alpha, \beta \in \mathcal{F}_{\sim}$, $\alpha \vdash_{\mathcal{L}_S} \beta$ if and only if $\alpha \vDash_{\mathcal{A}_S} \beta$ if and only if $\alpha \vDash_{\mathcal{S}B^{[2]}} \beta$ if and only if $\alpha \vDash_{\mathcal{S}(\mathcal{P}(U))^{[2]}} \beta$.*
2. For $\alpha, \beta \in \mathcal{F}_{\neg}$, $\alpha \vdash_{\mathcal{L}_{DS}} \beta$ if and only if $\alpha \vDash_{\mathcal{A}_{DS}} \beta$ if and only if $\alpha \vDash_{\mathcal{DS}B^{[2]}} \beta$ if and only if $\alpha \vDash_{\mathcal{DS}(\mathcal{P}(U))^{[2]}} \beta$.

Now, let us define the following semantic consequence relations.

Definition 11 1. Let $\alpha, \beta \in \mathcal{F}_{\sim}$.
(i) $\alpha \vDash^S_1 \beta$ if and only if, for all valuations v in $\mathbf{3}_{\sim}$ if $v(\alpha) = 1$ then $v(\beta) = 1$ (Truth preservation).
(ii) $\alpha \vDash^S_0 \beta$ if and only if, for all valuations v in $\mathbf{3}_{\sim}$ if $v(\beta) = 0$ then $v(\alpha) = 0$ (Falsity preservation).
(iii) $\alpha \vDash^S_{1,0} \beta$ if and only if, $\alpha \vDash^S_1 \beta$ *and* $\alpha \vDash^S_0 \beta$.
2. Let $\alpha, \beta \in \mathcal{F}_{\neg}$.
(i) $\alpha \vDash^{DS}_1 \beta$ if and only if, for all valuations v in $\mathbf{3}_{\neg}$ if $v(\alpha) = 1$ then $v(\beta) = 1$ (Truth preservation).
(ii) $\alpha \vDash^{DS}_0 \beta$ if and only if, for all valuations v in $\mathbf{3}_{\neg}$ if $v(\beta) = 0$ then $v(\alpha) = 0$ (Falsity preservation).
(iii) $\alpha \vDash^{DS}_{1,0} \beta$ if and only if, $\alpha \vDash^{DS}_1 \beta$ *and* $\alpha \vDash^{DS}_0 \beta$.

Proposition 3 *[1] Let $\mathcal{S} = (S, \vee, \wedge, \sim, 0, 1)$ and $\mathcal{DS} = (DS, \vee, \wedge, \neg, 0, 1)$ be Stone and dual Stone algebra, respectively. Then for $a, b \in S$ and $x, y \in DS$*
(i) $\sim\sim (a \vee b) = \sim\sim a \vee \sim\sim b$ and $\sim\sim (a \wedge b) = \sim\sim a \wedge \sim\sim b$.
(ii) $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$ and $\neg\neg(x \vee y) = \neg\neg x \vee \sim\sim y$.

Lemma 2 *1. For $\alpha, \beta \in \mathcal{F}_{\sim}$, if $\alpha \vDash^S_1 \beta$ then $\alpha \vDash^S_0 \beta$.*
2. For $\alpha, \beta \in \mathcal{F}_{\neg}$, if $\alpha \vDash^{DS}_0 \beta$ then $\alpha \vDash^{DS}_1 \beta$.

