On Low-Dimensional Complex ω-Lie Superalgebras

Jia Zhou
Shanghai Normal University
Liangyun Chen (chenly640@nenu.edu.cn)
Northeast Normal University

Research Article

Keywords: ω-Lie superalgebra, derivations, automorphisms, representations

Posted Date: May 25th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-554746/v1

On Low-Dimensional Complex ω-Lie Superalgebras

Jia zhou and Liangyun Chen

Abstract. Let $(g,\ [-,-],\ \omega)$ be a finite-dimensional complex ω-Lie superalgebra. In this paper, we introduce the notions of derivation superalgebra $\mathrm{Der}(g)$ and the automorphism group $\mathrm{Aut}(g)$ of $(g,\ [-,-],\ \omega)$. We study $\mathrm{Der}^{\omega}(g)$ and $\mathrm{Aut}^{\omega}(g)$, which are superalgebra of $\mathrm{Der}(g)$ and subgroup of $\mathrm{Aut}(g)$, respectively. For any 3-dimensional or 4-dimensional complex ω-Lie superalgebra g, we explicitly calculate $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$, and obtain Jordan standard forms of elements in the two sets. We also study representation theory of ω-Lie superalgebras and give a conclusion that all nontrivial non-ω-Lie 3-dimensional and 4-dimensional ω-Lie superalgebras are multiplicative, as well as we show that any irreducible respresentation of the 4-dimensional ω-Lie superalgebra $P_{2,k}(k \neq 0, -1)$ is 1-dimensional.

Keywords. ω-Lie superalgebra, derivations, automorphisms, representations.

1. Introduction

In 2007, Nurowski [6] introduced the notion of ω-Lie algebras, which is related to the study of isoparametric hypersufaces in Riemannian geometry (See Bobienski-Nurowski [1] and Nurowski [7]). A fundamental development of ω-Lie algebras was made by Zusmanovich [12], whose results (See [12], Section 9, Theorem 1 and Theorem 2) show that nontrivial finite-dimensional ω-Lie algebras are either low-dimensional or have a very degenerate structure. Nurowski [6] first gave a classification of 3-dimensional ω-Lie algebras over the field of real numbers. In 2014, Chen-Liu-Zhang [2,3] obtained a classification of 3-dimensional and 4-dimensional complex ω-Lie algebras. With the classification, Chen-Zhang-Zhang [4] calculated the derivation algebras and automorphism groups of low-dimensional complex ω-Lie algebras, gave a sufficient and necessary condition for an ω-Lie algebra with 1-dimensional module, and showed that any irreducible respresentation of the 3-dimensional ω-Lie algebra $C_{\alpha}(\alpha \neq 0, -1)$ is 1-dimensional.

The notation of ω-Lie superalgebras (See Definition 1.1) was introduced by Zhou-Chen-Ma in [11]. As a $\mathbb{Z}_2$-graded vector space, an ω-Lie superalgebra

2020 *Mathematics Subject Classification.* 17B60; 17A30.
Corresponding author(L. Chen): chenly640@nenu.edu.cn.

$g = g_{\bar{0}} \oplus g_{\bar{1}}$ is an ω-Lie algebra if $g_{\bar{1}} = 0$ i.e., $g = g_{\bar{0}}$. An ω-Lie superalgebra, with the bilinear form ω, becomes a Lie superalgebra if $\omega \equiv 0$. Hence we usually call Lie superalgebras trivial ω-Lie superalgebras. Therefore, ω-Lie superalgebras generalize both ω-Lie algebras and Lie superalgebras.

The purpose of this article is to study representation theory, derivation algebra $\mathrm{Der}(g)$ and automorphism group $\mathrm{Aut}(g)$ for a finite-dimensional ω-Lie superalgebra g. We introduce the notations of ω-derivations and ω-automorphisms and write $\mathrm{Der}^{\omega}(g)$ and $\mathrm{Aut}^{\omega}(g)$ for the sets consisting of ω-derivations and ω-automorphisms, respectively. We calculate $\mathrm{Der}(g)$, $\mathrm{Aut}(g)$, $\mathrm{Der}^{\omega}(g)$ and $\mathrm{Aut}^{\omega}(g)$, study some Lie superalgebra properties of $\mathrm{Der}(g)$ as well as Lie group properties of $\mathrm{Aut}(g)$, and obtain all Jordan standard forms of elements in $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$, for any 3-dimensional or 4-dimensional complex ω-Lie superalgebra g. We also study multiplicative ω-Lie superalgebras, showing that any 3-dimensional or 4-dimensional nontrivial non-ω-Lie complex ω-Lie superalgebra has 1-dimensional module, and prove that the only irreducible representation of the 4-dimensional ω-Lie superalgebra $P_{2,k}(k \neq 0, -1)$ is 1-dimensional.

We proceed as follows. In Section 2, we introduce $\mathrm{Der}(g)$, $\mathrm{Aut}(g)$, $\mathrm{Der}^{\omega}(g)$ and $\mathrm{Aut}^{\omega}(g)$ of ω-Lie superalgebra g and observe that $\mathrm{Der}^{\omega}(g)$ (resp. $\mathrm{Aut}^{\omega}(g)$) is a subalgebra (resp. subgroup) of $\mathrm{Der}(g)$ (resp. $\mathrm{Aut}(g)$). In section 3, for the only nontrivial non-ω-Lie 3-dimensional complex ω-Lie superalgebra H, we give the detailed arguments about $\mathrm{Der}(H)$ and $\mathrm{Aut}(H)$. We show that $\mathrm{Aut}(H)$ is a connected matrix Lie group and $\exp(\mathrm{Der}_{\bar{0}}(H)) = \mathrm{Aut}(H)$. Section 4 is devoted to calculating $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$ when g is a 4-dimensional complex ω-Lie superalgebra. Here, we study $\mathrm{Der}(P_{1,k})$ and $\mathrm{Aut}(P_{1,k})$ in detail, see Proposition 4.2 and Proposition 4.4. We also discuss the elementary properties such as nilpotency, solvability and commutativity of Lie superalgebra $\mathrm{Der}(g)$ and Lie group $\mathrm{Aut}(g)$. In Section 5, the Jordan standard forms about elements in $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$ for any 3-dimensional or 4-dimensional complex ω-Lie superalgebra g are calculated and listed in the tables. In Section 6, we give a sufficient and necessary condition for an ω-Lie superalgebra with 1-dimensional module, and prove that all 3-dimensional and 4-dimensional nontrivial non-ω-Lie complex ω-Lie superalgebras are multiplicative. Particularly, we show that any irreducible respresentation of the 4-dimensional ω-Lie superalgebra $P_{2,k}(k \neq 0, -1)$ is 1-dimensional.

Throughout this paper we assume that the ground field is the complex field $\mathbb{C}$. All representations, modules and vector spaces are finite-dimensional over $\mathbb{C}$. An element $x \in g$ is called homogeneous if x is in $g_{\bar{0}}$ or $g_{\bar{1}}$. For any homogeneous element x we shall use the standard notation $|x| \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ to indicate its degree.

2. Basic concepts and properties

Definition 2.1. [11] An ω-Lie superalgebra g is a $\mathbb{Z}_2$-graded vector space $g = g_{\bar{0}} \oplus g_{\bar{1}}$ with a multiplication $[\cdot,\cdot] : g \times g \to g$ over $\mathbb{K}$ and a bilinear form $\omega : g \times g \to \mathbb{K}$ satisfying

(1) $[g_\alpha, g_\beta] \subseteq g_{\alpha+\beta}, \quad \forall\ \alpha, \beta \in \mathbb{Z}_2$,

(2) $[x, y] = -(-1)^{|x||y|}[y, x]$, (graded skew-symmetric)

(3) $(-1)^{|x||y|}[[y, z], x] + (-1)^{|y||z|}[[z, x], y] + (-1)^{|x||z|}[[x, y], z] = (-1)^{|x||y|}\omega(y, z)x + (-1)^{|y||z|}\omega(z, x)y + (-1)^{|x||z|}\omega(x, y)z$, (graded ω-Jacobi identity)

(4) $\omega(g_{\bar{0}}, g_{\bar{1}}) = 0$, where x, y, z are homogeneous elements of g.

Definition 2.2. Let g be an ω-Lie superalgebra. An homogeneous linear map $d \in \mathrm{End}_\theta(g)$ is said to be a derivation of degree θ if

$$d([x, y] = [d(x), y] + (-1)^{\theta|x|}[x, d(y)],\ \forall x, y \in g.$$

We denote the set of all derivations of degree θ by $\mathrm{Der}_\theta(g)$. $\mathrm{Der}(g) = \mathrm{Der}_{\bar{0}}(g) \bigoplus \mathrm{Der}_{\bar{1}}(g)$ provided with the Lie-super commutator is a subalgebra of $\mathrm{End}(g)$ and is called the derivation algebra of g.

Definition 2.3. Let g be an ω-Lie superalgebra. A linear map $\rho : g \to g$ of even degree is called an automorphism of g if $\rho([x, y]) = [\rho(x), \rho(y)],\ \forall x, y \in g$.

Remark 2.4. The set $\mathrm{Aut}(g)$ of all automorphisms of g forms a closed Lie subgroup of the general linear group $\mathrm{GL}(g)$, which means that $\mathrm{Aut}(g)$ is a matrix Lie group, see Sagle-Walde ([8], Proposition 7.1). We call $\mathrm{Aut}(g)$ the automorphism group of g.

Definition 2.5. Let g be an ω-Lie superalgebra. A derivation $d \in \mathrm{Der}_\theta(g)$ is called an ω-derivation of degree θ if

$$\omega(d(x), y) + (-1)^{\theta|x|}\omega(x, d(y)) = 0,\ \forall x, y \in g.$$

A automorphism $\rho \in \mathrm{Aut}(g)$ is called an ω-automorphism of g if

$$\omega(x, y) = \omega(\rho(x), \rho(y)),\ \forall x, y \in g.$$

The set consisting of all ω-derivations is denoted by $\mathrm{Der}^\omega(g)$, and the set of all ω-automorphisms of g is called $\mathrm{Aut}^\omega(g)$.

Clearly, $\mathrm{Der}^\omega(g) \subseteq \mathrm{Der}(g)$ and $\mathrm{Aut}^\omega(g) \subseteq \mathrm{Aut}(g)$. However, the converse are not all true, see Chen et al. ([4], Proposition 5.1).

We also have the following two properities.

Proposition 2.6. $\mathrm{Aut}^\omega(g)$ *is a subgroup of* $\mathrm{Aut}(g)$.

Proof. Similar to Proposition 2.3 in [4]. □

Proposition 2.7. $\mathrm{Der}^\omega(g)$ *is a subalgebra of Lie superalgebra* $\mathrm{Der}(g)$.

