Cluster Synchronization in a Network of Nonlinear Systems with Directed Topology and Competitive Relationships

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Cluster synchronization in a network of nonlinear systems with directed topology and competitive relationships

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Abstract This paper studies the cluster synchronization problem of coupled nonlinear systems with directed topology and competitive relationships. We assume that nodes within the same cluster have the same intrinsic dynamics, whereas node dynamics between different clusters differ. In the same cluster, there only exist cooperative relationships, and there may have competitive relationships among nodes belonging to different clusters. Under the assumptions that each node satisfies one-sided Lipschitz condition, and the digraphs of each cluster are strongly connected, some sufficient conditions for cluster synchronization in the cases of linear coupling and nonlinear coupling are obtained respectively. The obtained conditions are presented as some algebraic conditions which are easy to solve. Finally, our results are validated by two numerical simulations.

Keywords Cluster synchronization · nonlinear systems · one-sided Lipschitz · competitive interactions

1 Introduction

The synchronization (consensus) phenomenon often exists in biology, ecology, engineering, physics and social sciences. Due to the influence of different structures within the network, the phenomenon of complete synchronization is relatively rare in the real world, and cluster (group) synchronization is more general. Thus, in the recent years, the cluster (group) synchronization problem has drawn significant attention from researchers due to its more accurate depiction of the real network [1–5].

For a network in real world, competition and cooperation often exist among different nodes [6–11]. Signed graph is an appropriate tool to describe competition and cooperation in a network, where the edges of positive and negative weights indicate

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that the relationship between the nodes is cooperative and competitive, respectively. If the nodes within the same cluster are cooperative and the nodes between different clusters are competitive (structurally balance), then the network will show some new synchronization phenomena, such as bipartite synchronization [9, 12–14], modulus synchronization [15], interval bipartite consensus [16].When cooperation and competition may exist simultaneously between different clusters (structurally unbalanced), the network will show more collective behaviours, and the cluster synchronization is a common occurrence in this setting. [17, 18].

To enable the system realize cluster synchronization, the pinning control is often used [1–5]. In essence, pinning control can be thought of as adding a virtual leader to each cluster to drive the node states of each cluster onto the same trajectory. In [3], by using pinning control technology, the cluster synchronization for fully-state coupled identical oscillator is considered, and a scheme of adaptively adjusting the coupling is presented. In [4], cluster synchronization in linear systems with partial state coupling is considered via pinning control. Similar method was extended for general linear and nonlinear network with external disturbance [2]. The problem of leaderless cluster synchronization is studied in [19–21]. For the cluster synchronization problem under a leaderless framework, one necessity is that there is a invariant cluster synchronization manifold in the network. Based on this, the authors in [22,23] propose the cluster-input-equivalence condition that ensure the invariance of cluster synchronous manifold. By considering the leaderless cluster synchronization problem, the authors the authors in [19] established the group synchronization results for heterogeneous systems with linear and nonlinear coupling, respectively, and the authors provided some corresponding algebraic conditions. In [2–4, 19, 21], however, all theoretical analysis are based on a particular hypothesis that the sum of the input degrees of other cluster nodes received by each cluster node is zero (in-degree balanced condition), which is a simple case of the cluster-input-equivalence condition.

Motivated by above observation, this article aims to establish the results of cluster synchronization analysis for coupling network with competitive relationships and directed topology in a leaderless framework. The nodes between different clusters have different nonlinear dynamics, and the dynamics of all nodes satisfy the one-sided Lipshitz condition. The contribution of this article can be summarized as follows:

1) For the network including competitive relationships and directed topology, we will further relax the in-degree balanced condition in [2–4, 19, 21] to make the cluster synchronous manifold hold.

2) In general, the calculation of generalized algebraic connectivity is extremely difficult [19]. We will present an algebraic condition which is easier to calculate than the result in [19].

3) We also consider the cluster synchronization for the nonlinearly coupled nonlinear systems with directed topology which is an extension of the work in [19].

4) The authors in [1, 19] just considered the case that the nonlinear systems satisfy the Lipschitz condition. We will investigate the cluster synchronization problem based on the premise that it is more general and challenging one-sided Lipschitz condition and quadratic inner-boundedness condition.

The rest of this paper is organized as follows. Section 2 presents the signed graph theory and describes the system model. Section 3 shows some theoretical results about cluster synchronization for linear coupling. Section 4 provides the results about cluster synchronization for nonlinear coupling. Section 5 gives two numerical simulations results about neural networks. Section 6 makes some summaries.

2 Preliminaries and problem statement

2.1 Notations

The following notations are used throughout this article. Let $\|y\|$ stands for the norm of vector y. $\otimes$ denotes the Kronecker product. I_n denotes the identity matrix of dimension n. Let $\mathbf{1}_n = [1,1,\cdots,1]^\top \in \mathbb{R}^n$. $\Re(\beta)$ is the real part of the complex number β. $\lambda_{\max}(\Xi)$ and $\lambda_{\min}(\Xi)$ represent the maximum and minimum eigenvalues of matrix Ξ, respectively. For two square matrix E and A with appropriate dimensions, $H(E,A) = EA + A^\top E^\top$. diag$\{\Lambda_1,\cdots,\Lambda_n\}$ represents a diagonal matrix with diagonal terms of $\Lambda_1,\cdots,\Lambda_n$.

2.2 Signed graph

A weighted directed signed graph is represented by $\mathscr{G} = (\mathscr{V},\mathscr{E},\mathscr{A})$, where $\mathscr{V} = \{1,\cdots,N\}$, $\mathscr{E} \subseteq \mathscr{V}\times\mathscr{V}$ are the vertex set and the edge set, respectively, and $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ represents the weighted connectivity matrix of the digraph $\mathscr{G}$. $a_{ij} \neq 0$ if there is a interaction from vertex j to i , $a_{ij} = 0$ otherwise (assume $a_{ii} = 0$). When the vertex i and j are cooperative (competitive) interactions, then $a_{ij} > 0$ ($a_{ij} < 0$). The Laplacian matrix of $\mathscr{G}$ is represented by $L = [l_{ij}]$, where $l_{ij} = -a_{ij}, i \neq j$ and $l_{ii} = \sum_{j=1}^N a_{ij}$. A signed digraph is called strongly connected if every vertex is reachable from any other vertex. If $\sum_{j=1}^N a_{ij} = \sum_{j=1}^N a_{ji}$, then the signed digraph $\mathscr{G}$ is called weight balanced.

2.3 Problem statement

Consider the following coupling network with N nonidentical nonlinear systems

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^N c_{ij}a_{ij}\Gamma(x_j - x_i), \quad i = 1,2,\cdots,N \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state variable of ith node, Γ is the diffusion matrix. $f_i : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector function, which represents the nonlinear dynamics of node i. $c_{ij} > 0$ indicates the coupling weight between nodes i and j.

Suppose that the node $\mathscr{V} = \{1,2,\cdots,N\}$ is divided into K subsets $\mathscr{V}_k, k = 1,\cdots,K$ in which all clusters just include cooperative interactions and in every cluster the intrinsic dynamics are the same. The cluster $\mathscr{V}_k$ contains m_k nodes and $\sum_{i=1}^K m_i = N$ ($\bigcup_{k=1}^K \mathscr{V}_k = \mathscr{V}$). Without compromising generality, we assume that the cluster $\mathscr{V}_k, k = 1,\cdots,K$ contains nodes $z_{k-1}+1,\cdots,z_{k-1}+m_k$, where $z_0 = 0$, $z_k = \sum_{j=1}^k m_j$. Let $\hat{\imath}$

represent the subscript of the cluster to which the node i belongs, i.e., $i \in \mathscr{V}_{\hat{i}}$, $\hat{i} \in \{1, \cdots, K\}$.