Proof 1. Let $\alpha \vDash^S_1 \beta$, and v be a valuation in $\mathbf{3}_{\sim}$ such that $v(\beta) = 0$. As $\alpha \vDash^S_1 \beta$, so $v(\alpha) \neq 1$. If $v(\alpha) = 0$, then our work is done. So, assume that $v(\alpha) = a$. Define a map $v^* : \mathcal{F}_{\sim} \to \mathbf{3}_{\sim}$ as:

$$v^*(\gamma) = \sim\sim v(\gamma).$$

Let us show that v^* is indeed a valuation in $\mathbf{3}_{\sim}$. For this, we have to show that $v^*(\gamma \wedge \delta) = v^*(\gamma) \wedge v^*(\delta)$, $v^*(\gamma \vee \delta) = v^*(\gamma) \vee v^*(\delta)$, $v^*(\sim \gamma) = \sim v^*(\gamma)$, $v^*(\bot) = 0$ and $v^*(\top) = 1$, but this follows immediately from Proposition 3.
Hence, v^* is a valuation and $v^*(\alpha) = 1$ and $v^*(\beta) = 0$ but this contradicts to the fact that $\alpha \vDash^S_1 \beta$. So, $\alpha \vDash^S_1 \beta$ implies $\alpha \vDash^S_0 \beta$.
2. Now, let $\alpha \vDash^{DS}_0 \beta$, and v be a valuation in $\mathbf{3}_{\neg}$ such that $v(\alpha) = 1$. As $\alpha \vDash^{DS}_0 \beta$, so $v(\beta) \neq 0$. If $v(\beta) = 1$, then our work is done. So, assume that $v(\beta) = a$. In a similar fashion as in the previous case, define a map $v^* : \mathcal{F}_{\neg} \to \mathbf{3}_{\neg}$ as:

$$v^*(\gamma) = \neg\neg v(\gamma).$$

Similar to the previous case, using Proposition 3, we can easily establish that v^* is indeed a valuation in $\mathbf{3}_{\neg}$. This arises a contradiction to $\alpha \vDash^{DS}_0 \beta$.

□

Note that converse of the above statements are not true, for example $\sim\sim \alpha \vDash^S_0 \alpha$ but $\sim\sim \alpha \nvDash^S_1 \alpha$ and $\beta \nvDash^{DS}_0 \neg\neg\beta$ but $\beta \vDash^{DS}_1 \neg\neg\beta$. This is contrary to the Dunn's "De Morgan consequence relations $\vDash_0$, $\vDash_1$ and $\vDash_{0,1}$" where all these three turn out be equivalent.

Theorem 6 *1. $\alpha \vDash_{\mathcal{SP}(U))^{[2]}} \beta$ if and only if $\alpha \vDash^S_1 \beta$, for any $\alpha, \beta \in \mathcal{F}_{\sim}$.*
2. $\alpha \vDash_{\mathcal{DSP}(U)^{[2]}} \beta$ if and only if $\alpha \vDash^{DS}_0 \beta$, for any $\alpha, \beta \in \mathcal{F}_{\neg}$.

Proof 1. Let $\alpha \vDash_{\mathcal{SP}(U))^{[2]}} \beta$, and $v : \mathcal{F}_{\sim} \to \mathbf{3}$ be a valuation. By Theorem 4, $\mathbf{3}_{\sim}$ is embedded to a Stone algebra of $\mathcal{P}(U)^{[2]}$ for some set U. If this embedding is denoted by ϕ, $\phi \circ v$ is a valuation in $\mathcal{P}(U)^{[2]}$. Then $(\phi \circ v)(\alpha) \leq (\phi \circ v)(\beta)$ implies $v(\alpha) \leq v(\beta)$. Thus if $v(\alpha) = 1$, we have $v(\beta) = 1$.
Now, let $\alpha \vDash^S_1 \beta$. Let U be a set, and $\mathcal{P}(U)^{[2]}$ be the corresponding Stone algebra. Let v be a valuation on $\mathcal{P}(U)^{[2]}$ – we need to show $v(\alpha) \leq v(\beta)$. For any $\gamma \in \mathcal{F}_{\sim}$ with $v(\gamma) := (A, B)$, $A, B \subseteq U$, and for each $x \in U$, define a map $v_x : \mathcal{F}_{\sim} \to \mathbf{3}_{\sim}$ as

$$v_x(\gamma) := \begin{cases} 1 \text{ if } x \in A \\ a \text{ if } x \in B \setminus A \\ 0 \text{ if } x \notin B. \end{cases}$$