Proof. We need only to show that $[d, d'] = d \cdot d' - (-1)^{\theta\mu} d' \cdot d \in \mathrm{Der}^{\omega}_{\theta+\mu}(g)$, where $d \in \mathrm{Der}^{\omega}_{\theta}(g)$, $d' \in \mathrm{Der}^{\omega}_{\mu}(g)$. Indeed, for any x, $y \in g$, we have

$$\begin{aligned}\omega([d,d'](x),y) &= \omega(d\cdot d'(x)-(-1)^{\theta\mu}d'\cdot d(x),y)\\ &= \omega(d\cdot d'(x),y)-(-1)^{\theta\mu}\omega(d'\cdot d(x),y)\\ &= -(-1)^{\theta(\mu+|x|)}\omega(d'(x),d(y))+(-1)^{\mu|x|}\omega(d(x),d'(y)).\end{aligned}$$

On the other hand,

$$\begin{aligned}\omega(x,[d,d'](y)) &= \omega(x,d\cdot d'(y)-(-1)^{\theta\mu}d'\cdot d(y))\\ &= \omega(x,d\cdot d'(y))-(-1)^{\theta\mu}\omega(x,d'\cdot d(y))\\ &= -(-1)^{\theta|x|}\omega(d(x),d'(y))+(-1)^{\mu(|x|+\theta)}\omega(d'(x),d(y)).\end{aligned}$$

Thus

$$\omega([d,d'](x),y)+(-1)^{(\theta+\mu)|x|}\omega(x,[d,d'](y))=0.$$

So $[d, d'] \in \mathrm{Der}^{\omega}_{\theta+\mu}(g)$. □

3. Derivations and Automorphisms of 3-dimensional ω-Lie superalgebras

Theorem 3.1. *[11] Any nontrivial complex 3-dimensional ω-Lie superalgebra must be isomorphic to one the following algebras:*

$$L_1 : [x_1,x_2]=x_2,\ [x_2,x_3]=x_3,\ [x_1,x_3]=0,\ \omega(x_1,x_2)=1.$$

$$L_2 : [x_1,x_2]=0,\ [x_1,x_3]=x_2,\ [x_2,x_3]=x_3,\ \omega(x_1,x_3)=1.$$

$$A_k : [x_1,x_2]=x_1,\ [x_1,x_3]=x_1+x_2,\ [x_2,x_3]=kx_1+x_3,\ \omega(x_2,x_3)=-1.$$

$$B : [x_1,x_2]=x_2,\ [x_1,x_3]=x_2+x_3,\ [x_2,x_3]=x_1,\ \omega(x_2,x_3)=2.$$

$$C_k : [x_1,x_2]=x_2,\ [x_1,x_3]=kx_3,\ [x_2,x_3]=x_1,\ \omega(x_2,x_3)=1+k,\ k\neq 0,-1.$$

$$H : [x_1,x_2]=x_1,\ [x_1,y]=y,\ [x_2,y]=[y,y]=0,\ \omega(x_1,x_2)=1,\ \omega(y,y)=0,$$

where $x_1, x_2, x_3 \in g_{\bar{0}},\ y \in g_{\bar{1}},\ k \in \mathbb{C}$.

The derivations and automorphisms of 3-dimensional ω-Lie algebras(L_1, L_2, A_k, B, C_k) have been computed explicitly in Chen et al. ([4], Table 1, Table 2), so here we need only to consider $\mathrm{Der}(H)$ and $\mathrm{Aut}(H)$.

We first study the derivations of H. As in Theorem 2.1, H has a basis $\{x_1, x_2, y\}$, where $x_1, x_2 \in g_{\bar{0}},\ y \in g_{\bar{1}}$, and H is defined by $[x_1, x_2] = x_1,\ [x_1, y] = y,\ [x_2, y] = [y, y] = 0,\ \omega(x_1, x_2) = 1,\ \omega(y, y) = 0$. Let $E_{i,j}$ be the $n \times n$ matrix in which the (i, j)-entry is 1 and other entries are zero. It is clearly that the $\{E_{i,j} \mid 1 \le i, j \le n\}$ is a basis for the linear space $\mathrm{gl_n}(\mathbb{C})$.

Proposition 3.2.

$$(1)\ \mathrm{Der}(H) = \left\{ d \in \mathrm{gl}_3(\mathbb{C}) \middle|\ d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b & a \end{pmatrix}, a, b \in \mathbb{C} \right\}.$$

(2) $\mathrm{Der}(H)$ *is a 2-dimensional soluble (but not nilpotent) Lie superalgebra.*

(3) $\mathrm{Der}^{\omega}(H) = \mathrm{Der}(H)$.

Proof. (1) For any $d \in \mathrm{Der}(H)$, one can get $d = d_{\bar{0}} + d_{\bar{1}}$, where $d_{\bar{0}} \in \mathrm{Der}_{\bar{0}}(H)$ and $d_{\bar{1}} \in \mathrm{Der}_{\bar{1}}(H)$.

For $d_{\bar{0}} \in \mathrm{Der}_{\bar{0}}(H)$, we assume that $d_{\bar{0}} = (a_{i,j}) \in \mathrm{gl}_3(\mathbb{C})$. Suppose

$$d_{\bar{0}}(x_1) = a_{11}x_1 + a_{21}x_2,$$
$$d_{\bar{0}}(x_2) = a_{12}x_1 + a_{22}x_2,$$
$$d_{\bar{0}}(y) = a_{33}y.$$

Then we have that

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 = d_{\bar{0}}(x_1) = d_{\bar{0}}([x_1, x_2]) &= [d_{\bar{0}}(x_1), x_2] + [x_1, d_{\bar{0}}(x_2)] \\ &= [a_{11}x_1 + a_{21}x_2, x_2] + [x_1, a_{12}x_1 + a_{22}x_2] \\ &= a_{11}x_1 + a_{22}x_1. \end{aligned}$$

Since x_1, x_2 are linearly independent, we have $a_{21} = a_{22} = 0$.

Analogously, we have that

$$\begin{aligned} a_{33}y = d_{\bar{0}}(y) = d_{\bar{0}}([x_1, y]) &= [d_{\bar{0}}(x_1), y] + [x_1, d_{\bar{0}}(y)] \\ &= [a_{11}x_1 + a_{21}x_2, y] + [x_1, a_{33}y] = (a_{11} + a_{33})y \end{aligned}$$

and

$$\begin{aligned} 0 = d_{\bar{0}}([x_2, y]) &= [d_{\bar{0}}(x_2), y] + [x_2, d_{\bar{0}}(y)] \\ &= [a_{12}x_1 + a_{22}x_2, y] + [x_2, a_{33}y] = a_{12}y. \end{aligned}$$

This implies that $a_{11} = a_{12} = 0$. So $d_{\bar{0}} = diag\{0, 0, 0, a_{33}\}$, where $a_{33} \in \mathbb{C}$.

For $d_{\bar{1}} \in \mathrm{Der}_{\bar{1}}(H)$, we assume that $d_{\bar{1}} = (b_{i,j}) \in \mathrm{gl}_3(\mathbb{C})$. Suppose

$$d_{\bar{1}}(x_1) = b_{31}y,$$
$$d_{\bar{1}}(x_2) = b_{32}y,$$
$$d_{\bar{1}}(y) = b_{13}x_1 + b_{23}x_2.$$

We have

$$\begin{aligned} b_{31}y = d_{\bar{1}}(x_1) = d_{\bar{1}}([x_1, x_2]) &= [d_{\bar{1}}(x_1), x_2] + [x_1, d_{\bar{1}}(x_2)] \\ &= [b_{31}y, x_2] + [x_1, b_{32}y] = b_{32}y. \end{aligned}$$

Thus, $b_{31} = b_{32} = 0$.

Similarly,

$$\begin{aligned} b_{13}x_1 + b_{23}x_2 = d_{\bar{1}}(y) = d_{\bar{1}}([x_1, y]) &= [d_{\bar{1}}(x_1), y] + [x_1, d_{\bar{1}}(y)] \\ &= [b_{31}y, y] + [x_1, b_{13}x_1 + b_{23}x_2] = b_{23}x_1. \end{aligned}$$

Since x_1, x_2 are linearly independent, we have $b_{13} = b_{23} = 0$.

We also know that

$$\begin{aligned} 0 = d_{\bar{1}}([x_2, y]) &= [d_{\bar{1}}(x_2), y] + [x_2, d_{\bar{1}}(y)] \\ &= [b_{32}y, y] + [x_2, b_{13}x_1 + b_{23}x_2] = -b_{13}x_1 \end{aligned}$$

and

$$\begin{aligned} 0 = d_{\bar{1}}([y, y]) &= [d_{\bar{1}}(y), y] - [y, d_{\bar{1}}(y)] \\ &= 2([b_{13}x_1 + b_{23}x_2, u]) = b_{13}y, \end{aligned}$$

Thus $b_{13} = 0$. Consequently,

$$d_{\bar{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{31} & 0 \end{pmatrix}, \text{ where } b_{31} \in \mathbb{C}.$$

(2) Clearly, $\{E_{33}\}$ is a basis of $\mathrm{Der}_{\bar{0}}(H)$, $\{E_{31} + E_{32}\}$ is a basis of $\mathrm{Der}_{\bar{1}}(H)$. Therefore, $\{E_{33}, E_{31} + E_{32}\}$ is a basis of $\mathrm{Der}(H)$, which means $\mathrm{Der}(H)$ is a 2-dimensional Lie superalgebra. Note that $[E_{33}, E_{33}] = [E_{31} + E_{32}, E_{31} + E_{32}] = 0$, $[E_{33}, E_{31} + E_{32}] = E_{31} + E_{32} \neq 0$, so $\mathrm{Der}_{\bar{0}}(H)$ is abelian, while $\mathrm{Der}(H)$ is not abelian. We can get the conclusion that $\mathrm{Der}(H)$ is soluble but not nilpotent from the direct calculation and definitions of nilpotent and soluble Lie superalgebras.

Note that $\mathrm{Der}(H)$ is isomorphic to $L_2^2 : [x, x] = 0, [x, y] = y, [y, y] = 0$, x, y is a basis with $x \in (L_2^2)_{\bar{0}}$ and $y \in (L_2^2)_{\bar{1}}$, see Wang ([10], Chapter 3).