The subdigraph of node $\mathscr{V}_i$ is represented by $\mathscr{G}_{\mathscr{V}_i} = (\mathscr{V}_i, \mathscr{E}_i)$, and $\mathscr{G}_{\mathscr{V}_i}$ describes the interaction within each cluster. The $\mathscr{E}_i$ represents the set of interactive edges within the cluster $\mathscr{V}_i$. Then

$$\mathscr{G}(\mathscr{A}) = \left(\bigcup_{i=1}^{K} \mathscr{G}_{\mathscr{V}_i} \right) \bigcup \bar{\mathscr{G}},$$

where $\bar{\mathscr{G}} = (\mathscr{V}, \mathscr{E} \backslash \bigcup_{i=1}^{K} \mathscr{E}_i)$ is the signed graph describing the interaction between different clusters. Let $L_{\mathscr{V}_i}$ denote the Laplacian matrix of $\mathscr{G}_{\mathscr{V}_i}$ and $\bar{L}$ denote the Laplacian matrix of $\bar{\mathscr{G}}$. Hence, the Laplacian matrix of $\mathscr{G}(\mathscr{A})$ can be written as $L = L_{\mathscr{V}} + \bar{L}$, where $L_{\mathscr{V}} = \text{diag}\left\{L_{\mathscr{V}_1}, \cdots, L_{\mathscr{V}_K}\right\}$.

Generally, similar to the linear coupling network (1), the corresponding nonlinear coupling network can be presented as

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{N} c_{ij} a_{ij} \psi(x_j - x_i), \quad i = 1, 2, \cdots, N \tag{2}$$

where $x_i \in \mathbb{R}^n$ is the state variable of ith node. $\psi : \mathbb{R}^n \to \mathbb{R}^n$, $\psi(0) = 0$ is a continuous nonlinear coupling function.

For the networks (1) and (2), we assume that $c_{ij} = c_k > 0$ when $\hat{i} = \hat{j} = k$, otherwise, $c_{ij} = 1$ when $\hat{i} \neq \hat{j}$. Moreover, $f_i = f_{\hat{i}}, \forall i \in \mathscr{V}_{\hat{i}}$. This paper focuses on analyzing under what conditions the coupling networks (1) and (2) can achieve cluster synchronization (see the Definition 1).

Definition 1 The coupling network (1) and (2) are said to reach cluster synchronization if for any initial states $x_i(0)$, there $\lim_{t\to\infty} \left\|x_i(t) - x_j(t)\right\| = 0, \forall \hat{i} = \hat{j}$, $i, j = 1, \cdots, N$.

Assumption 1 *Suppose that the coupled systems satisfy the cluster-input-equivalence condition, that is,*

$$\mathscr{R}_{\mathscr{V}_{k_1}\mathscr{V}_{k_2}} := \sum_{l \in \mathscr{V}_{k_2}} a_{il} = \sum_{l \in \mathscr{V}_{k_2}} a_{jl}, \ \forall i, j \in \mathscr{V}_{k_1},$$

where $k_1 \neq k_2 \in \{1, \cdots, K\}$.

According to references [19, 22, 23], for the networks (1) and (2), when the Assumption 1 holds, the following cluster manifold are invariant:

$$S = \{x \in \mathbb{R}^{nN} \mid x_1 = \cdots = x_{m_1}, \cdots, x_{z_{K-1}+1} = \cdots = x_N\}.$$

However, in [18, 19, 24, 25], $\mathscr{R}_{\mathscr{V}_{k_1}\mathscr{V}_{k_2}} = 0$ is required, which is a simple case. In this article, we consider the case that $\mathscr{R}_{\mathscr{V}_{k_1}\mathscr{V}_{k_2}}$ may be positive, negative or zero.

Assumption 2 *The nonlinear function* $f_k(x)$ *in (1) and (2) satisfies the one-sided Lipschitz condition, that is,* $\forall x_a, x_b \in \mathbb{R}^n$,

$$(x_a - x_b)^\top (f_k(x_a) - f_k(x_b)) \leq \eta_k \left\|x_a - x_b\right\|^2,$$

where $\eta_k \in \mathbb{R}$ *is called the one-sided Lipschitz constant.*

Assumption 3 *The nonlinear function $f_k(x)$ in (1) and (2) satisfies the quadratic inner-boundedness condition, that is, there exist $\sigma_k, \gamma_k \in \mathbb{R}$ such that $\forall x_a, x_b \in \mathbb{R}^n$,*

$$\begin{aligned}&(f_k(x_a) - f_k(x_b))^\top (f_k(x_a) - f_k(x_b)) \\ &\leq \sigma_k \|x_a - x_b\|^2 + \gamma_k (x_a - x_b)^\top (f_k(x_a) - f_k(x_b)).\end{aligned}$$

Remark 1 Note that one-sided Lipschitz constant may be positive, negative or zero. However, in Lipschitz condition the Lipschitz constant can only be positive. A function that satisfies the Lipschitz condition also satisfies the one-sided Lipschitz condition. But the reverse is not necessarily correct. Furthermore, if the function f_k satisfies the Lipschitz condition, then it is also quadratically inner-bounded with $\sigma_k > 0$ and $\gamma_k = 0$. Similarly, the converse is not true. In general, γ_k can choose any real number. The functions satisfying the Lipschitz condition are continuous. However, the function satisfying the one-sided Lipschitz condition may be discontinuous. In fact, recent studies have shown that any function satisfying the quadratic inner-boundedness constraint, is also Lipschitz continuous [26]. Therefore, the one-sided Lipschitz condition are more general and challenging.

2.4 Some lemmas for unsigned digraph

This subsection will recall some lemmas about the unsigned digraph in which all edges are positive. Note that $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$ are unsigned graphs, therefore, these lemmas can be applied to $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$.

Lemma 1 *[27] Consider a digraph $\mathscr{G}(\mathscr{A})$ to be strongly connected with Laplacian matrix L. Let $\boldsymbol{\xi} = [\xi_1, \cdots, \xi_n]^\top \in \mathbb{R}^{n\times 1}$ satisfy $\boldsymbol{\xi}^\top L = 0$. Then,*

1) there exists at least one $\boldsymbol{\xi} > 0$.

2) all eigenvalues of matrix $EL + L^T E$ are non-negative real numbers and can be represented as $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, where $E = \mathrm{diag}\{\xi_1, \cdots, \xi_n\} \succ 0$.