Consider any $\gamma, \delta \in \mathcal{F}_{\sim}$, with $v(\gamma) := (A, B)$ and $v(\delta) := (C, D)$, $A, B, C, D \subseteq U$. It is easy to show that (for a complete proof, we refer to [2]), $v_x(\gamma \wedge \delta) = v_x(\gamma) \wedge v_x(\delta)$, $v_x(\gamma \vee \delta) = v_x(\gamma) \vee v_x(\delta)$. Let us show the following: $v_x(\sim \gamma) = \sim v_x(\gamma)$.
Note that $v(\sim \gamma) = (B^c, B^c)$.
<u>Case 1</u> $v_x(\gamma) = 1$: Then $x \in A$, i.e. $x \notin A^c$ and so $x \notin B^c$. Hence, $v_x(\sim \gamma) = 0 = \sim v_x(\gamma)$.
<u>Case 2</u> $v_x(\gamma) = a$: $x \notin A$ but $x \in B$, so $x \notin B^c$. Hence, $v_x(\sim \gamma) = 0 = \sim v_x(\gamma)$.

Case 3 $v_x(\gamma) = 0$: $x \notin B$, i.e. $x \in B^c$. So $v_x(\sim \gamma) = 1 = \sim v_x(\gamma)$.
Hence, v_x is a valuation in $\mathbf{3}_\sim$. Now let us show that $v(\alpha) \leq v(\beta)$. Let $v(\alpha) := (A', B')$, $v(\beta) := (C', D')$, and $x \in A'$. Then $v_x(\alpha) = 1$, and as $\alpha \vDash_1^S \beta$, by definition, $v_x(\beta) = 1$. This implies $x \in C'$, whence $A' \subseteq C'$.
On the other hand, if $x \notin D'$, $v_x(\beta) = 0$. Then using Lemma 2, we have $v_x(\alpha) = 0$, so that $x \notin B'$, giving $B' \subseteq D'$.

2. We prove second part only. For this let $\alpha \vDash_0^{DS} \beta$. Let U be a set, and $\mathcal{P}(U)^{[2]}$ be the corresponding dual Stone algebra. Let v be a valuation on $\mathcal{P}(U)^{[2]}$, we show that $v(\alpha) \leq v(\beta)$. Very similar to the previous case, for any $\gamma \in \mathcal{F}_\neg$ with $v(\gamma) := (A, B)$, $A, B \subseteq U$, and for each $x \in U$, define a map $v_x : \mathcal{F}_\neg \to \mathbf{3}_\neg$ as
$$v_x(\gamma) := \begin{cases} 1 \text{ if } x \in A \\ a \text{ if } x \in B \setminus A \\ 0 \text{ if } x \notin B. \end{cases}$$
Consider any $\gamma, \delta \in \mathcal{F}_\neg$, with $v(\gamma) := (A, B)$ and $v(\delta) := (C, D)$, $A, B, C, D \subseteq U$. Let us show the following: $v_x(\neg\gamma) = \neg v_x(\gamma)$.
Note that $v(\neg\gamma) = (A^c, A^c)$.
Case 1 $v_x(\gamma) = 1$: Then $x \in A$, i.e. $x \notin A^c$. Hence, $v_x(\neg\gamma) = 0 = \neg v_x(\gamma)$.
Case 2 $v_x(\gamma) = a$: $x \notin A$ but $x \in B$, so $x \in A^c$. Hence, $v_x(\neg\gamma) = 0 = \neg v_x(\gamma)$.
Case 3 $v_x(\gamma) = 0$: $x \notin B$, and so $x \notin A$, i.e. $x \in A^c$. So $v_x(\neg\gamma) = 1 = \neg v_x(\gamma)$.
Hence, v_x is a valuation in $\mathbf{3}_\neg$. To complete the proof let us show that $v(\alpha) \leq v(\beta)$. Let $v(\alpha) := (A', B')$, $v(\beta) := (C', D')$, and $x \in A'$. Then $v_x(\alpha) = 1$, and as $\alpha \vDash_1^{DS} \beta$, by Lemma 2, $v_x(\beta) = 1$. This implies $x \in C'$, whence $A' \subseteq C'$.
On the other hand, if $x \notin D'$, $v_x(\beta) = 0$. Then by our assumption $\alpha \vDash_0^{DS} \beta$, we have $v_x(\alpha) = 0$, so that $x \notin B'$, giving $B' \subseteq D'$.