(3) For any $d \in \mathrm{Der}_\theta(H)$, since ω is bilinear, we need only to show for any basic element $x, y \in H$,

$$\omega(d(x), y) + (-1)^{\theta|x|}\omega(x, d(y)) = 0. \tag{3.1}$$

Let $d_{\bar{0}} \in \mathrm{Der}_{\bar{0}}(H)$, from (2) we have $d_{\bar{0}}(x_1) = d_{\bar{0}}(x_2) = 0, d_{\bar{0}}(y) = a_{33}y$, thus

$$\begin{aligned} &\omega(d_{\bar{0}}(x_1), x_2) + \omega(x_1, d_{\bar{0}}(x_2)) = \omega(0, x_2) + \omega(x_1, 0) = 0, \\ &\omega(d_{\bar{0}}(x_1), y) + \omega(x_1, d_{\bar{0}}(y)) = \omega(0, y) + \omega(x_1, a_{33}y) = 0, \\ &\omega(d_{\bar{0}}(x_2), y) + \omega(x_2, d_{\bar{0}}(y)) = \omega(0, y) + \omega(x_2, a_{33}y) = 0, \\ &\omega(d_{\bar{0}}(y), y) + \omega(y, d_{\bar{0}}(y)) = \omega(a_{33}y, y) + \omega(y, a_{33}y) = 0. \end{aligned}$$

Let $d_{\bar{1}} \in \mathrm{Der}_{\bar{1}}(H)$, since $d_{\bar{1}}(x_1) = b_{31}y, d_{\bar{1}}(x_2) = b_{32}y$ and $d_{\bar{1}}(y) = 0$, we have

$$\begin{aligned} &\omega(d_{\bar{1}}(x_1), x_2) + \omega(x_1, d_{\bar{1}}(x_2)) = \omega(b_{31}y, x_2) + \omega(x_1, b_{31}y) = 0, \\ &\omega(d_{\bar{1}}(x_1), y) + \omega(x_1, d_{\bar{1}}(y)) = \omega(b_{31}y, y) + \omega(x_1, 0) = 0, \\ &\omega(d_{\bar{1}}(x_2), y) + \omega(x_2, d_{\bar{1}}(y)) = \omega(b_{32}y, y) + \omega(x_2, 0) = 0, \\ &\omega(d_{\bar{1}}(y), y) - \omega(y, d_{\bar{1}}(y)) = \omega(d_{\bar{1}}(y), y) - \omega(d_{\bar{1}}(y), y) = 0. \end{aligned}$$

For other cases not mentioned above, the Equation (3.1) still holds from the fact that ω is graded-anti-symmetric. So $\mathrm{Der}^\omega(H) = \mathrm{Der}(H)$. □

Proposition 3.3. (1) $\mathrm{Aut}(H) = \left\{ d \in \mathrm{gl}_3(\mathbb{C}) \middle| \, d = \mathrm{diag}\{1,1,a\}, 0 \neq a \in \mathbb{C} \right\}$.

(2) $\mathrm{Aut}(H) = \mathrm{Aut}^{\omega}(H)$.

(3) $\mathrm{Aut}(H) = \exp(\mathrm{Der}_{\bar{0}}(H))$, *where* $\exp(-)$ *denotes the matrix exponential.*

(4) $\mathrm{Aut}(H)$ *is a connected, abelian matrix Lie group.*

Proof. (1) For any $\sigma \in \mathrm{Aut}(H)$, we assume that $\sigma = (a_{i,j}) \in \mathrm{gl}_3(\mathbb{C})$ with $\det(\sigma) \neq 0$. Suppose

$$\begin{aligned} \sigma(x_1) &= a_{11}x_1 + a_{21}x_2, \\ \sigma(x_2) &= a_{12}x_1 + a_{22}x_2, \\ \sigma(y) &= a_{33}y. \end{aligned}$$

Firstly, we have

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 = \sigma(x_1) = \sigma([x_1,x_2]) &= [\sigma(x_1), \sigma(x_2)] \\ &= [a_{11}x_1 + a_{21}x_2, a_{12}x_1 + a_{22}x_2] \\ &= a_{11}a_{22}x_1 - a_{12}a_{21}x_1 \end{aligned}$$

$\Rightarrow a_{11} = a_{11}a_{22}, \ a_{21} = 0.$

And

$$a_{33}y = \sigma(y) = \sigma([x_1,y]) = [\sigma(x_1), \sigma(y)] = [a_{11}x_1 + a_{21}x_2, a_{33}y] = a_{11}a_{33}y$$

$\Rightarrow a_{33} = a_{11}a_{33}.$

Finally,

$$0 = \sigma([x_2,y]) = [\sigma(x_2), \sigma(y)] = [a_{12}x_1 + a_{22}x_2, a_{33}y] = a_{12}a_{33}y \Rightarrow a_{12}a_{33} = 0.$$

A direct computation shows that $\sigma = \mathrm{diag}\{1,1,a\}$, where $a \neq 0$, $a \in \mathbb{C}$.

(2) For any $\sigma \in \mathrm{Aut}(H)$, since ω is bilinear, for basic elements $x, y \in H$, we prove that

$$\omega(\sigma(x), \sigma(y)) = \omega(x,y). \tag{3.2}$$

For $\forall \sigma \in \mathrm{Aut}(H)$, since $\sigma(x_1) = x_1$, $\sigma(x_2) = x_2$, $\sigma(y) = ay$, we have

$$\begin{aligned} \omega(\sigma(x_1), \sigma(x_2)) &= \omega(x_1, x_2) = 1, \\ \omega(\sigma(x_1), \sigma(y)) &= \omega(x_1, ay) = 0, \\ \omega(\sigma(x_2), \sigma(y)) &= \omega(x_2, ay) = 0, \\ \omega(\sigma(y), \sigma(y)) &= \omega(ay, ay) = 0. \end{aligned}$$

For other cases not mentioned above, the Equation (3.2) still holds from the fact that ω is graded-anti-symmetric. So $\mathrm{Aut}^{\omega}(H) = \mathrm{Aut}(H)$.

(3) Since $\mathrm{Aut}(H)$ is a closed Lie subgroup of the general linear group $\mathrm{GL}(3,\mathbb{C})$, see Sagle-Walde ([8], Proposition 7.1), so it is a matrix Lie group, according to Hall ([5], Definition 1.4).

Recall that the matrix exponential exp: $\mathrm{gl}_3(\mathbb{C}) \to \mathrm{GL}(3,\mathbb{C})$ was given by $X \mapsto \sum_{k=0}^{\infty} \frac{X^k}{k!}$ for any 3×3 complex matrix X. By Sagle-Walde ([8], Proposition 7.3(a)), we have $\exp(tX) \in \mathrm{Aut}(H)$ for any $t \in \mathbb{R}$ and any derivation $X \in \exp(\mathrm{Der}_{\bar{0}}(H))$, which means $\exp(\mathrm{Der}_{\bar{0}}(H)) \subseteq \mathrm{Aut}(H)$. To show $\exp(\mathrm{Der}_{\bar{0}}(H)) = \mathrm{Aut}(H)$, we need only to show that $\exp(\mathrm{Der}_{\bar{0}}(H)) \supseteq \mathrm{Aut}(H)$.

For any $\sigma = \mathrm{diag}\{1, 1, a\} \in \mathrm{Aut}(H)$, since $a \neq 0$, it follows from elementary analysis that $e^x = a$ has a nonzero solution in $\mathbb{C}$, i.e., there is a complex number $0 \neq a_0 \in \mathbb{C}$ such that $e^{a_0} = a$. We consider $d_{\bar{0}} = \mathrm{diag}\{0, 0, a_0\} \in \mathrm{Der}_{\bar{0}}(H)$ and show that

$$\begin{aligned}\exp(d_{\bar{0}}) &= I_3 + d_{\bar{0}}/1! + d_{\bar{0}}^2/2! + d_{\bar{0}}^3/3! + ... \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & a_0/1! + a_0^2/2! + a_0^3/3! + ...\end{pmatrix} \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1 + 1 + a_0/1! + a_0^2/2! + a_0^3/3! + ...\end{pmatrix} \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & e^{a_0} - 1\end{pmatrix} \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{a_0}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & a\end{pmatrix} = \sigma.\end{aligned}$$

Therefore, $\sigma \in \exp(\mathrm{Der}_{\bar{0}}(H))$.

(4) It follows from Hall ([5], §2.5.2) that the Lie algebra of a matrix Lie group $\mathrm{GL}(3, \mathbb{C})$ is just $\mathrm{gl}_3(\mathbb{C})$. Hence, as the subgroup of $\mathrm{GL}(3, \mathbb{C})$, and with $\mathrm{Der}_{\bar{0}}(H)$ as its Lie algebra, see Sagle-Walde ([8], Proposition 7.3(b)), $\mathrm{Aut}(H)$ is a connected matrix Lie group, according to Hall ([5], Definition 3.12).

Since $\mathrm{Aut}(H)$ is connected and its Lie algebra $\mathrm{Der}_{\bar{0}}(H)$ is abelian, $\mathrm{Aut}(H)$ is abelian by Sagle-walde ([8], §5, Exercise(2)). □

Remark 3.4. Consider the relationship between $\mathrm{Aut}(H)$ and $\exp(\mathrm{Der}(H))$.

For $d \in \mathrm{Der}(H)$, $d = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ b & b & a\end{pmatrix}$. Suppose that $a \neq 0,\ b \neq 0$, by induction on k, we can show that $d^k = a^{k-1}d$ for all $k \in \mathbb{N}^+$. Thus

$$\begin{aligned}\exp(d) &= I_3 + d/1! + d^2/2! + d^3/3! + ... \\ &= I_3 + \frac{d}{a}(a + a^2/2! + a^3/3! + ...) \\ &= I_3 + \frac{d}{a}(-1 + 1 + a + a^2/2! + a^3/3! + ...) \\ &= I_3 + \frac{d}{a}(e^a - 1) \\ &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ b & b & e^a - 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ b & b & e^a\end{pmatrix} \notin \mathrm{Aut}(H).\end{aligned}$$

Hence $\mathrm{Aut}(H) \subset \exp(\mathrm{Der}(H))$ but $\mathrm{Aut}(H) \neq \exp(\mathrm{Der}(H))$.

4. Derivations and Automorphisms of 4-dimensional ω-Lie superalgebras

The classification of 4-dimensional complex ω-Lie algebras has been derived from Chen et al. ([3], Theorem 1.5). With the similar method appeared in [3], we obtained a classification of non-ω-Lie 4-dimensional complex ω-Lie superalgebras, see ([11], Theorem 3.3).

Theorem 4.1. [3] *Any nontrivial 4-dimensional ω-Lie algebra must be isomorphic to one of the following algebras:*

$$\{L_{1,1}, \cdots, L_{1,8}, L_{2,1}, L_{2,2}, L_{2,3}, L_{2,4}, \tilde{B}, E_{1,k}(0,-1 \neq k), F_{1,k}(0,-1 \neq k),$$
$$G_{1,k}, H_{1,k}, \tilde{A}_k, \tilde{C}_k(0,-1 \neq k)\}.$$

For all cases, the parameter $k \in \mathbb{C}$.