Lemma 2 *[27] Consider a vector $\varsigma \in \mathbb{R}^n$ with $\langle \varsigma, \mathbf{1} \rangle = 1$, and the variable vector x is orthogonal to the vector ς . Then*

$$\min_{\forall x \in \mathbb{R}^n, x\neq 0, \langle x, \varsigma \rangle = 0} = \frac{x^T[(EL + L^T E) \otimes I_n]x}{x^T x} \geq \frac{\mu_2}{n\|\varsigma\|^2}.$$

In this paper, if the digraphs $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$ are strongly connected, then let $\boldsymbol{\xi}_k = [\xi_{z_{k-1}+1}, \cdots, \xi_{z_{k-1}+m_k}]^\top \in \mathbb{R}^{m_k \times 1}$ be the positive left eigenvector of $L_{\mathscr{V}_k}$ corresponding to eigenvalue 0. Further, assume that $\sum_{i=1}^{m_k} \xi_{z_{k-1}+i} = 1$. Let $E_k = \mathrm{diag}\{\xi_{z_{k-1}+1}, \cdots, \xi_{z_{k-1}+m_k}\}$ and $E = \mathrm{diag}\{E_1, \cdots, E_k\}$. Thus, the eigenvalues of $H(E_k, L_{\mathscr{V}_k})$ can be represented as $0 = \mu_{k1} \leq \mu_{k2} \leq \cdots \leq \mu_{km_k}$.

3 Cluster synchronization for linearly coupled nonlinear systems

In this part, we concentrates on the cluster synchronization analysis of coupling network (1). Suppose that the diffusion matrix Γ is diagonalizable, that is, there exists an invertible matrix P such that $P\Gamma P^{-1} = \Phi := \text{diag}\{\beta_1, \cdots, \beta_n\}$, where $\beta_1, \cdots, \beta_n$ are eigenvalues of Γ. Let $\beta_{\min} = \min_{i=1,\cdots,n} \Re(\beta_i)$, $\beta_{\max} = \max_{i=1,\cdots,n} \Re(\beta_i)$ and $\tilde{L}_{\mathscr{V}} = \text{diag}\left\{c_1 L_{\mathscr{V}_1}, \cdots, c_K L_{\mathscr{V}_K}\right\}$. In this paper, we assume $\beta_{\min} > 0$.

Theorem 1 *Consider the coupling network (1) with Assumptions 1-3. If all digraphs $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$ are strongly connected, and the following conditions hold:*

1) there exists positive constant θ_k satisfying

$$\gamma_k + 2\theta_k \geq 0, \tag{3}$$

$$\tilde{\lambda}(Q)(\theta_k^2 - 1) \leq \theta_k^2, \tag{4}$$

2) for cluster $\mathscr{V}_k, k = 1, \cdots, K$ the intra-cluster coupling strength c_k satisfies

$$c_k > \max\left\{ \frac{\left(\bar{\xi}_k\left(\tau_k \tilde{\lambda}(Q) - 1\right) - \zeta\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k \beta_{\min} \mu_{k2}}, \frac{\left(\underline{\xi}_k\left(\tau_k - \bar{\lambda}(Q)\right) - \zeta\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k \beta_{\min} \mu_{k2}}, 0 \right\}, \tag{5}$$

where

$$\zeta = 0.5(1 + sgn(\bar{\zeta}))\beta_{\min}\bar{\zeta} + 0.5(1 - sgn(\bar{\zeta}))\beta_{\max}\bar{\zeta},$$

$\bar{\xi}_k = \max \xi_{\mathscr{V}_k}$, $\underline{\xi}_k = \min \xi_{\mathscr{V}_k}$, $\xi_{\mathscr{V}_k} = \{\xi_{z_{k-1}+1}, \cdots, \xi_{z_{k-1}+m_k}\}$, $\bar{\zeta} = \lambda_{\min}(H(E, \bar{L}))$, $\tau_k = \sigma_k + 1 + \eta_k(2\theta_k + \gamma_k)$, $\tilde{\lambda}(Q) = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}$, $\bar{\lambda}(Q) = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}$, $Q = PP^\top$*, then the coupling network (1) achieves cluster synchronization.*

Proof Let $\bar{x} = \left[[\mathbf{1}_{m_1} \otimes \bar{x}_1]^\top \cdots [\mathbf{1}_{m_K} \otimes \bar{x}_K]^\top\right]^\top$, where $\bar{x}_k = \sum_{i=1}^{m_k} \xi_{z_{k-1}+i} x_{z_{k-1}+i}$. We then define:

$$e(t) = x(t) - \bar{x}(t),$$

where $e(t) = [e_{\mathscr{V}_1}^\top, \cdots, e_{\mathscr{V}_K}^\top]^\top$ and $e_{\mathscr{V}_k} = [e_{z_{k-1}+1}^\top, \cdots, e_{z_{k-1}+m_k}^\top]^\top = [(x_{z_{k-1}+1} - \bar{x}_k)^\top, \cdots, (x_{z_{k-1}+m_k} - \bar{x}_k)^\top]^\top$.

For $\forall i \in \mathscr{V}_k, k = 1, \cdots, K$, we have

$$\begin{aligned}
\dot{e}_i =& \dot{x}_i - \dot{\bar{x}}_{\hat{i}} \\
=& f_i(t, x_i) + \sum_{j=1}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i) - \sum_{j=z_{k-1}+1}^{z_{k-1}+m_k} \xi_j \left(\sum_{m=1}^{N} c_{jm} a_{jm} \Gamma(x_m - x_j) + f_{\hat{j}}(t, x_j) \right) \\
=& f_i(t, x_i) - f_{\hat{i}}(t, \bar{x}_{\hat{i}}) + \sum_{j=1}^{N} c_{ij} a_{ij} \Gamma(x_j - x_i) + \Delta(x),
\end{aligned}$$

where

$$\begin{aligned}
\Delta(x) = &- \sum_{j=z_{k-1}+1}^{z_{k-1}+m_k} \xi_j \left(\sum_{m=1}^{N} c_{jm} a_{jm} \Gamma(x_m - x_j) + f_{\hat{j}}(t, x_j) \right) \\
&+ f_{\hat{i}}(t, \bar{x}_{\hat{i}}).
\end{aligned}$$

Let $\hat{e}_i(t) = Pe_i(t)$. By the above equation about e_i, we can get that

$$\dot{\hat{e}}_i = P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] + \sum_{j=1}^{N} c_{ij}a_{ij}P\Gamma(x_j - x_i) + P\Delta(x). \tag{6}$$

Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^{N} \frac{1}{2}\xi_i \hat{e}_i^\top \hat{e}_i.$$

Then, calculate the time derivative of $V(t)$ along (6) is given by

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N} \xi_i \hat{e}_i^\top \dot{\hat{e}}_i \\ &= \sum_{i=1}^{N} \xi_i \hat{e}_i^\top P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] + \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i c_{ij}a_{ij}\hat{e}_i^\top P\Gamma(x_j - x_i) + \sum_{i=1}^{N} \xi_i \hat{e}_i^\top P\Delta(x).\end{aligned}$$

Not that $\sum_{i=z_{k-1}+1}^{z_{k-1}+m_k} \xi_i \hat{e}_i = P\sum_{i=z_{k-1}+1}^{z_{k-1}+m_k} \xi_i(x_i - \bar{x}_{\hat{i}}) = 0$, therefore, we have

$$\begin{aligned}\dot{V}(t) &= \sum_{i=1}^{N} \xi_i \hat{e}_i^\top P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] + \sum_{i=1}^{N}\sum_{j=1}^{N} \xi_i c_{ij}a_{ij}\hat{e}_i^\top P\Gamma P^{-1}P(e_j - e_i) \\ &= \sum_{i=1}^{N} \xi_i \hat{e}_i^\top P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] - \frac{1}{2}\hat{e}^\top[H(E,\tilde{L}_{\mathscr{V}}) \otimes \Phi]\hat{e} - \frac{1}{2}\hat{e}^\top[H(E,\bar{L}) \otimes \Phi]\hat{e}.\end{aligned}$$