$\square$

Finally, we have the following 3-valued semantics of the logics $\mathcal{L_S}$ and $\mathcal{L_{DS}}$.

Theorem 7 *(3-valued semantics) For $\alpha, \beta \in \mathcal{F}_\sim$ and $\alpha', \beta' \in \mathcal{F}_\neg$*

1. *$\alpha \vdash_{\mathcal{L_S}} \beta$ if and only if $\alpha \vDash_1^S \beta$.*
2. *$\alpha' \vdash_{\mathcal{L_{DS}}} \beta'$ if and only if $\alpha' \vDash_0^{DS} \beta'$.*

4 Rough set models for 3-valued logics

For an approximation space (U, R), $\mathcal{RS} \subseteq \mathcal{P}(U) \times \mathcal{P}(U)$. So $\mathcal{RS}$ has a natural ordering $' \leq '$ (inherited from $\mathcal{P}(U) \times \mathcal{P}(U)$).

In [26], J. Pomykała and J.A. Pomykała showed that $(\mathcal{RS}, \leq)$ is a Stone algebra. In [27], M. Gehrke and E. Walker characterized the lattice structure of rough sets. They showed that $(\mathcal{RS}, \leq) \cong \mathbf{2}^I \times \mathbf{3}^J$ for some appropriate index sets I and J. In [28], S.D. Comer proved that for any index sets I and J, there is an approximation space (U, R) such that the lattices $\mathbf{2}^I \times \mathbf{3}^J$ and $\mathcal{RS}$ are isomorphic. Hence, any Stone (dual Stone) algebra is embeddable into Stone (dual Stone) algebra formed by rough sets. Alternatively, we can also prove this assertion by using Theorem 4.

An easy consequence we get the following rough set semantic for the logic $\mathcal{L_S}$ ($\mathcal{L_{DS}}$).

Theorem 8 *1. Let $\mathcal{A_{SRS}}$ denote the class of all Stone algebras formed by $\mathcal{RS}$. Then we have for $\alpha, \beta \in \mathcal{F}_\sim$: $\alpha \vdash_{\mathcal{L_S}} \beta$ if and only if $\alpha \vDash_{\mathcal{A_{SRS}}} \beta$.*
2. Let $\mathcal{A_{DSRS}}$ denote the class of all dual Stone algebras formed by $\mathcal{RS}$. Then we have for $\alpha, \beta \in \mathcal{F}_\neg$: $\alpha \vdash_{\mathcal{L_{DS}}} \beta$ if and only if $\alpha \vDash_{\mathcal{A_{DSRS}}} \beta$.

In [13], the interpretations 1, 2 and 3 have been captured through the logic compatible with Kleene algebras. Now, we follow the same approach to capture the interpretations 1, 2 and 3 through the logics $\mathcal{L_S}$ and $\mathcal{L_{DS}}$. Let us define the following semantic consequence relations.