Theorem 4.2. [11] *Any nontrivial non-ω-Lie 4-dimensional ω-Lie superalgebra must be isomorphic to one of the following algebras:*

$$P_{1,k} : [x_1, x_2] = x_2,\ [x_2, x_3] = x_3,\ [x_1, x_3] = [x_2, y] = [x_3, y] = [y, y] = 0,$$
$$[x_1, y] = ky,\ \omega(x_1, x_2) = 1,\ \omega(y, y) = 0.$$

$$N_k : [x_1, x_2] = 0,\ [x_1, x_3] = x_2,\ [x_2, x_3] = x_3,\ [x_1, y] = ky, [x_2, y] = y,$$
$$[x_3, y] = [y, y] = 0,\ [x_2, y] = y,\ \omega(x_1, x_3) = 1,\ \omega(y, y) = 0.$$

$$M_k : [x_1, x_2] = x_1,\ [x_1, x_3] = x_1 + x_2,\ [x_2, x_3] = kx_1 + x_3,\ [x_3, y] = -y,$$
$$[x_1, y] = [x_2, y] = [y, y] = 0,\ \omega(x_2, x_3) = -1,\ \omega(y, y) = 0.$$

$$Q : [x_1, x_2] = x_2,\ [x_1, x_3] = x_2 + x_3,\ [x_2, x_3] = x_1,\ [x_1, y] = 2y,$$
$$[x_2, y] = [x_3, y] = [y, y] = 0,\ \omega(x_2, x_3) = 2,\ \omega(y, y) = 0.$$

$$P_{2,k} : [x_1, x_2] = x_2,\ [x_1, x_3] = kx_3,\ [x_2, x_3] = x_1,\ [x_1, y] = (1 + k)y,$$
$$[x_2, y] = [x_3, y] = [y, y] = 0,\ \omega(x_2, x_3) = 1 + k,\ \omega(y, y) = 0,\ k \neq 0, -1.$$

$$S_k : [x_1, y_1] = ky_1 + y_2, [x_2, y_1] = y_1, [y_1, y_1] = [y_1, y_2] = [y_2, y_2] = 0, [x_1, x_2] = x_2,$$
$$[x_1, y_2] = ky_2, [x_2, y_2] = y_2, \omega(x_1, x_2) = 1, \omega(y_1, y_1) = \omega(y_1, y_2) = \omega(y_2, y_2) = 0.$$

$$T_k : [x_1, y_1] = (k-1)y_1, [x_2, y_1] = y_1 + y_2, [y_1, y_1] = [y_1, y_2] = [y_2, y_2] = 0, [x_1, x_2] = x_2,$$
$$[x_1, y_2] = ky_2, [x_2, y_2] = y_2, \omega(x_1, x_2) = 1, \omega(y_1, y_1) = \omega(y_1, y_2) = \omega(y_2, y_2) = 0.$$

$$R_{t,k} : [x_1, y_1] = ty_1, [x_2, y_1] = y_1, [y_1, y_1] = [y_1, y_2] = [y_2, y_2] = 0, [x_1, x_2] = x_2,$$
$$[x_1, y_2] = ky_2, [x_2, y_2] = y_2, \omega(x_1, x_2) = 1, \omega(y_1, y_1) = \omega(y_1, y_2) = \omega(y_2, y_2) = 0.$$

where $x_1, x_2, x_3 \in g_{\bar{0}},\ y, y_1, y_2 \in g_{\bar{1}},\ k, t \in \mathbb{C}$.

With the classification, Chen et al. (see [4], Table 3 and Table 4) summarized the characterizations of $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$ of all 4-dimensional complex ω-Lie algebras. In this section, we compute derivations and automorphisms for all nontrivial non-ω-Lie 4-dimensional ω-Lie superalgebras. We only give detailed proofs for the cases of $P_{1,k}$. Similar arguments can be applied to the remaining cases, so we omit them.

As in Theorem 4.2, $P_{1,k}$ has a basis $\{x_1, x_2, x_3, y\}$, where $x_1, x_2, x_3 \in g_{\bar{0}}$, $y \in g_{\bar{1}}$, and $P_{1,k}$ is defined by $[x_1, x_2] = x_2$, $[x_2, x_3] = x_3$, $[x_1, y] = ky$, $[x_1, x_3] = [x_2, y] = [x_3, y] = [y, y] = 0$, $\omega(x_1, x_2) = 1$, $\omega(y, y) = 0$.

We have the following results about $\mathrm{Der}(P_{1,k})$ and $\mathrm{Aut}(P_{1,k})$.

Proposition 4.3. (1)

$$\mathrm{Der}(P_{1,1}) = \left\{ d \in \mathrm{gl}_4(\mathbb{C}) \,\middle|\, d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & -a & b & 0 \\ e & f & 0 & c \end{pmatrix}, a, b, c, e, f \in \mathbb{C} \right\}.$$

$$\mathrm{Der}(P_{1,k}) = \left\{ d \in \mathrm{gl}_4(\mathbb{C}) \,\middle|\, d = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & -a & b & 0 \\ e & 0 & 0 & c \end{pmatrix}, a, b, c, e, f \in \mathbb{C} \right\}, (k \neq 1).$$

(2) $\mathrm{Der}(P_{1,k})$ *is a soluble (but not nilpotent) Lie superalgebra.*
(3) $\mathrm{Der}^{\omega}(P_{1,k}) = \mathrm{Der}(P_{1,k})$.

Proof. (1) For any $d \in \mathrm{Der}(H)$, one can get $d = d_{\bar{0}} + d_{\bar{1}}$, where $d_{\bar{0}} \in \mathrm{Der}_{\bar{0}}(H)$ and $d_{\bar{1}} \in \mathrm{Der}_{\bar{1}}(H)$.

Let $d_{\bar{0}} = (a_{i,j}) \in \mathrm{gl}_4(\mathbb{C})$. Suppose

$$\begin{aligned} d_{\bar{0}}(x_1) &= a_{11}x_1 + a_{21}x_2 + a_{31}x_3, \\ d_{\bar{0}}(x_2) &= a_{12}x_1 + a_{22}x_2 + a_{32}x_3, \\ d_{\bar{0}}(x_3) &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3, \\ d_{\bar{0}}(y) &= a_{44}y. \end{aligned}$$

we get from the definition of derivations that

$$\begin{aligned} a_{12}x_1 + a_{22}x_2 + a_{32}x_3 &= d_{\bar{0}}(x_2) = d_{\bar{0}}([x_1, x_2]) \\ &= [d_{\bar{0}}(x_1), x_2] + [x_1, d_{\bar{0}}(x_2)] \\ &= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3, x_2] + [x_1, a_{12}x_1 + a_{22}x_2 + a_{32}x_3] \\ &= a_{11}x_2 - a_{31}x_3 + a_{22}x_2. \end{aligned}$$

Since x_1, x_2, x_3 are linearly independent, we have $a_{11} = a_{12} = 0$, $a_{32} + a_{31} = 0$.

Analogously,

$$\begin{aligned} a_{13}x_1 + a_{23}x_2 + a_{33}x_3 &= d_{\bar{0}}(x_3) = d_{\bar{0}}([x_2, x_3]) = [d_{\bar{0}}(x_2), x_3] + [x_2, d_{\bar{0}}(x_3)] \\ &= [a_{12}x_1 + a_{22}x_2 + a_{32}x_3, x_3] + [x_2, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] \\ &= a_{22}x_3 - a_{13}x_2 + a_{33}x_3. \end{aligned}$$

This leads to $a_{13} = a_{22} = 0, a_{23} + a_{13} = 0$. Note that

$$\begin{aligned} 0 = d_{\bar{0}}([x_1, x_3]) &= [d_{\bar{0}}(x_1), x_3] + [x_1, d_{\bar{0}}(x_3)] \\ &= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3, x_3] + [x_1, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] \\ &= a_{21}x_3 + a_{23}x_2. \end{aligned}$$

Thus $a_{21} = 0, a_{23} = 0$. Together with the above we can conclude that

$$d_{\bar{0}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{31} & -a_{31} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}.$$

Next we consider $d_{\bar{1}} \in \mathrm{Der}_{\bar{1}}(H)$. Let $d_{\bar{1}} = (b_{i,j}) \in \mathrm{gl}_4(\mathbb{C})$ be such that

$$\begin{aligned} d_{\bar{1}}(x_1) &= b_{41}y, \\ d_{\bar{1}}(x_2) &= b_{42}y, \\ d_{\bar{1}}(x_3) &= b_{43}y, \\ d_{\bar{1}}(y) &= b_{14}x_1 + b_{24}x_2 + b_{34}x_3. \end{aligned}$$

Using the notation of derivations, we find that

$$\begin{aligned} b_{42}y = d_{\bar{1}}(x_2) = d_{\bar{1}}([x_1, x_2]) &= [d_{\bar{1}}(x_1), x_2] + [x_1, d_{\bar{1}}(x_2)] \\ &= [b_{41}y, x_2] + [x_1, b_{42}y] = kb_{42}y. \end{aligned}$$

Thus, $(1 - k)b_{42} = 0$.

Analogously,

$$\begin{aligned} b_{43}y = d_{\bar{1}}(x_3) = d_{\bar{1}}([x_2, x_3]) &= [d_{\bar{1}}(x_2), x_3] + [x_2, d_{\bar{1}}(x_3)] \\ &= [b_{42}y, x_3] + [x_2, b_{43}y] = 0. \end{aligned}$$

Then we have that $b_{43} = 0$.

Since

$$\begin{aligned} 0 = d_{\bar{1}}([x_2, y]) &= [d_{\bar{1}}(x_2), y] + [x_2, d_{\bar{1}}(y)] \\ &= [b_{42}y, y] + [x_2, b_{14}x_1 + b_{24}x_2 + b_{34}x_3] = -b_{14}x_2 + b_{34}x_3, \end{aligned}$$

and

$$\begin{aligned} 0 = d_{\bar{1}}([x_3, y]) &= [d_{\bar{1}}(x_3), y] + [x_3, d_{\bar{1}}(y)] \\ &= [b_{43}y, y] + [x_3, b_{14}x_1 + b_{24}x_2 + b_{34}x_3] = -b_{24}x_3, \end{aligned}$$

we find that $b_{14} = b_{24} = b_{34} = 0$. So

$$d_{\bar{1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{41} & 0 & 0 & 0 \end{pmatrix} \ (k \neq 1), \ \textit{or} \ d_{\bar{1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & 0 & 0 \end{pmatrix} \ (k = 1).$$

(2) Clearly, $\{E_{31} - E_{32}, E_{33}, E_{44}\}$ is a basis of $\mathrm{Der}_{\bar{0}}(P_{1,k})$, $\{E_{41}\}$ is a basis of $\mathrm{Der}_{\bar{1}}(P_{1,k})$ for $k \neq 1$ and $\{E_{41}, E_{42}\}$ a basis for $k = 1$. Note that $[E_{31} - E_{32}, E_{44}] = [E_{33}, E_{44}] = 0$ while $[E_{31} - E_{32}, E_{33}] = E_{31} - E_{32}$, hence

$\mathrm{Der}_{\bar{0}}(P_{1,k})$ is soluble but not nilpotent from the definitions of nilpotent and soluble Lie algebras, so is Lie superalgebra $\mathrm{Der}(P_{1,k})$ with similar arguments.

(3) For any $d \in \mathrm{Der}_\theta(P_{1,k})$, since ω is bilinear, we need only to show that for any basic element x, y,

$$\omega(d(x), y) + (-1)^{\theta|x|}\omega(x, d(y)) = 0. \tag{4.1}$$

The Equation (4.1) follows by direct calculations, so we omit them. Therefore, $\mathrm{Der}^\omega(P_{1,k}) = \mathrm{Der}(P_{1,k})$. □

Proposition 4.4. $\mathrm{Der}(g) = \mathrm{Der}^\omega(g)$ *holds for any nontrivial non-ω-Lie 4-dimensional ω-Lie superalgebra g.*

Proof. Similar arguments with Proposition 4.2(3). □

Proposition 4.5.

(1) $$\mathrm{Aut}(P_{1,k}) = \left\{ d \in \mathrm{gl}_4(\mathbb{C}) \,\middle|\, d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, 0 \neq b,\ 0 \neq c \right\}.$$

(2) $\mathrm{Aut}(P_{1,k}) = \mathrm{Aut}^\omega(P_{1,k})$.