Let $\bar{f}_k = f_k(t,x_i) - f_k(t,\bar{x}_k)$. For $\forall i \in \mathscr{V}_k, k = 1,\cdots,K$, there exists positive constant θ_k that makes the following inequality holds

$$\begin{aligned}e_i^\top Q\bar{f}_k =& \frac{1}{2\theta_k}(e_i + \theta_k\bar{f}_k)^\top Q(e_i + \theta_k\bar{f}_k) - \frac{1}{2\theta_k}e_i^\top Qe_i - \frac{\theta_k}{2}\bar{f}_k^\top Q\bar{f}_k \\ \le& \frac{\lambda_{\max}(Q)}{2\theta_k}\left\|e_i + \theta_k\bar{f}_k\right\|^2 - \frac{\lambda_{\min}(Q)}{2\theta_k}(\|e_i\|^2 + \theta_k^2\left\|\bar{f}_k\right\|^2).\end{aligned} \tag{7}$$

Using the quadratical inner-bounded condition, we have

$$\left\|e_i + \theta_k\bar{f}_k\right\|^2 \le (\sigma_k + 1)\|e_i\|^2 + (2\theta_k + \gamma_k)e_i^\top\bar{f}_k + (\theta_k^2 - 1)\left\|\bar{f}_k\right\|^2. \tag{8}$$

Using one-side Lipschitz condition and $\gamma_k + 2\theta_k \ge 0$, we can obtain from (8) that

$$\left\|e_i + \theta_k\bar{f}_k\right\|^2 \le [\sigma_k + 1 + \eta_k(2\theta_k + \gamma_k)]\|e_i\|^2 + \left(\theta_k^2 - 1\right)\left\|\bar{f}_k\right\|^2. \tag{9}$$

Combining (7) and (9), one has

$$\begin{aligned}e_i^\top Q\bar{f}_k &\le \frac{1}{2\theta_k}[\tau_k\lambda_{\max}(Q) - \lambda_{\min}(Q)]\|e_i\|^2 + \frac{s(\theta_k)}{2\theta_k}\left\|\bar{f}_k\right\|^2 \\ &\le \frac{1}{2\theta_k}[\tau_k\lambda_{\max}(Q) - \lambda_{\min}(Q)]\|e_i\|^2.\end{aligned}$$

where $s(\theta_k) = \lambda_{\max}(Q)(\theta_k^2 - 1) - \lambda_{\min}(Q)\theta_k^2$, the last inequality is obtained by the observation condition (4).

Hence,

$$\begin{aligned}
&\sum_{i=1}^{N} \xi_i e_i^\top P^\top P[f_{\bar{i}}(t,x_i) - f_{\bar{i}}(t,\bar{x}_{\bar{i}})] \\
&= \sum_{k=1}^{K} \sum_{i=1}^{m_k} \xi_{z_{k-1}+i} e_{z_{k-1}+i}^\top P^\top P[f_k(t,x_{z_{k-1}+i}) - f_k(t,\bar{x}_k)] \\
&\leq \sum_{k=1}^{K} \sum_{i=1}^{m_k} \frac{\xi_{z_{k-1}+i}}{2\theta_k} [\tau_k \lambda_{\max}(Q) - \lambda_{\min}(Q)] \left\| e_{z_{k-1}+i} \right\|^2 .
\end{aligned} \tag{10}$$

If $\tau_k \lambda_{\max}(Q) - \lambda_{\min}(Q) \geq 0$, we have

$$\sum_{i=1}^{N} \xi_i e_i^\top P^\top P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] \leq \sum_{k=1}^{K} \bar{\xi}_k \frac{1}{2\theta_k} [\tau_k \frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)} - 1] \left\| \hat{e}_{\mathscr{V}_k} \right\|^2 . \tag{11}$$

If $\tau_k \lambda_{\max}(Q) - \lambda_{\min}(Q) < 0$, we have

$$\sum_{i=1}^{N} \xi_i e_i^\top P^\top P[f_{\hat{i}}(t,x_i) - f_{\hat{i}}(t,\bar{x}_{\hat{i}})] \leq \sum_{k=1}^{K} \underline{\xi}_k \frac{1}{2\theta_k} [\tau_k - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}] \left\| \hat{e}_{\mathscr{V}_k} \right\|^2 . \tag{12}$$

If the real matrix Γ has complex eigenvalue β_{i1}, then there must exist a conjugate eigenvalue β_{i2}. Consider that $H(E_k, L_{\mathscr{V}_k})$ is a real symmetric and positive semidefinite matrix, then, one can get

$$\hat{e}_{\mathscr{V}_k}^\top [H(E_k, L_{\mathscr{V}_k}) \otimes \Phi] \hat{e}_{\mathscr{V}_k} \geq \beta_{\min} \hat{e}_{\mathscr{V}_k}^\top [H(E_k, L_{\mathscr{V}_k}) \otimes I_n] \hat{e}_{\mathscr{V}_k} .$$

Since $\sum_{i=1}^{m_k} \xi_{ki} \hat{e}_{z_{k-1}+i} = 0$ and $\sum_{i=1}^{m_k} \xi_{ki} = 1$, then one can obtain from Lemma 2 that

$$\hat{e}_{\mathscr{V}_k}^\top [H(E_k, L_{\mathscr{V}_k}) \otimes I_n] \hat{e}_{\mathscr{V}_k} \geq \frac{\mu_{k2}}{m_k \left\| \boldsymbol{\xi}_k \right\|^2} \left\| \hat{e}_{\mathscr{V}_k} \right\|^2 . \tag{13}$$

Hence,

$$\begin{aligned}
\hat{e}^\top [H(E, \tilde{L}_{\mathscr{V}}) \otimes \Phi] \hat{e} &= \sum_{k=1}^{K} c_k \hat{e}_{\mathscr{V}_k}^\top [H(E_k, L_{\mathscr{V}_k}) \otimes \Phi] \hat{e}_{\mathscr{V}_k} \\
&\geq \beta_{\min} \sum_{k=1}^{K} c_k \hat{e}_{\mathscr{V}_k}^\top [H(E_k, L_{\mathscr{V}_k}) \otimes I_n] \hat{e}_{\mathscr{V}_k} \\
&\geq \sum_{k=1}^{K} \frac{c_k \mu_{k2} \beta_{\min}}{m_k \left\| \boldsymbol{\xi}_k \right\|^2} \left\| \hat{e}_{\mathscr{V}_k} \right\|^2 .
\end{aligned} \tag{14}$$

In addition, let $\bar{\mu}_1$ be the smallest eigenvalue of $H(E, \bar{L})$, where $\bar{\mu}_1$ may be positive, negative or zero. Using the similar method as in (14), it is derived that

$$\begin{aligned}
\hat{e}^\top [H(E, \bar{L}) \otimes \Phi] \hat{e} &\geq \sum_{i=1}^{N} \zeta \left\| \hat{e}_i \right\|^2 \\
&= \zeta \sum_{k=1}^{K} \left\| \hat{e}_{\mathscr{V}_k} \right\|^2 .
\end{aligned} \tag{15}$$