Definition 12 1. Let α be a formula in $\mathcal{F}_\sim$ and v be a valuation in $\mathcal{RS}_\sim$ for some approximation space (U, R) such that $v(\alpha) := (\mathsf{L}A, \mathsf{U}A)$, $A \subseteq U$. Then for $x \in U$,
$v, x \vDash_1^{\mathcal{RS}_\sim} \alpha$ if and only if $x \in \mathsf{L}A$.
$v, x \vDash_0^{\mathcal{RS}_\sim} \alpha$ if and only if $x \notin \mathsf{U}A$.
$v, x \vDash_u^{\mathcal{RS}_\sim} \alpha$ if and only if $x \notin \mathsf{L}A, x \in \mathsf{U}A$.
2. Let α be a formula in $\mathcal{F}_\neg$ and v be a valuation in $\mathcal{RS}_\neg$ for some approximation space (U, R) such that $v(\alpha) := (\mathsf{L}A, \mathsf{U}A)$, $A \subseteq U$. Then for $x \in U$,
$v, x \vDash_1^{\mathcal{RS}_\neg} \alpha$ if and only if $x \in \mathsf{L}A$.
$v, x \vDash_0^{\mathcal{RS}_\neg} \alpha$ if and only if $x \notin \mathsf{U}A$.
$v, x \vDash_u^{\mathcal{RS}_\neg} \alpha$ if and only if $x \notin \mathsf{L}A, x \in \mathsf{U}A$.

The relation $v, x \vDash_1^{\mathcal{RS}_\sim} \alpha$ can be interpreted as α is certainly true at x under the valuation v in approximation space (U, R), hence captures the interpretation 1. $v, x \vDash_0^{\mathcal{RS}_\sim} \alpha$ captures the interpretation 2 and can be interpreted as α is certainly false at x under the valuation v in approximation space (U, R). Finally, $v, x \vDash_u^{\mathcal{RS}_\sim} \alpha$ captures the interpretation 3.

Now, let us define the notions of validity.

Definition 13 Let $\alpha, \beta \in \mathcal{F}_\sim$ and $\gamma, \delta \in \mathcal{F}_\neg$,

1. $\alpha \vDash_1^{\mathcal{RS}_\sim} \beta$ if and only if $v, x \vDash_1^{\mathcal{RS}_\sim} \alpha$ implies $v, x \vDash_1^{\mathcal{RS}_\sim} \beta$, for all valuations v in $\mathcal{RS}_\sim$ and $x \in U$.
$\alpha \vDash_0^{\mathcal{RS}_\sim} \beta$ if and only if $v, x \vDash_0^{\mathcal{RS}_\sim} \beta$ implies $v, x \vDash_0^{\mathcal{RS}_\sim} \alpha$, for all valuations v in $\mathcal{RS}_\sim$ and $x \in U$.
$\alpha \vDash_{1,0}^{\mathcal{RS}_\sim} \beta$ if and only if $\alpha \vDash_1^{\mathcal{RS}_\sim} \beta$ and $\alpha \vDash_0^{\mathcal{RS}_\sim} \beta$.

2. $\gamma \models_1^{\mathcal{RS}_\neg} \delta$ if and only if $v', y \models_1^{\mathcal{RS}_\neg} \gamma$ implies $v', y \models_1^{\mathcal{RS}_\neg} \delta$, for all valuations v' in $\mathcal{RS}_\neg$ and $y \in U'$.
   $\gamma \models_0^{\mathcal{RS}_\neg} \delta$ if and only if $v', y \models_0^{\mathcal{RS}_\neg} \delta$ implies $v', y \models_0^{\mathcal{RS}_\neg} \gamma$, for all valuations v' in $\mathcal{RS}_\neg$ and $y \in U'$.
   $\gamma \models_{1,0}^{\mathcal{RS}_\neg} \delta$ if and only if $\gamma \models_1^{\mathcal{RS}_\neg} \delta$ and $\gamma \models_0^{\mathcal{RS}_\neg} \delta$.

Now we link the syntax and semantics through following Definition and Theorem.