(3) $\mathrm{Aut}(P_{1,k}) = \exp(\mathrm{Der}_{\bar{0}}(P_{1,k}))$, *where* $\exp(-)$ *denotes the matrix exponential.*

(4) $\mathrm{Aut}(P_{1,k})$ *is a connected, soluble matrix Lie group.*

Proof. (1) For any $\sigma \in \mathrm{Aut}(H)$, we assume that $\sigma = (a_{i,j}) \in \mathrm{gl}_3(\mathbb{C})$ with $\det(\sigma) \neq 0$.

Suppose

$$\begin{aligned} \sigma(x_1) &= a_{11}x_1 + a_{21}x_2 + a_{31}x_3, \\ \sigma(x_2) &= a_{12}x_1 + a_{22}x_2 + a_{32}x_3, \\ \sigma(x_3) &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3, \\ \sigma(y) &= a_{44}y. \end{aligned}$$

Using the automorphism rule we obtain

$$\begin{aligned} a_{12}x_1 + a_{22}x_2 + a_{32}x_3 &= \sigma(x_2) = \sigma([x_1, x_2]) = [\sigma(x_1), \sigma(x_2)] \\ &= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3, a_{12}x_1 + a_{22}x_2 + a_{32}x_3] \\ &= a_{11}a_{22}x_2 - a_{12}a_{21}x_2 + a_{21}a_{32}x_3 - a_{22}a_{31}x_3 \end{aligned}$$

$\Rightarrow a_{12} = 0,\ a_{22} = a_{11}a_{22},\ a_{32} = a_{21}a_{32} - a_{22}a_{31}$. Since

$$\begin{aligned} a_{13}x_1 + a_{23}x_2 + a_{33}x_3 &= \sigma(x_3) = \sigma([x_2, x_3]) = [\sigma(x_2), \sigma(x_3)] \\ &= [a_{12}x_1 + a_{22}x_2 + a_{32}x_3, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] \\ &= a_{12}a_{23}x_2 - a_{13}a_{22}x_2 + a_{22}a_{33}x_3 - a_{23}a_{32}x_3, \end{aligned}$$

and

$$\begin{aligned}0 &= \sigma(x_3) = \sigma([x_1, x_3]) = [\sigma(x_1), \sigma(x_3)] \\ &= [a_{11}x_1 + a_{21}x_2 + a_{31}x_3, a_{13}x_1 + a_{23}x_2 + a_{33}x_3] \\ &= a_{11}a_{23}x_2 - a_{13}a_{21}x_2 + a_{21}a_{33}x_3 - a_{23}a_{31}x_3,\end{aligned}$$

by comparing the coefficients of x_1, x_2 and x_3, we find that $a_{11}a_{23} = a_{13}a_{21}$, $a_{21}a_{33} = a_{23}a_{31}$, $a_{13} = 0$, $a_{23} = a_{12}a_{23}$ and $a_{33} = a_{22}a_{33} - a_{23}a_{32}$.

From a direct computation, we show that

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & -a_{31} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix}, \text{ where } a_{33} \neq 0,\ a_{44} \neq 0.$$

(2) For any $\sigma \in \mathrm{Aut}(P_{1,k})$, recall that ω is bilinear, we need only to show

$$\omega(\sigma(x), \sigma(y)) = \omega(x, y) \tag{4.2}$$

holds for all basic elements $x, y \in P_{1,k}$. For $\forall \sigma \in \mathrm{Aut}(P_{1,k})$, since $\sigma(x_1) = x_1 + a_{31}x_3$, $\sigma(x_2) = x_2 - a_{31}x_3$, $\sigma(x_3) = a_{33}x_3$, $\sigma(y) = a_{44}(y)$, we have that

$$\begin{aligned}&\omega(\sigma(x_1), \sigma(x_2)) = \omega(x_1 + a_{31}x_3, x_2 - a_{31}x_3) = 1 = \omega(x_1, x_2), \\ &\omega(\sigma(x_1), \sigma(x_3)) = \omega(x_1 + a_{31}x_3, a_{33}x_3) = 0 = \omega(x_1, x_3), \\ &\omega(\sigma(x_2), \sigma(x_3)) = \omega(x_2 - a_{31}x_3, a_{33}x_3) = 0 = \omega(x_2, x_3), \\ &\omega(\sigma(x_1), \sigma(y)) = \omega(x_1 + a_{31}x_3, a_{44}y) = 0 = \omega(x_1, y), \\ &\omega(\sigma(x_2), \sigma(y)) = \omega(x_2 - a_{31}x_3, a_{44}y) = 0 = \omega(x_2, y), \\ &\omega(\sigma(x_3), \sigma(y)) = \omega(a_{33}x_3, a_{44}y) = 0 = \omega(x_3, y), \\ &\omega(\sigma(y), \sigma(y)) = \omega(a_{44}y, a_{44}y) = 0 = \omega(y, y).\end{aligned}$$

For other cases not mentioned above, the Equation (4.2) still holds from the fact that ω is graded-anti-symmetric. So $\mathrm{Aut}^{\omega}(H) = \mathrm{Aut}(H)$.

(3) We see that $\mathrm{Aut}(P_{1,k})$ is a matrix Lie group because it is closed in the general linear group $\mathrm{GL}(4, \mathbb{C})$, see Sagle-Walde ([8], Proposition 7.1).

Consider the catrix exponential exp: $\mathrm{gl}_4(\mathbb{C}) \to \mathrm{GL}(4, \mathbb{C})$ given by

$$X \mapsto \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

for any 4×4 complex matrix X. From Sagle-Walde ([8], Proposition 7.3(a)), we know that $\exp(tX) \in \mathrm{Aut}(P_{1,k})$ for any $t \in \mathbb{R}$ and any derivation $X \in \exp(\mathrm{Der}_{\bar{0}}(P_{1,k}))$, which means $\exp(\mathrm{Der}_{\bar{0}}(P_{1,k})) \subseteq \mathrm{Aut}(P_{1,k})$. In order to show $\exp(\mathrm{Der}_{\bar{0}}(P_{1,k})) = \mathrm{Aut}(P_{1,k})$, we now prove that $\exp(\mathrm{Der}_{\bar{0}}(P_{1,k})) \supseteq \mathrm{Aut}(P_{1,k})$.

For any $\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix} \in \mathrm{Aut}(P_{1,k})$, σ can be seen as a quasi-diagonal matrix $\sigma = \begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix}$ with $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & -a & b \end{pmatrix}$ and $c \in \mathbb{C}$. Since $c \neq 0$, it follows from elementary analysis that $e^x = c$ has a nonzero solution in $\mathbb{C}$, i.e., there is a complex number $0 \neq c_0 \in \mathbb{C}$ such that $e^{c_0} = c$. Acorrding to Chen et al. ([4], Proposition 4.1(2)), for any 3×3 invertible matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & -a & b \end{pmatrix}$, there exists a 3×3 complex matrix $A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_0 & -a_0 & b_0 \end{pmatrix}$ such that $\exp(A_0) = A$. Therefore, $\sigma = \begin{pmatrix} e^{A_0} & 0 \\ 0 & e^{c_0} \end{pmatrix}$. Let $d = \begin{pmatrix} A_0 & 0 \\ 0 & c_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_0 & -a_0 & b_0 & 0 \\ 0 & 0 & 0 & c_0 \end{pmatrix}$, we obtain that $\sigma = \exp(d)$, see Sagle-Walde([8], §2.2.1). Since $d \in \mathrm{Der}_{\bar{0}}(P_{1,k})$, we have $\sigma \in \exp(\mathrm{Der}_{\bar{0}}(H))$.

(4) Similar arguments with Proposition 2.3 (4) show that $\mathrm{Aut}(P_{1,k})$ is a connected matrix Lie group. Since the Lie algebra $\mathrm{Der}_{\bar{0}}(P_{1,k})$ is soluble, it follows from Sagle-walde ([8], Theorem 10.9(b)) that $\mathrm{Aut}(P_{1,k})$ is soluble. □

For the remaining nontrival non-ω-Lie 4-dimensional ω-Lie superalgebras g, we summarize the characterizations about $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$ in Table 3 and Table 4, see **Appendix**.

5. Jordan standard forms of elements in $\mathrm{Der}(g)$ and $\mathrm{Aut}(g)$

In this section, we compute the Jordan standard forms of elements in derivation algebras and automorphism groups about 3-dimensional ω-Lie superalgebra H and all 4-dimensional ω-Lie superalgebras mentioned in Theorem 3.2. We only give the detailed proofs for the cases of P_{21} and R_{kk} since the arguments for the remaining cases of 4-dimensional ω-Lie superalgebras are similar.

Proposition 5.1. *For any $d \in \mathrm{Der}(H)$, the Jordan standard form of d is one of the following:*

$$(1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}; \ (2) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ where } a \in \mathbb{C}.$$

Proof. Since the characteristic polynomial of d is $f(\lambda) = \lambda^2(\lambda - a)$. Our arguments will be separated into two cases: $a = 0$ and $a \neq 0$.

Case 1 $a = 0$, which means $f(\lambda) = \lambda^3$. For $\lambda = 0$, let

$$B = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b & 0 \end{pmatrix}.$$

For the first case where $b = 0$, we see that $J = 0$.

For the second case where $b \neq 0$, we see that $r(B) = 1$, $r(B^2) = r(B^3) = 0$, therefore,

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case 2 $a \neq 0$. In this case, $f(\lambda) = \lambda^2(\lambda - a)$. For $\lambda = 0$, let $B = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b & a \end{pmatrix}$. Since $r(B) = r(B^2) = 1$, we obtain that $J = \mathrm{diag}\{0, 0, a\}$. □

Proposition 5.2. *For any* $\sigma \in \mathrm{Aut}(H)$*, the Jordan standard form of* σ *is* $\mathrm{diag}\{1, 1, a\}$*, where* $a \in \mathbb{C}$*.*

Proof. It is clearly since the representation matrix of any $\sigma \in \mathrm{Aut}(H)$ on some basis is a diagonal matrix. □

Proposition 5.3. *For any* $d \in \mathrm{Der}(P_{21})$*, the Jordan standard form of* d *is one of the following:*

$$(1) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \ (2) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \ \textit{where } a, b \in \mathbb{C}.$$

Proof. The characteristic polynomial of d is $f(\lambda) = \lambda(\lambda - b)(\lambda^2 - a^2 - ec)$.

Case 1 Assume that $a^2 + ec = 0$, which means $f(\lambda) = \lambda^3(\lambda - b)$. Our arguments will be separated into two subcases: $b = 0$, and $b \neq 0$.

Case 1.1 If $b = 0$, then $f(\lambda) = \lambda^4$. For $\lambda = 0$, let

$$B = A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & e & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the first case where $r(B) = 0$, since $a = e = c = 0$, we see that $J = 0$.

For the second case where $r(B) = 1$, direct calculations show that $r(B^2) = 0$ and $r(B^3) = 0$, therefore, $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Case 1.2 If $b \neq 0$, then $f(\lambda) = \lambda^3(\lambda - b)$. For $\lambda = 0$, let

$$B = A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & e & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$$

For the first case where $r(B) = 1$, since $a = e = c = 0$, we obtain $J = \mathrm{diag}\{0, 0, 0, b\}$.

For the second case where $r(B) = 2$, we see that $r(B^2) = 1$, $r(B^3) = 1$ and hence $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$.