If $\tau_k\lambda_{\max}(Q)-\lambda_{\min}(Q)\geq 0$, combining (11), (14) and (15), it then follows that

$$\begin{aligned}\dot{V}(t)\leq&\sum_{k=1}^{K}\frac{\bar{\xi}_k}{2\theta_k}\left[\tau_k\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}-1\right]\left\|\hat{e}_{\mathscr{V}_k}\right\|^2-\beta_{\min}\sum_{k=1}^{K}\frac{c_k\mu_{k2}}{2m_k\left\|\boldsymbol{\xi}_k\right\|^2}\left\|\hat{e}_{\mathscr{V}_k}\right\|^2\\&-\frac{\zeta}{2}\sum_{k=1}^{K}\left\|\hat{e}_{\mathscr{V}_k}\right\|^2,\end{aligned}\tag{16}$$

in this case,

$$\frac{\left(\bar{\xi}_k\left(\tau_k\tilde{\lambda}(Q)-1\right)-\zeta\theta_k\right)m_k\left\|\boldsymbol{\xi}_k\right\|^2}{\theta_k\beta_{\min}\mu_{k2}}>\frac{\left(\underline{\xi}_k\left(\tau_k-\bar{\lambda}(Q)\right)-\zeta\theta_k\right)m_k\left\|\boldsymbol{\xi}_k\right\|^2}{\theta_k\beta_{\min}\mu_{k2}}.$$

If $\tau_k\lambda_{\max}(Q)-\lambda_{\min}(Q)<0$, combining (12), (14) and (15), it then follows that

$$\begin{aligned}\dot{V}(t)\leq&\sum_{k=1}^{K}\frac{\underline{\xi}_k}{2\theta_k}\left[\tau_k-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(Q)}\right]\left\|\hat{e}_{\mathscr{V}_k}\right\|^2-\beta_{\min}\sum_{k=1}^{K}\frac{c_k\mu_{k2}}{2m_k\left\|\boldsymbol{\xi}_k\right\|^2}\left\|\hat{e}_{\mathscr{V}_k}\right\|^2\\&-\frac{\zeta}{2}\sum_{k=1}^{K}\left\|\hat{e}_{\mathscr{V}_k}\right\|^2,\end{aligned}\tag{17}$$

in this case,

$$\frac{\left(\bar{\xi}_k\left(\tau_k\tilde{\lambda}(Q)-1\right)-\zeta\theta_k\right)m_k\left\|\boldsymbol{\xi}_k\right\|^2}{\theta_k\beta_{\min}\mu_{k2}}<\frac{\left(\underline{\xi}_k\left(\tau_k-\bar{\lambda}(Q)\right)-\zeta\theta_k\right)m_k\left\|\boldsymbol{\xi}_k\right\|^2}{\theta_k\beta_{\min}\mu_{k2}}.$$

Hence, if the condition (5) holds, then $\hat{e}(t)\to 0$ as $t\to\infty$. Because $\hat{e}(t)=(I_N\otimes P^{-})e(t)$, hence, $e(t)\to 0$ as $t\to\infty$. That is, the coupled nonlinear systems (1) achieves cluster synchronization.

Remark 2 In [19], the condition $\mathscr{R}_{\mathscr{V}_{k_1}\mathscr{V}_{k_2}}=0$ is required, and the nonlinear function satisfies the Lipschitz condition. In the Theorem 1, the function f_i are one-sided Lipschitz. When the nonlinear dynamics of nodes satisfy the Lipschitz condition, then $\tau_k>1$. Hence, the result from Theorem 1 requires that the coupling strength must satisfy $\beta_{\min}\frac{c_k\mu_{k2}}{m_k\|\boldsymbol{\xi}_k\|^2}+\zeta>0$. Compared with the results in [19], the results of Theorem 1 do not require calculation of generalized algebraic connectivity. Thus, the Theorem 1 is easier to verify and solve, and the lower bound is less conservative.

In particular, we next consider that Γ is positive definite. Then, we can deduce the following result from Theorem 1:

Corollary 1 *Consider the coupling network (1) with Assumptions 1-3. If all digraphs $\mathscr{G}_{\mathscr{V}_k}, k=1,\cdots,K$ are strongly connected, and the following conditions hold:*
1) there exists positive constant θ_k satisfying

$$\gamma_k+2\theta_k\geq 0,\tag{18}$$

2) for cluster $\mathscr{V}_k, k=1,\cdots,K$ the intra-cluster coupling strength c_k satisfies

$$c_k > \max\left\{ \frac{\left(\bar{\xi}_k\tilde{\tau}_k - \zeta\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k\beta_{\min}\mu_{k2}}, \frac{\left(\underline{\xi}_k\tilde{\tau}_k - \zeta\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k\beta_{\min}\mu_{k2}}, 0 \right\}, \tag{19}$$

where

$$\zeta = 0.5(1+sgn(\bar{\zeta}))\beta_{\min}\bar{\zeta} + 0.5(1-sgn(\bar{\zeta}))\beta_{\max}\bar{\zeta},$$

$\bar{\xi}_k = \max \xi_{\mathscr{V}_k}$, $\underline{\xi}_k = \min \xi_{\mathscr{V}_k}$, $\xi_{\mathscr{V}_k} = \{\xi_{z_{k-1}+1}, \cdots, \xi_{z_{k-1}+m_k}\}$, $\bar{\zeta} = \lambda_{\min}(H(E,\bar{L}))$, $\tilde{\tau}_k = \sigma_k + \eta_k(2\theta_k + \gamma_k)$*, then the coupling network (1) achieves cluster synchronization.*

Proof When Γ is symmetrically positive definite matrix, then this implies that $P\Gamma P^{-1} = P\Gamma P^\top = \Phi$ and $Q = PP^T = I_n$. Thus, we have

$$\tau_k\tilde{\lambda}(Q) - 1 = \tau_k - \bar{\lambda}(Q) = \tau_k - 1 = \tilde{\tau}_k,$$

and

$$\tilde{\lambda}(Q)(\theta_k^2 - 1) = (\theta_k^2 - 1) < \theta_k^2.$$

Then the Corollary 1 is easy to obtain from Theorem 1.

When Γ is a real positive definite matrix and all subdigraphs $\mathscr{G}_{\mathscr{V}_k}, k=1,\cdots,K$ are weight balanced, we can get the following results:

Corollary 2 *Consider the coupling network (1) with Assumptions 1-3. If all digraphs $\mathscr{G}_{\mathscr{V}_k}, k=1,\cdots,K$ are strongly connected and weight balanced, and the following conditions hold:*

1) there exists positive constant θ_k satisfying

$$\gamma_k + 2\theta_k \geq 0, \tag{20}$$

2) for cluster $\mathscr{V}_k, k=1,\cdots,K$ the intra-cluster coupling strength c_k satisfies

$$c_k > \max\left\{ \frac{\tilde{\tau}_k - m_k\zeta\theta_k}{m_k\theta_k\beta_{\min}\mu_{k2}}, 0 \right\}, \tag{21}$$

where

$$\zeta = 0.5(1+sgn(\bar{\zeta}))\beta_{\min}\bar{\zeta} + 0.5(1-sgn(\bar{\zeta}))\beta_{\max}\bar{\zeta},$$

$\bar{\zeta} = \lambda_{\min}(H(E,\bar{L}))$, $\tilde{\tau}_k = \sigma_k + \eta_k(2\theta_k + \gamma_k)$*, then the coupling network (1) achieves cluster synchronization.*

Proof If subdigraphs $\mathscr{G}_{\mathscr{V}_k}, k=1,\cdots,K$ are all weight balanced, then $\boldsymbol{\xi}_k = \frac{1}{m_k}\mathbf{1}_{m_k}, k=1,\cdots,K$. Hence,

$$\begin{aligned} m_k\|\boldsymbol{\xi}_k\|^2 &= m_k\frac{1}{m_k} \\ &= 1, \end{aligned}$$

and $\bar{\xi}_k = \underline{\xi}_k = 1/m_k$. Then the Corollary 2 is easy to obtain from Corollary 1.