Definition 14 1. Let $\alpha, \beta \in \mathcal{F}_\sim$ and $\alpha \vdash \beta$ be a consequent.
   - $\alpha \vdash \beta$ is valid in an approximation space (U, R), if and only if $\alpha \models_1^{\mathcal{RS}_\sim} \beta$.
   - $\alpha \vdash \beta$ is valid in a class $\mathbb{F}$ of approximation spaces if and only if $\alpha \vdash \beta$ is valid in all approximation spaces $(U, R) \in \mathbb{F}$.
2. Let $\alpha, \beta \in \mathcal{F}_\neg$ and $\alpha \vdash \beta$ be a consequent.
   - $\alpha \vdash \beta$ is valid in an approximation space (U, R), if and only if $\alpha \models_0^{\mathcal{RS}_\neg} \beta$.
   - $\alpha \vdash \beta$ is valid in a class $\mathbb{F}$ of approximation spaces if and only if $\alpha \vdash \beta$ is valid in all approximation spaces $(U, R) \in \mathbb{F}$.

Theorem 9 *Let $\alpha, \beta \in \mathcal{F}_\sim$ and $\gamma, \delta \in \mathcal{F}_\neg$. Then*

1. $\alpha \models_{\mathcal{A}_{S\mathcal{RS}}} \beta$ if and only if $\alpha \vdash \beta$ is valid in the class of all approximation spaces.
2. $\gamma \models_{\mathcal{A}_{DS\mathcal{RS}}} \delta$ if and only if $\gamma \vdash \delta$ is valid in the class of all approximation spaces.

Proof 1. Let $\alpha \models_{\mathcal{A}_{S\mathcal{RS}}} \beta$. Let (U, R) be an approximation space, and v be a valuation in $\mathcal{RS}_\sim$ with $v(\alpha) := (\mathsf{L}A, \mathsf{U}A)$ and $v(\beta) := (\mathsf{L}B, \mathsf{U}B)$, $A, B \subseteq U$. By the assumption, $\mathsf{L}A \subseteq \mathsf{L}B$ and $\mathsf{U}A \subseteq \mathsf{U}B$. Now, let us show that $\alpha \models_1^{\mathcal{RS}_\sim} \beta$. So, let $x \in U$ and $v, x \models_1^{\mathcal{RS}_\sim} \alpha$, i.e., $x \in \mathsf{L}A$. But we have $\mathsf{L}A \subseteq \mathsf{L}B$, hence $v, x \models_1^{\mathcal{RS}_\sim} \beta$.
   Now, suppose $\alpha \vdash \beta$ is valid in the class of all approximation spaces. We want to show that $\alpha \models_{\mathcal{A}_{S\mathcal{RS}}} \beta$. Let v be a valuation in $\mathcal{RS}_\sim$ as taken above. We have to show that $\mathsf{L}A \subseteq \mathsf{L}B$ and $\mathsf{U}A \subseteq \mathsf{U}B$. Let $x \in \mathsf{L}A$, i.e., $v, x \models_1^{\mathcal{RS}_\sim} \alpha$. Hence, by our assumption, $v, x \models_1^{\mathcal{RS}} \beta$, i.e., $x \in \mathsf{L}B$. So $\mathsf{L}A \subseteq \mathsf{L}B$. Now, let $y \notin \mathsf{U}B$, using Lemma 2, we have $v, y \models_0^{\mathcal{RS}_\sim} \beta$. By our assumption, $v, y \models_0^{\mathcal{RS}_\sim} \alpha$, i.e., $y \notin \mathsf{U}A$.
2. The proof of this part is very similar to that of part 1 which uses lemma 2.

□

5 Conclusions

This paper presents a relationship between Stone algebras, rough sets and 3-valued logics. We have drawn a line parallel to the line of *Boolean algebra - 2-valued Boolean algebra - Stone's representation theorem - classical propositional logic*. We have shown that the logic $\mathcal{L}_\mathcal{S}$ ($\mathcal{L}_{\mathcal{DS}}$) is truly a 3-valued logic via a 3-valued semantics. Further this 3-valued semantics of the logic $\mathcal{L}_\mathcal{S}$ can be interpreted in rough set theory, where the third value can be treated as not certain but possible.