Case 2 Assume that $a^2 + ec \neq 0$. In this case $f(\lambda) = \lambda(\lambda - b)(\lambda - \sqrt{a^2 + ec})(\lambda + \sqrt{a^2 + ec})$. Our arguments will be separated into two subcases: $b = 0$, and $b \neq 0$.

Case 2.1 If $b = 0$, then $f(\lambda) = \lambda^2(\lambda - \sqrt{a^2 + ec})(\lambda + \sqrt{a^2 + ec})$. For $\lambda = 0$, let $B = A$. Since $r(B) = r(B^2) = 2$, so $J = \mathrm{diag}\{0, -\sqrt{a^2 + ec}, \sqrt{a^2 + ec}, 0\}$.

Case 2.2 If $b \neq 0$, then $f(\lambda) = \lambda(\lambda - b)(\lambda - \sqrt{a^2 + ec})(\lambda + \sqrt{a^2 + ec})$.

For the first case where $a^2 + ec = b^2$, we see that $f(\lambda) = \lambda(\lambda - b)^2(\lambda + b)$. For $\lambda = b$, let $B = bE - A = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & -a + b & -c & 0 \\ 0 & -e & a + b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, direct calculations show that $B^2 = \begin{pmatrix} b^2 & 0 & 0 & 0 \\ 0 & -2b^2 - 2ab & -2cb & 0 \\ 0 & -2eb & 2b^2 + 2ab & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $r(B) = 2$, $r(B^2) = 2$, we have $J = \mathrm{diag}\{0, -b, b, b\}$.

For the second case where $a^2 + ec \neq b^2$, since $f(\lambda) = \lambda(\lambda - b)(\lambda - \sqrt{a^2 + ec})(\lambda + \sqrt{a^2 + ec})$, we obtain that $J = \mathrm{diag}\{0, -\sqrt{a^2 + ec}, \sqrt{a^2 + ec}, b\}$ □

Proposition 5.4. *For any $\sigma \in \mathrm{Aut}(P_{21})$, the Jordan standard form of σ is one of the following:*

$$(1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & h \end{pmatrix}; \ (2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & h \end{pmatrix}; \ (3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & h \end{pmatrix},$$

where $0 \neq a, h \in \mathbb{C}$.

Proof. The characteristic polynomial of σ is $f(\lambda) = (\lambda - 1)(\lambda - h)(\lambda^2 - (a + f)\lambda + 1)$.

Case 1 Assume that $a + f = 2$. In this case $f(\lambda) = (\lambda - 1)^3(\lambda - h)$. Our arguments will be separated into two subcases: $h = 1$ and $h \neq 1$.

Case 1.1 If $h=1$, then $f(\lambda)=\lambda^4$. For $\lambda=1$, let

$$B=A-E=\begin{pmatrix}0&0&0&0\\0&a-1&b&0\\0&e&f-1&0\\0&0&0&h-1\end{pmatrix}.$$

For the first case where $r(B)=0$, we see that $a=f=1$, $e=b=0$ and hence $J=\mathrm{diag}\{1,1,1,1\}$.

For the second case where $r(B)=1$, it can be derived from $r(B^2)=r(B^3)=0$ that $J=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{pmatrix}$.

Case 1.2 If $h\neq 1$, then $f(\lambda)=(\lambda-1)^3(\lambda-h)$. For $\lambda=1$, let $B=A-E$.

For the first case where $r(B)=1$, we have $a=f=1$, $e=b=0$. It follows from $r(B^2)=1$ that $J=\mathrm{diag}\{1,1,1,h\}$.

For the second case where $r(B)=2$, we see that $r(B^2)=1$, $r(B^3)=1$ and hence $J=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&h\end{pmatrix}$.

Case 2 Assume that $a+f\neq 2$. In this case $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2})(\lambda-h)$. Our arguments will be separated into three subcases: $h=1$, $h=-1$, and $h\neq -1,1$.

Case 2.1 $h=1$, which means $f(\lambda)=(\lambda-1)^2(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2})$.

Case 2.1.1 If $(a+f)^2=4$, then $a+f=-2$ and $f(\lambda)=(\lambda-1)^2(\lambda+1)^2$.

For $\lambda=1$, let $B=A-E$, we have $r(B)=2$, $r(B^2)=2$; for $\lambda=-1$, let $C=A+E=\begin{pmatrix}2&0&0&0\\0&a+1&b&0\\0&e&f+1&0\\0&0&0&2\end{pmatrix}$.

For the first case where $r(C)=2$, we obtain $b=e=0$, $a=f=-1$, and hence $r(C^2)=2$, which gives $J=\mathrm{diag}\{1,-1,-1,1\}$.

For the second case where $r(C)=3$, it follows from $r(C^2)=r(C^3)=2$ that $J=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&1&-1&0\\0&0&0&1\end{pmatrix}$.

Case 2.1.2 If $(a+f)^2\neq 4$, which means

$$f(\lambda)=(\lambda-1)^2(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2}).$$

For $\lambda=1$, let $B=A-E$, then we obtain $r(B)=2$ and $r(B^2)=2$, which gives $J=\mathrm{diag}\{1,\frac{a+f+\sqrt{(a+f)^2-4}}{2},\frac{a+f-\sqrt{(a+f)^2-4}}{2},1\}$.

Case 2.2 $h=-1$, so we have $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2})(\lambda+1)$.

Case 2.2.1 Assume that $(a+f)^2=4$, which means $a+f=-2$ and $f(\lambda)=(\lambda-1)(\lambda+1)^3$. For $\lambda=-1$, let $B=A+E=\begin{pmatrix}2&0&0&0\\0&a+1&b&0\\0&e&f+1&0\\0&0&0&0\end{pmatrix}$, $B^2=\mathrm{diag}\{4,(a+1)^2+eb,(f+1)^2+eb,0\}$,

For the first case where $r(B)=1$, we see that $b=e=0$, $a=f=-1$. Direct calculations show $r(B^2)=1$, so $J=\mathrm{diag}\{1,-1,-1,-1\}$.

For the second case where $r(B)=2$, we note that $r(B^2)=r(B^3)=1$ and hence $J=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&1&-1&0\\0&0&0&-1\end{pmatrix}$.

Case 2.2.2 Assume that $(a+f)^2\neq 4$, which means $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f\pm\sqrt{(a+f)^2-4}}{2})(\lambda+1)$. So $J=\mathrm{diag}\{1,\frac{a+f+\sqrt{(a+f)^2-4}}{2},\frac{a+f-\sqrt{(a+f)^2-4}}{2},-1\}$.

Case 2.3 $h\neq -1,1$, in this case we obtain that $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2})(\lambda-h)$. In this case, our arguments will be separated into two subcases: $(a+f)^2=4$ and $(a+f)^2\neq 4$.

Case 2.3.1 $(a+f)^2=4$, which means $a+f=-2$ and $f(\lambda)=(\lambda-1)(\lambda+1)^2(\lambda-h)$. For $\lambda=-1$, let $B=A+E=\begin{pmatrix}2&0&0&0\\0&a+1&b&0\\0&e&f+1&0\\0&0&0&h+1\end{pmatrix}$,

For the first case where $r(B)=2$, we see that $r(B^2)=2$ and hence $J=\mathrm{diag}\{1,-1,-1,h\}$.

For the second case where $r(B)=3$, it follows from $r(B^2)=r(B^3)=2$ that $J=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&1&-1&0\\0&0&0&h\end{pmatrix}$.

Case 2.3.2 $(a+f)^2\neq 4$, Note that $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f\pm\sqrt{(a+f)^2-4}}{2})(\lambda-h)$.

For the first case where $h^2-(a+f)h+1=0$, we have $f(\lambda)=(\lambda-1)(\lambda-h)^2(\lambda-1/h)$. For $\lambda=h$, let $B=A-hE=\begin{pmatrix}1-h&0&0&0\\0&a-h&b&0\\0&e&f-h&0\\0&0&0&0\end{pmatrix}$. We obtain that $r(B)=2$ and $r(B^2)=2$, so $J=\mathrm{diag}\{1,h,1/h,h\}$.

For the second case where $h^2-(a+f)h+1\neq 0$, we see that $f(\lambda)=(\lambda-1)(\lambda-\frac{a+f+\sqrt{(a+f)^2-4}}{2})(\lambda-\frac{a+f-\sqrt{(a+f)^2-4}}{2})(\lambda-h)$ and this identity implies $J=\mathrm{diag}\{1,\frac{a+f+\sqrt{(a+f)^2-4}}{2},\frac{a+f-\sqrt{(a+f)^2-4}}{2},h\}$. □

Proposition 5.5. *For any $d\in \mathrm{Der}(R_{kk})$, the Jordan standard form of d is one of the following:*

$$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&b\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&b&0\\0&0&1&b\end{pmatrix};\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&a&0\\0&0&0&b\end{pmatrix},$$

where $a,b\in\mathbb{C}$.

Proof. The characteristic polynomial of d is $f(\lambda)=\lambda^2(\lambda^2-(a+b)\lambda+ab-ec)$.

Case 1 Assume that $ab=ec$ and $a+b=0$. In this case $f(\lambda)=\lambda^4$. For $\lambda=0$, let $B^2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\(k-1)fa+(k-1)he&fa+he&a^2+ec&(a+b)e\\(k-1)fc+(k-1)hb&fc+hb&(a+b)c&b^2+ec\end{pmatrix}$, $B=A=\begin{pmatrix}0&0&0&0\\0&0&0&0\\(k-1)f&f&a&e\\(k-1)h&h&c&b\end{pmatrix}$. Our arguments will be separated into two subcases: $ec=0$ and $ec\neq 0$.

Case 1.1 $ec=0$, which means $a=b=0$.

For the first case where $r(B)=0$, we see that $a=b=e=c=f=h=0$, and hence $J=0$.

For the second case where $r(B)=1$, it follows from $r(B^2)=r(B^3)=0$ that $J=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$.

For the third case where $r(B)=2$, direct calculations show $r(B^2)=1$, hence we obtain that $J=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$.

Case 1.2 If $ec\neq 0$, then $a\neq 0$, $b\neq 0$ and $a+b=0$. For $\lambda=0$, let $B=A$.

For the first case where $r(B)=1$, we obtain that $\frac{f}{h}=\frac{a}{c}=\frac{e}{b}$. It follows from $r(B^2)=r(B^3)=0$ that $J=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}$.

For the second case where $r(B) = 2$, we see that $\frac{f}{h} \neq \frac{a}{c} = \frac{e}{b}$, where f and h are not all zero. Since $r(B^2) = r(B^3) = 1$, we have $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Case 2 Assume that $ab = ec$ and $a + b \neq 0$, then $f(\lambda) = \lambda^3(\lambda - (a + b))$. For $\lambda = 0$, let $B = A$. In this case, our arguments will be separated into two subcases: $r(B) = 1$ and $r(B) = 2$.

Case 2.1 If $r(B) = 1$, then $r(B^2) = 1$ and $J = \mathrm{diag}\{0, 0, 0, a + b\}$.