Remark 3 It follows from Theorem 1, Corollaries 1 and 2 that, when the interactive topology satisfies certain conditions, then the coupling network (1) can realize cluster synchronization as long as the coupling strength c_k is large enough.

4 Cluster synchronization for nonlinearly coupled nonlinear systems

We next focus on the cluster synchronization convergence analysis of nonlinear coupling network (2). First we provide an assumption about nonlinear function ψ.

Assumption 4 *The nonlinear function $\psi(\cdot)$ in network (2) satisfies the Lipschitz condition, and $\forall x_a, x_b \in \mathbb{R}^n$,*

1) there exist $\underline{\varepsilon}, \bar{\varepsilon} > 0$ such that

$$\underline{\varepsilon}(x_a - x_b)^\top (x_a - x_b) \le (x_a - x_b)^\top (\psi(a) - \psi(b)) \le \bar{\varepsilon}(x_a - x_b)^\top (x_a - x_b),$$

2) $\psi(x_a - x_b) = -\psi(x_b - x_a)$.

Remark 4 As in [19], this assumption guarantees the uniqueness of the solution of (2). Sector conditions 1) and 2) can be regarded as an extension of the dissipative condition, which can simplify the analysis of the problem.

Theorem 2 *Consider the coupling network (2) with Assumptions 1-4. Suppose that all subdigraphs $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$ are strongly connected, and the following conditions hold:*

1) there exists positive constant θ_k satisfying

$$\gamma_k + 2\theta_k \ge 0, \tag{22}$$

2) for cluster $\mathscr{V}_k, k = 1, \cdots, K$ the intra-cluster coupling strength c_k satisfies

$$c_k > \max\left\{ \frac{\underline{\varepsilon}\left(\bar{\xi}_k \tilde{\tau}_k - \tilde{\zeta}\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k \mu_{k2}}, \frac{\underline{\varepsilon}\left(\underline{\xi}_k \tilde{\tau}_k - \tilde{\zeta}\theta_k\right) m_k \|\boldsymbol{\xi}_k\|^2}{\theta_k \mu_{k2}}, 0 \right\}, \tag{23}$$

where $\bar{\xi}_k = \max \xi_{\mathscr{V}_k}$, $\underline{\xi}_k = \min \xi_{\mathscr{V}_k}$, $\xi_{\mathscr{V}_k} = \{\xi_{z_{k-1}+1}, \cdots, \xi_{z_{k-1}+m_k}\}$, $\tilde{\tau}_k = \sigma_k + \eta_k(2\theta_k + \gamma_k)$, $\tilde{\zeta} = \lambda_{\min}(H(E, \bar{L}^*))$, $\bar{L}'$ is the Laplacian matrix constructed by a'_{ij} , where $a'_{ij} = \bar{\varepsilon} a_{ij}$ if $a_{ij} < 0$; $a'_{ij} = \underline{\varepsilon} a_{ij}$ if $a_{ij} > 0$, $\hat{\imath} \ne \hat{\jmath}$. Then the coupling network (2) achieves cluster synchronization.

Proof Similarly, let $e(t) = x(t) - \bar{x}(t)$. Then, it is easy to get

$$\begin{aligned}
\dot{e}_i =& \dot{x}_i - \dot{\bar{x}}_{\hat{\imath}} \\
=& f_i(t, x_i) + \sum_{j=1}^{N} c_{ij} a_{ij} \psi(x_j - x_i) - \sum_{j=z_{k-1}+1}^{z_{k-1}+m_k} \xi_j \left(\sum_{m=1}^{N} c_{jm} a_{jm} \psi(x_m - x_j) + f_{\hat{\jmath}}(t, x_j) \right) \\
=& f_i(t, x_i) - f_{\hat{\imath}}(t, \bar{x}_{\hat{\imath}}) + \sum_{j=1}^{N} c_{ij} a_{ij} \psi(x_j - x_i) + D(x),
\end{aligned}$$

where

$$D(\bar{x}) = - \sum_{j=z_{k-1}+1}^{z_{k-1}+m_k} \xi_j \left(\sum_{m=1}^{N} c_{jm} a_{jm} \psi(x_m - x_j) + f_{\hat{\jmath}}(t, x_j) \right) + f_{\hat{\imath}}(t, \bar{x}_{\hat{\imath}}).$$

Assign a Lyapunov function as follows:

$$V(t)=\sum_{i=1}^{N}\frac{1}{2}\xi_i e_i^\top e_i.$$

By calculating the derivatives of $V(t)$, we can get

$$\begin{aligned}\dot V(t)=&\sum_{i=1}^{N}\xi_i e_i^\top \dot e_i\\ =&\sum_{i=1}^{N}\xi_i e_i^\top\left[f_{\hat i}(t,x_i)-f_{\hat i}(t,\bar x_{\hat i})\right]+\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i c_{ij}a_{ij}e_i^\top\psi(x_j-x_i)+\sum_{i=1}^{N}\xi_i e_i^\top D(x).\end{aligned}\tag{24}$$

According to $\sum_{i\in\mathscr{V}_k}\xi_i e_i=0, k=1,\cdots,K$, one obtains

$$\dot V(t)=\sum_{i=1}^{N}\xi_i e_i^\top(f_{\hat i}(t,x_i)-f_{\hat i}(t,\bar x_{\hat i}))+\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i c_{ij}a_{ij}e_i^\top(\psi(x_j-x_i)-\psi(\bar x_{\hat j}-\bar x_{\hat i})).\tag{25}$$

Again using $\psi(x_a-x_b)=-\psi(x_b-x_a)$ and $e_j-e_i=(x_j-x_i)-(\bar x_{\hat j}-\bar x_{\hat i})$, we have

$$\begin{aligned}&\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i c_{ij}a_{ij}e_i^\top(\psi(x_j-x_i)-\psi(\bar x_{\hat j}-\bar x_{\hat i}))\\ \le&\sum_{k=1}^{K}\sum_{i\in\mathscr{V}_k}\sum_{j\in\mathscr{V}_k}\xi_i\underline{\varepsilon}c_k a_{ij}e_i^\top(e_j-e_i)+\sum_{k=1}^{K}\sum_{i\in\mathscr{V}_k}\sum_{j\notin\mathscr{V}_k}\xi_i a'_{ij}e_i^\top(e_j-e_i)\\ =&-\frac{1}{2}\sum_{k=1}^{K}\underline{\varepsilon}c_k e_{\mathscr{V}_k}^\top H(E_k,L_{\mathscr{V}_k})e_{\mathscr{V}_k}-\frac{1}{2}e^\top H(E,\bar L')e\\ \le&-\underline{\varepsilon}\frac{1}{2}\sum_{k=1}^{K}\frac{c_k\mu_{k2}}{m_k\|\boldsymbol{\xi}_k\|^2}\left\|e_{\mathscr{V}_k}\right\|^2-\lambda_{\min}(H(E,\bar L'))\frac{1}{2}\sum_{k=1}^{K}\left\|e_{\mathscr{V}_k}\right\|^2.\end{aligned}\tag{26}$$