In [29], Kumar and Banerjee analyzed the Stone and dual Stone negations in perp frames [25,30,31], where negations are viewed as modal operators. We introduced Stone and dual Stone frames and showed that the logics $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_{\mathcal{DS}}$ are sound and complete respectively in these classes of frames. Thus, the perp semantics of the logics $\mathcal{L}_\mathcal{S}$ and $\mathcal{L}_{\mathcal{DS}}$ are established. Hence, in view of Theorems 5, 6, 7, 8 and 9 we can conclude that algebraic, 3-valued, rough set and perp semantics of the logic $\mathcal{L}_\mathcal{S}$ ($\mathcal{L}_{\mathcal{DS}}$) are all equivalent.

In future, we would like to discuss the following.

1. In Düntsch and Orłowska [32], discrete duality for Stone algebras have been obtained. So naturally it would be interesting to investigate the relationship between the frame defined there, the logic $\mathcal{L}_\mathcal{S}$, $3-$ *valued* consequence relation $\models_1^S$ and the Stone frames defined in [29].
2. There has been a lot of study on Topological Boolean algebras (TBAs). Similarly, can we define Topological Stone algebras? Can we obtain representation results of these Topological Stone algebras in terms of $B^{[2]}$ and $\mathcal{RS}$?
3. Hilbert style axiomatization of the logic of Stone algebras.
4. In [33], Zhou and Zhao studied the Stone-like representation theorems of 3-valued Łukasiewicz algebras. Naturally, it would be interesting to investigate the Stone-like representation theorems for the class of Stone algebras determined by rough sets.
5. Applications of the logic $\mathcal{L}_\mathcal{S}$ in approximate reasoning.

Compliance with ethical standards

Conflict of interest Author Arun Kumar declares that he is no conflict of interest. Author Shilpi Kumari declares that she has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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Figures

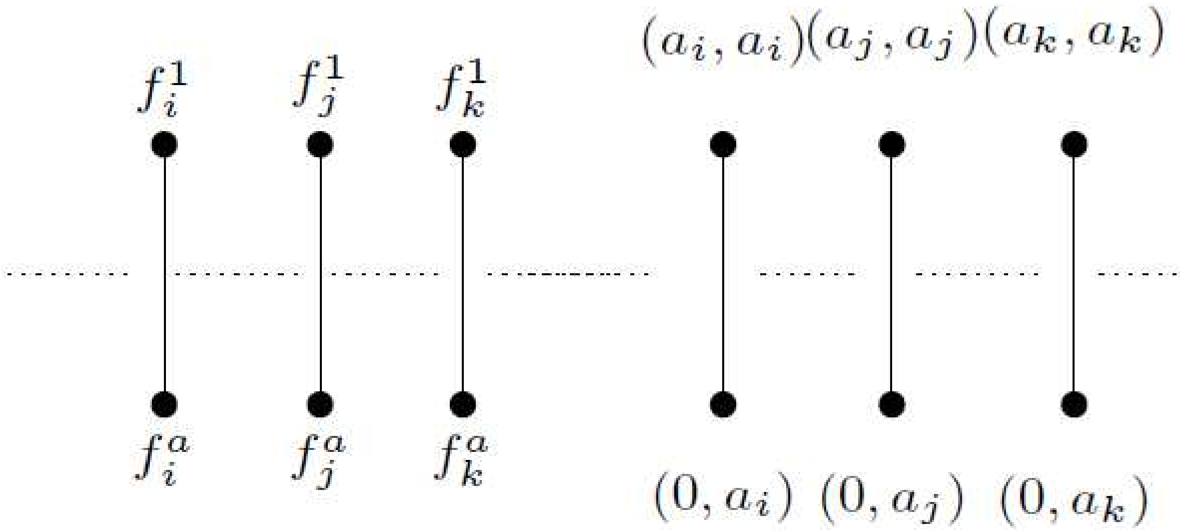

Figure 1

Hasse diagram of J3I