Case 2.2 If $r(B) = 2$, then $\frac{f}{h} \neq \frac{a}{c} = \frac{e}{b}$. since $r(B^2) = 1$, we obtain that

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a+b \end{pmatrix}.$$

Case 3 Assume that $ab \neq ec$. In this case we obtain that $f(\lambda) = \lambda^2(\lambda - \frac{a+b+\sqrt{(a-b)^2+4ec}}{2})(\lambda - \frac{a+b-\sqrt{(a-b)^2+4ec}}{2})$. We could also separate our arguments into two subcases: $(a - b)^2 + 4ec = 0$ and $(a - b)^2 + 4ec \neq 0$.

Case 3.1 If $(a - b)^2 + 4ec = 0$, then $f(\lambda) = \lambda^2(\lambda - \frac{a+b}{2})^2$.

For $\lambda = 0$, let $B = A$, we obtain that $r(B) = 2$, $r(B^2) = 2$. For $\lambda = \frac{a+b}{2}$, let $C = A - \frac{a+b}{2}E$, from direct calculations, we have

$$C = \begin{pmatrix} -\frac{a+b}{2} & 0 & 0 & 0 \\ 0 & -\frac{a+b}{2} & 0 & 0 \\ (k-1)f & -f & \frac{a-b}{2} & -e \\ (k-1)h & -h & -c & \frac{b-a}{2} \end{pmatrix},$$

and

$$C^2 = \begin{pmatrix} \frac{(a+b)^2}{4} & 0 & 0 & 0 \\ 0 & \frac{(a+b)^2}{4} & 0 & 0 \\ -(k-1)fb - (k-1)eh & fb + eh & \frac{(a-b)^2}{4} + ec & 0 \\ -(k-1)fc - (k-1)ah & fc + ah & 0 & \frac{(a-b)^2}{4} + ec \end{pmatrix}.$$

For the first case where $r(C) = 2$, it follows from $r(C^2) = 2$ that $J = \mathrm{diag}\{0, 0, \frac{a+b}{2}, \frac{a+b}{2}\}$.

For the second case where $r(C) = 3$, we see that $r(C^2) = r(C^3) = 2$, therefore, $J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a+b}{2} & 0 \\ 0 & 0 & 1 & \frac{a+b}{2} \end{pmatrix}$.

Case 3.2 If $(a-b)^2+4ec \neq 0$, then $f(\lambda) = \lambda^2(\lambda - \frac{a+b+\sqrt{(a-b)^2+4ec}}{2})(\lambda - \frac{a+b-\sqrt{(a-b)^2+4ec}}{2})$. For $\lambda = 0$, let $B = A$, note that $r(B) = 2$ and $r(B^2) = 2$, so we obtain the fact $J = \mathrm{diag}\{0, 0, \frac{a+b+\sqrt{(a-b)^2+4ec}}{2}, \frac{a+b-\sqrt{(a-b)^2+4ec}}{2}\}$. $\square$

Proposition 5.6. *For any $\sigma \in \mathrm{Aut}(R_{kk})$, the Jordan standard form of σ is one of the following:*

$$(1)\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&f\end{pmatrix};\ (2)\ \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&1&a\end{pmatrix},\ \textit{where } 0\neq a,f\in\mathbb{C}.$$

Proof. The characteristic polynomial of σ is $f(\lambda)=(\lambda-1)^2(\lambda^2-(a+f)\lambda+af-eb)$. Our arguments will be separated into three subcases: $af-eb=1$, $a+f=2$; $af-eb=a+f-1$, $a+f\neq 2$ and $af-eb\neq a+f-1$.

Case 1 Assume that $af-eb=1$ and $a+f=2$, in this case $f(\lambda)=(\lambda-1)^4$. For $\lambda=1$, let $B=A-E=\begin{pmatrix}0&0&0&0\\0&a-1&0&0\\0&0&a-1&b\\0&0&e&f-1\end{pmatrix}$, immediately, $B^2=\mathrm{diag}\{0,0,(a-1)^2+eb,(f-1)^2+eb\}$.

For the first case where $r(B)=0$, note that $a=f=1$, $e=b=0$, we have $J=\mathrm{diag}\{1,1,1,1\}$.

For the second case where $r(B)=1$, we see that $r(B^2)=0$. Therefore, $J=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{pmatrix}$.

Case 2 Assume that $af-eb=a+f-1$ and $a+f\neq 2$, which means $f(\lambda)=(\lambda-1)^3(\lambda-(a+f-1))$. For $\lambda=1$, let $B=A-E$, it is straightforward to show that $r(B)=1$ and $r(B^2)=1$. Therefore, $J=\mathrm{diag}\{1,1,1,a+f-1\}$.

Case 3 Assume that $af-eb\neq a+f-1$. In this case $f(\lambda)=(\lambda-1)^2(\lambda-\frac{a+f+\sqrt{(a-f)^2+4eb}}{2})(\lambda-\frac{a+f-\sqrt{(a-f)^2+4eb}}{2})$. Our arguments will be separated into two subcases: $(a-f)^2+4eb=0$ and $(a-f)^2+4eb\neq 0$.

Case 3.1 If $(a-f)^2+4eb=0$, then $f(\lambda)=(\lambda-1)^2(\lambda-\frac{a+f}{2})^2$ $(a+f\neq 2)$. For $\lambda=1$, let $B=A-E$, we get $r(B)=2$, $r(B^2)=2$; For $\lambda=\frac{a+f}{2}$, let $C=A-\frac{a+f}{2}E=\begin{pmatrix}1-\frac{a+f}{2}&0&0&0\\0&1-\frac{a+f}{2}&0&0\\0&0&\frac{a-f}{2}&b\\0&0&e&\frac{f-a}{2}\end{pmatrix}$, clearly, $C^2=\mathrm{diag}\{(1-\frac{a+f}{2})^2,(1-\frac{a+f}{2})^2,0,0\}$.

For the first case where $r(C)=2$, it implies that $r(C^2)=2$, so $J=\mathrm{diag}\{1,1,\frac{a+f}{2},\frac{a+f}{2}\}$.

For the second case where $r(C)=3$, since $r(C^2)=2$ and $r(C^3)=2$, we obtain that $J=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\frac{a+f}{2}&0\\0&0&1&\frac{a+f}{2}\end{pmatrix}$.

Case 3.2 If $(a-f)^2+4eb\neq 0$, then we see that $f(\lambda)=(\lambda-1)^2(\lambda-\frac{a+f+\sqrt{(a-f)^2+4eb}}{2})(\lambda-\frac{a+f-\sqrt{(a-f)^2+4eb}}{2})$. For $\lambda=1$, let $B=A-E$, we have $r(B)=r(B^2)=2$, so $J=\text{diag}\{1,1,\frac{a+f+\sqrt{(a+f)^2+4eb}}{2},\frac{a+f-\sqrt{(a+f)^2+4eb}}{2}\}$. □

The Jordan standard forms about elements in $\text{Der}(g)$ and $\text{Aut}(g)$ for the remaining nontrival non-ω-Lie 4-dimensional ω-Lie superalgebras are summarized in Table 5 and Table 6, see **Appendix**.

6. Representations of ω-Lie superalgebras

This section will give a sufficient and necessary condition for an ω-Lie superalgebra with 1-dimensional module, prove that all 3-dimensional and 4-dimensional nontrivial non-ω-Lie complex ω-Lie superalgebras are multiplicative, and show that any irreducible respresentation of the 4-dimensional ω-Lie superalgebra $P_{2,k}(k\neq 0,-1)$ is 1-dimensional, steming from our previous work [11] in which fundamental results on representations and semidirect products of ω-Lie superalgebras have been formulated.

Definition 6.1. [11] A representation of an ω-Lie superalgebra g on a $\mathbb{Z}_2$-graded vector space V is an even-graded linear homomorphism $\varphi: g\to End(V)$ satisfying $\varphi([x,y])=\varphi(x)\varphi(y)-(-1)^{|x||y|}\varphi(y)\varphi(x)+\omega(x,y)\text{id}$, for all homogeneous element $x,y\in g$.

Theorem 6.2. *[11] Let g be an ω-Lie superalgebra, V be a $\mathbb{Z}_2$-graded vector space and φ a representation of g on V. The $\mathbb{Z}_2$-graded vector space $g\oplus V$, where $(g\oplus V)_\gamma=g_\gamma\oplus V_\gamma$, for $\gamma\in\mathbb{Z}_2$, provided with the following bracket and a bilinear form ω defined respectively by*

$$[(x_1,v_1),(x_2,v_2)]=([x_1,x_2],\varphi(x_1)v_2-(-1)^{|x_1||x_2|}\varphi(x_2)v_1),$$

$$\omega((x_1,v_1),(x_2,v_2))=\omega(x_1,x_2),\ \forall(x_1,v_1),(x_2,v_2)\in hg(g\oplus V),$$

is an ω-Lie superalgebra. we call $g\oplus V$ the semidirect product of the ω-Lie superalgebra g and V.

Proposition 6.3. *g is a finite-dimensional nontrivial ω-Lie superalgebra over $\mathbb{C}$. Then $\mathbb{C}$ is a 1-dimension g-module if and only if there exists a linear functional $\tau\in g^*$ such that $\omega(x,y)=\tau([x,y])$ for all $x,y\in g$.*

Proof. $\Rightarrow$ We define the $\mathbb{Z}_2$-gradation of vector space $\mathbb{C}$ by $\mathbb{C}_{\bar 0}=\mathbb{C}$, $\mathbb{C}_{\bar 1}=0$. Suppose $0\neq c\in\mathbb{C}$, since $\mathbb{C}$ is a 1-dimension g-module, then for any homogeneous element $x\in g$, there exists a $\tau: g\to\mathbb{C}$ such that $\varphi(x)(c)=\tau(x)\cdot c$ (Obviously, $\tau(g_{\bar 1})=0$). Since $\varphi(g)$ on $\mathbb{C}$ is bilinear, $\tau\in g^*$ (the definition of dual space could be seen in [9]) is a linear functional. Thus for any homogeneous element $x, y \in g$, we have

$$\begin{aligned}\tau([x,y])\cdot c &= \varphi([x,y])(c)\\ &= \varphi(x)(\varphi(y)(c)) - (-1)^{|x||y|}\varphi(y)(\varphi(x)(c)) + \omega(x,y)\cdot c\\ &= \tau(x)\cdot(\tau(y)\cdot c) - (-1)^{|x||y|}\tau(y)\cdot(\tau(x)\cdot c) + \omega(x,y)\cdot c\\ &= \omega(x,y)\cdot c.\end{aligned}$$

Hence, $\omega(x,y) = \tau([x,y])$ for all $x, y \in g$.

$\Leftarrow$ Since there exists a linear functional $\tau \in g^*$ such that $\omega(x,y) = \tau([x,y])$, then we could define a map $\varphi : g \to End(\mathbb{C})$ by $\varphi(x)(c) := \tau(x)\cdot c$ for any $x \in g$ and $c \in \mathbb{C}$. Clearly, φ is bilinear. Moreover,

$$\begin{aligned}\varphi([x,y])(c) &= \tau(x,y)\cdot c\\ &= \omega(x,y)\cdot c\\ &= \omega(x,y)\cdot c + \tau(x)\cdot(\tau(y)\cdot c) - (-1)^{|x||y|}\tau(y)\cdot(\tau(x)\cdot c)\\ &= \varphi(x)(\varphi(y)(c)) - (-1)^{|x||y|}\varphi(y)(\varphi(x)(c)) + \omega(x,y)\cdot c.\end{aligned}$$

Which means that $\mathbb{C}$ is a g-module. □

Remark 6.4. Due to Chen et al. [4], an ω-Lie algebra having 1-dimensional module is called multiplicative. Analogously, we call an ω-Lie superalgebra with 1-dimensional module a multiplicative ω-Lie superalgebra.