Use a technique similar to (9) of Theorem 1, if $\tau_k\lambda_{\max}(Q)-\lambda_{\min}(Q)\ge 0$, we have

$$\sum_{i=1}^{N}\xi_i e_i^\top[f_{\hat i}(t,x_i)-f_{\hat i}(t,\bar x_{\hat i})]\le\sum_{k=1}^{K}\bar\xi_k\frac{\tilde\tau_k}{2\theta_k}\left\|e_{\mathscr{V}_k}\right\|^2.\tag{27}$$

If $\tau_k\lambda_{\max}(Q)-\lambda_{\min}(Q)<0$, we have

$$\sum_{i=1}^{N}\xi_i e_i^\top[f_{\hat i}(t,x_i)-f_{\hat i}(t,\bar x_{\hat i})]\le\sum_{k=1}^{K}\underline{\xi}_k\frac{\tilde\tau_k}{2\theta_k}\left\|e_{\mathscr{V}_k}\right\|^2.\tag{28}$$

Combining (26) and (27), it then follows that

$$\dot V(t)\le\sum_{k=1}^{K}\frac{\bar\xi_k\tilde\tau_k}{2\theta_k}\left\|e_{\mathscr{V}_k}\right\|^2-\underline{\varepsilon}\sum_{k=1}^{K}\frac{c_k\mu_{k2}}{2m_k\|\boldsymbol{\xi}_k\|^2}\left\|e_{\mathscr{V}_k}\right\|^2-\frac{\lambda_{\min}(H(E,\bar L'))}{2}\sum_{k=1}^{K}\left\|e_{\mathscr{V}_k}\right\|^2.\tag{29}$$

Combining (26) and (28), one could get

$$\dot{V}(t) \leq \sum_{k=1}^{K} \frac{\underline{\xi}_k \tilde{\tau}_k}{2\theta_k} \left\| e_{\mathscr{V}_k} \right\|^2 - \underline{\varepsilon} \sum_{k=1}^{K} \frac{c_k \mu_{k2}}{2m_k \left\| \xi_k \right\|^2} \left\| e_{\mathscr{V}_k} \right\|^2 - \frac{\lambda_{\min}(H(E, \bar{L}'))}{2} \sum_{k=1}^{K} \left\| e_{\mathscr{V}_k} \right\|^2 . \tag{30}$$

Hence, when condition (23) is satisfied, it means $e(t) \to 0$ as $t \to \infty$. The coupled nonlinear systems (2) achieves cluster synchronization.

Corollary 3 *Consider the coupling network (2) with Assumptions 1-4. Suppose that all subdigraphs $\mathscr{G}_{\mathscr{V}_k}, k = 1, \cdots, K$ are strongly connected, weight balanced, and the following conditions hold:*

1) there exists positive constant θ_k satisfying

$$\gamma_k + 2\theta_k \geq 0, \tag{31}$$

2) for cluster $\mathscr{V}_k, k = 1, \cdots, K$ the intra-cluster coupling strength c_k satisfies

$$c_k > \max \left\{ \frac{\underline{\varepsilon}(\tilde{\tau}_k - m_k \tilde{\zeta} \theta_k)}{m_k \theta_k \mu_{k2}}, \, 0 \right\}, \tag{32}$$

where $\tilde{\tau}_k = \sigma_k + \eta_k(2\theta_k + \gamma_k)$, $\tilde{\zeta} = \lambda_{\min}(H(E, \bar{L}^*))$, $\bar{L}'$ is the Laplacian matrix constructed from a'_{ij} with $a'_{ij} = \bar{\varepsilon} a_{ij}$ if $a_{ij} < 0$; $a'_{ij} = \underline{\varepsilon} a_{ij}$ if $a_{ij} > 0$, $\hat{\imath} \neq \hat{\jmath}$. Then the coupling network (2) achieves cluster synchronization.

Remark 5 The proof of Corollary 3 is similar to the proof of Corollary 2, and we omit it for brevity. Note that the results in [19] about network (2) can be considered as a special case of Theorem 2 and Corollary 3.

5 Numerical example

In this part, two examples of simulation are provided to illustrate the accuracy of the established results.

Example 1: Consider the following coupled nonlinear system interacting by 3 clusters:

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{12} c_{ij} a_{ij} \Gamma (x_j - x_i), \; i = 1, \cdots, 12, \tag{33}$$

where $f_i(x_i)$ is a three-dimensional neural network with $f_i(x_i) = -v_i I_3 x_i + U g(x_i)$, where $g(x_i) = [\bar{g}(x_{i1}) \; \bar{g}(x_{i2}) \; \bar{g}(x_{i3})]^\top$, $\bar{g}(x_{ij}) = (\left|x_{ij} + 1\right| - \left|x_{ij} - 1\right|)/2, \; j = 1, 2, 3$ and

$$U = \begin{bmatrix} 1.1 & -0.2 & -0.2 \\ -0.2 & 1.1 & -0.4 \\ -0.2 & 0.4 & 1 \end{bmatrix}.$$

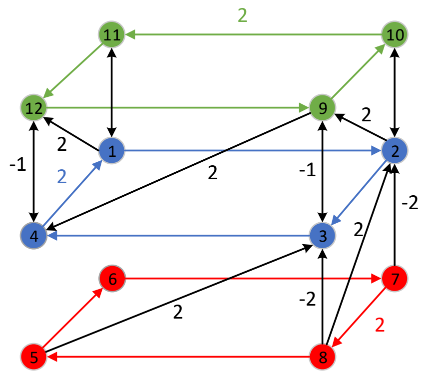

Fig. 1: The topology of network (33) and (34). Each cluster is strongly connected, and different clusters satisfy the cluster-input-equivalence condition.

Suppose that the diffusion matrix Γ is an identical matrix. Then, $\lambda_{\min}(\Gamma) = \lambda_{\max}(\Gamma) = 1$. Suppose that the interactive topology of the network (33) is illustrated in Fig. 1. Hence, $m_1 = m_2 = m_3 = 4$ and $\mathcal{G}_{\mathcal{V}_k}, k = 1,2,3$ are strongly connected. One can see from Fig. 1 that $v_i = \bar{v}_1 = 0.8, i = 1,\cdots,4$; $v_i = \bar{v}_2 = 1.2, i = 5,\cdots,8$; $v_i = \bar{v}_3 = 1.5, i = 9,\cdots,12$. For $\forall i \in \mathcal{V}_k, k = 1,2,3$, one has $f_k(x_i) = -\bar{v}_k I_3 x_i + U g(x_i)$.

As indicated by [28], the function f_k satisfies Assumption 2 with $\eta_k = 1.42$. For any $x_1, x_2 \in \mathbb{R}^n$, we have

$$\begin{aligned}
&[f_k(x_1) - f_k(x_2)]^\top [f_k(x_1) - f_k(x_2)] \\
&\leq -\|x_1 - x_2\|^2 - 2\bar{v}_k (x_1 - x_2)^\top [f(x_1) - f(x_2)] \\
&+ \lambda_{\max}(U^\top U)\,\|g(x_1) - g(x_2)\|^2 .
\end{aligned}$$

Since $\lambda_{\max}(U^\top U) = 1.9476$ and $\|g(x_1) - g(x_2)\|^2 \leq \|x_1 - x_2\|^2$, one can obtain

$$\begin{aligned}
&[f(x_1) - f(x_2)]^\top [f(x_1) - f(x_2)] \\
&\leq 1.9476\,\|x_1 - x_2\|^2 - 2\bar{v}_k (x_1 - x_2)^\top [f(x_1) - f(x_2)].
\end{aligned}$$

Hence, the function f_k satisfies quadratically inner-bounded condition with $\sigma_k = 1.9476$, $\gamma_k = -2\bar{v}_k$.

Note that the unmarked edges in Fig. 1 are all 1. Then, the left positive eigenvectors of $L_{\mathcal{V}_k}, k = 1,2,3$ corresponding to the eigenvalue 0 are calculated as

$$\begin{aligned}
\xi_1 &= [0.1429\ 0.2857\ 0.2857\ 0.2857]^\top, \\
\xi_2 &= [0.2857\ 0.2857\ 0.1429\ 0.2857]^\top, \\
\xi_3 &= [0.2857\ 0.1429\ 0.2857\ 0.2857]^\top.
\end{aligned}$$

Hence, $\|\xi_1\|^2 = 0.2653$ and $\zeta = \lambda_{\min}(H(E,\bar{L})) = -0.8764$. The eigenvalue of $H(E_1, L_{\mathcal{V}_1})$ are $\mu_{11} = 0$, $\mu_{12} = 0.5714$, $\mu_{13} = 0.5715$, $\mu_{14} = 1.1429$. If we choose $\theta_1 = 0.8$, then $\tau_1 = 1.9476$. When $c_1 > 2.9194$, condition (4) is satisfied. Similarly, one chooses

$\theta_2 = 1.2$ and $\theta_3 = 1.5$, it can be concluded that when $c_2 > 2.4888$ and $c_3 > 2.3165$, condition (4) is satisfied. We choose $c_1 = c_2 = c_3 = 3$, and Fig. 2 shows that the network (33) achieves the cluster synchronization.

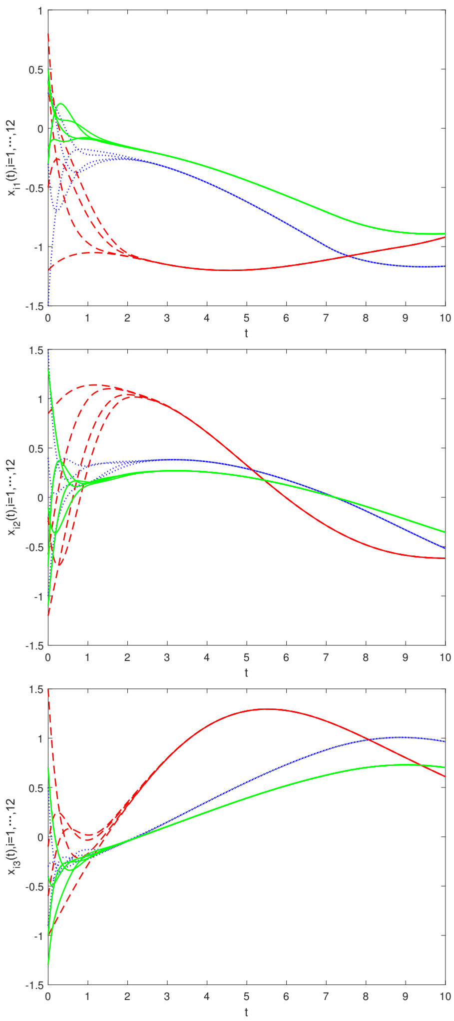

Fig. 2: Evolution trajectories of the coupling network (33) when $c_1 = c_2 = c_3 = 3$.

Example 2: Consider another nonlinear network that is nonlinearly coupled:

$$\dot{x}_i = f_i(x_i) + \sum_{j=1}^{12} c_{ij} a_{ij} \psi(x_j - x_i),\ i = 1, \cdots, 12, \tag{34}$$

where $\psi(x_i) = [x_{i1} + \tanh(x_{i1})\ x_{i2} + \tanh(x_{i2})\ x_{i3} + tanh(x_{i3})]^\top$, $f_i(x_i)$ is the same as in Example 1.

Likewise, the interaction topology of system (34) is also as shown in Fig. 1. From $\underline{\varepsilon} = 1$ and $\bar{\varepsilon} = 2$, we can get that $\zeta = \lambda_{\min}(H(E, \bar{L}')) = -2.0048$. By direct computation, when $c_1 > 5.0150$, $c_2 > 4.5844$ and $c_3 > 4.4122$ condition (34) is satisfied. We choose $c_1 = c_2 = c_3 = 5.1$, and Fig. 3 shows that the network (34) achieves the cluster synchronization.

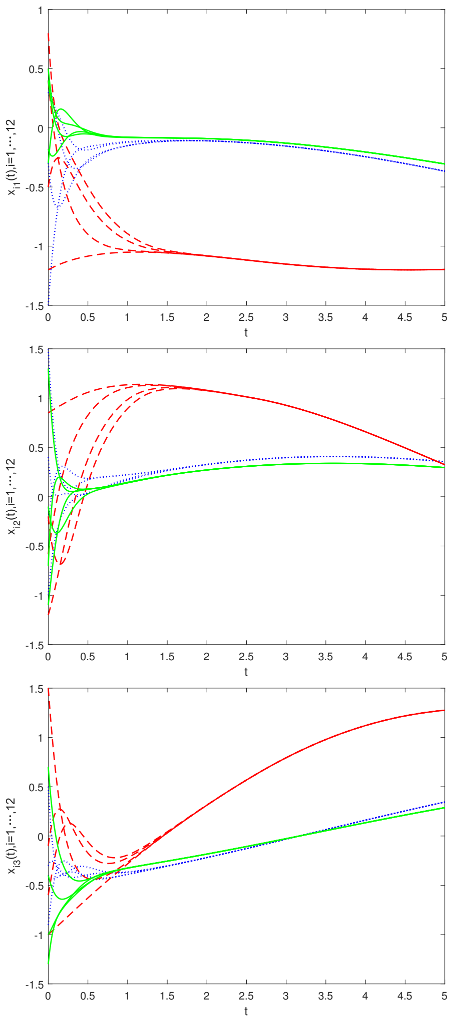

Fig. 3: Evolution trajectories of the nonlinear coupling network (34) when $c_1 = c_2 = c_3 = 5.1$.

6 Concluding remarks

In this article, cluster synchronization convergence results for coupled nonlinear systems with competitive relationship and directed topology are established. Under the assumption that the node dynamics satisfy satisfy one-side Lipschitz, the linear and nonlinear coupling cases were studied respectively. It is proved that if the coupling topology within each cluster is strongly connected and the in-cluster coupling strength is large enough, then the coupled nonlinear systems can achieve cluster synchronization. All obtained conditions are presented as algebraic conditions that are easy to solve. Finally, two examples of numerical simulation about neural networks are presented to verify the availability of the results obtained. Future work could consider cases of switching topologies.

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7 Data availability

The manuscript has no associated data.

8 Conflict of interest

The authors declare that they have no conflict of interest.

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