Proposition 6.5. *3-dimensional complex ω-Lie superalgebra H is multiplicative.*

Proof. Suppose $\{x_1, x_2, y\}$ is a basis for H and $\{x_1^*, x_2^*, y^*\}$ a dual basis of H^*. We can take $\tau = x_1^*$, and see $\tau([x_1, x_2]) = \omega(x_1, x_2) = 1$. □

From Chen et al. ([4], Proposition 6.4) and Proposition 6.5, one can check that all 3-dimensional complex ω-Lie superalgebras are multiplicative.

Proposition 6.6. *All nontrivial non-ω-Lie 4-dimensional ω-Lie superalgebras are multiplicative.*

Proof. Since the classification of nontrivial non-ω-Lie 4-dimensional ω-Lie superalgebras has been shown in ([11], Theorem 3.3), we need only to prove that $P_{1,k}$, N_k, M_k, Q, $P_{2,k}$, S_k, T_k, $R_{t,k}$ are multiplicative. From Proposition 6.3, it suffices to find a linear functional $\tau \in g^*$ such that $\omega(x,y) = \tau([x,y])$ for all $x, y \in g$. Actually, Let $\{x_1^*,\ x_2^*,\ x_3^*\ , y^*\}$ be a dual basis of g^* when g is $P_{1,k},\ N_k,\ M_k,\ Q,\ P_{2,k}$, and let $\{x_1^*,\ x_2^*,\ y_1^*\ , y_2^*\}$ be a dual basis when g is $S_k,\ T_k,\ R_{t,k}$. For $P_{1,k}$, N_k, S_k, T_k and $R_{t,k}$, such τ could be x_2^*. For M_k, we can take τ for $-x_3^*$. We can take $\tau = 2x_1^*$ for Q, and $\tau = (1+k)x_1^*$ for $P_{2,k}$. □

Theorem 6.7. *Any irreducible $P_{2,k}$-module is 1-dimensional.*

Proof. Recall that $P_{2,k}: [x_1, x_2] = x_2,\ [x_1, x_3] = kx_3,\ [x_2, x_3] = x_1,\ [x_1, y] = (1+k)y, [x_2, y] = [x_3, y] = [y, y] = 0,\ \omega(x_2, x_3) = 1 + k,\ \omega(y, y) = 0,\ k \neq 0, -1$. where $x_1, x_2, x_3 \in L_{\bar{0}}$, $y \in L_{\bar{1}}$. Since $P_{2,k} = C_k \oplus \tilde{P}$, where $\tilde{P} = \{ty \mid t \in \mathbb{C}\}$, and C_k is a 3-dimension ω-Lie algebra mentioned in Chen et al. ([4]. Theorem 2.1).

Suppose that V is an irreducible finite-dimensional $P_{2,k}$-module, so V is also a finite-dimensional C_k-module, which means that for every element $x \in C_k$, $\varphi(x)$ is a linear map from V to itself. Since the linear map $\varphi(x_2)$ is over $\mathbb{C}$, it must have an eigenvector $v_0 \neq 0$. Let λ_0 be the corresponding eigenvalue satisfying $\varphi(x_2)(v_0) = \lambda_0 v_0$. We claim that for $k \in \mathbb{N}^+$,

$$\varphi(x_1)(\varphi(x_2)^k(v_0)) = (\lambda_0 + k)\varphi(x_2)^k(v_0).$$

In fact, it follows from Definition 6.1 that

$$\begin{aligned}\varphi(x_1)(\varphi(x_2)(v_0)) &= \varphi([x_1, x_2])(v_0) + \varphi(x_2)(\varphi(x_1)(v_0)) - \omega(x_1, x_2)v_0 \\ &= \varphi(x_2)(v_0) + \varphi(x_2)(\lambda_0 v_0) \\ &= (\lambda_0 + 1)\varphi(x_2)(v_0).\end{aligned}$$

Suppose $\varphi(x_1)(\varphi(x_2)^{k-1}(v_0)) = (\lambda_0 + k - 1)\varphi(x_2)^{k-1}(v_0)$, by induction hyperthesis, we have

$$\begin{aligned}\varphi(x_1)(\varphi^k(x_2)(v_0)) &= \varphi(x_1)(\varphi(x_2)(\varphi^{k-1}(x_2)(v_0))) \\ &= \varphi([x_1, x_2])(\varphi^{k-1}(x_2)(v_0))) + \varphi(x_2)(\varphi(x_1)(\varphi^{k-1}(x_2)(v_0)))) \\ &\quad - \omega(x_1, x_2)\varphi^{k-1}(x_2)(v_0) \\ &= \varphi(x_2)(\varphi^{k-1}(x_2)(v_0))) + \varphi(x_2)(\varphi(x_1)(\varphi^{k-1}(x_2)(v_0)))) \\ &= \varphi^k(x_2)(v_0) + \varphi(x_2)((\lambda_0 + k - 1)\varphi^{k-1}(x_2)(v_0)) \\ &= (\lambda_0 + k)\varphi^k(x_2)(v_0).\end{aligned}$$

This means that the vectors $\{\varphi^k(x_2)(v_0) \mid k = 0, 1, 2......\}$ are either zero or eigenvectors of $\varphi(x_1)$. Note that any two eigenvalues in $\{\lambda_0 + k \mid k = 0, 1, 2......\}$ are distinct, and since the eigenvectors corresponding to different eigenvalues are linear indepenednt, we may find a minimal integer k_0 such that $\varphi^{k_0}(x_2)(v_0) \neq 0$ and $\varphi^{k_0+1}(x_2)(v_0) = 0$. Let $v = \varphi^{k_0}(x_2)(v_0)$, $\lambda = \lambda_0 + k_0$, we see that $\varphi(x_2)(v) = 0$ and $\varphi(x_1)(v) = \lambda v$. We also obtain $\lambda \neq 0$, $\varphi(x_3)(v) = 0$, according to Chen et al. ([4], Theorem 7.1).

From

$$\varphi([x_1, y])v = \varphi(x_1)(\varphi(y)v) - \varphi(y)(\varphi(x_1)v) + \omega(x_1, y)v.$$

$$\varphi([x_2, y])v = \varphi(x_2)(\varphi(y)v) - \varphi(y)(\varphi(x_2)v) + \omega(x_2, y)v,$$

and

$$\varphi([x_3, y])v = \varphi(x_3)(\varphi(y)v) - \varphi(y)(\varphi(x_3)v) + \omega(x_3, y)v,$$

we obtain $\varphi(x_1)(\varphi(y)v) = (1+k+\lambda)\varphi(y)v$, $\varphi(x_2)(\varphi(y)v) = \varphi(x_3)(\varphi(y)v) = 0$. Combining with the equation

$$\varphi([x_2, x_3])(\varphi(y)v) = \varphi(x_2)(\varphi(x_3)(\varphi(y)v)) - \varphi(x_3)(\varphi(x_2)(\varphi(y)v)) + \omega(x_2, x_3)\varphi(y)v,$$

we obtain the fact that $\varphi(y)v = 0$.

Let $V_0 = \{kv \mid k = 0, 1, 2......\}$, the subspace generated by v, then $V_0 \neq \{0\}$. To see that V_0 is a $P_{2,k}$-submodule of V, we only have to show that V_0 is stable under the action of $\varphi(x_1)$, $\varphi(x_2)$, $\varphi(x_3)$ and $\varphi(y)$. Obviously, we have $\varphi(x_1)(V_0) \subseteq V_0$, $\varphi(x_2)(V_0) = \varphi(x_3)(V_0) = \varphi(y)(V_0) = \{0\} \subseteq V_0$ from the arguments above. Therefore, V_0 is a nonzero $P_{2,k}$-submodule of V. Since V is irreducible, we have $V = V_0$, so $\dim V = \dim V_0 = 1$. □

7. Appendix

The characterizations about Der(g), Aut(g) and Jordan standard forms of elements in Der(g) and Aut(g) for the remaining nontrival non-ω-Lie 4-dimensional ω-Lie superalgebras are summarized in the following Table 3, Table 4, Table 5 and Table 6, without the detailed proofs.

Table 3: Derivations about 4-dimensional ω-Lie superalgebras

g	Elements in Der(g)	dimensions	properties
$P_{1,1}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & -a & b & 0 \\ e & f & 0 & c \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 3$ $\dim \mathrm{Der}(g) = 5$	$\mathrm{Der}_{\bar{0}}(g)$: Soluble $\mathrm{Der}(g)$: Soluble
$P_{1,k}$ $(k \neq 1)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & -a & b & 0 \\ e & 0 & 0 & c \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 3$ $\dim \mathrm{Der}(g) = 4$	$\mathrm{Der}_{\bar{0}}(g)$: Soluble $\mathrm{Der}(g)$: Soluble
N_0	$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & b & c \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 2$ $\dim \mathrm{Der}(g) = 3$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Soluble
N_k $(k \neq 0)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k^2b & kb & b & a \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 1$ $\dim \mathrm{Der}(g) = 2$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Soluble
M_k	$\begin{pmatrix} 0 & 2a & a & 0 \\ 0 & 0 & 2a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 2$ $\dim \mathrm{Der}(g) = 2$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Abelian
Q	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 2$ $\dim \mathrm{Der}(g) = 2$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Abelian
$P_{2,1}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & c & 0 \\ 0 & e & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 4$ $\dim \mathrm{Der}(g) = 4$	$\mathrm{Der}_{\bar{0}}(g)$: Not Soluble $\mathrm{Der}(g)$: Not Soluble
$P_{2,k}$ $(k \neq 1)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$	$\dim \mathrm{Der}_{\bar{0}}(g) = 2$ $\dim \mathrm{Der}(g) = 2$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Abelian

T_k	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (k-2)c & c & a & 0 \\ (k-1)b-(k-2)c & b & 0 & a \end{pmatrix}$	$\dim\mathrm{Der}_{\bar{0}}(g)=1$ $\dim\mathrm{Der}(g)=3$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Soluble
$R_{k,k}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (k-1)f & f & a & e \\ (k-1)h & h & c & b \end{pmatrix}$	$\dim\mathrm{Der}_{\bar{0}}(g)=4$ $\dim\mathrm{Der}(g)=6$	$\mathrm{Der}_{\bar{0}}(g)$: Not Soluble $\mathrm{Der}(g)$: Not Soluble
$R_{t,k}$ $(t\neq k)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (t-1)f & f & a & 0 \\ (k-1)h & h & 0 & b \end{pmatrix}$	$\dim\mathrm{Der}_{\bar{0}}(g)=2$ $\dim\mathrm{Der}(g)=4$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Soluble
S_1	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & a & 0 \\ c & e & b & a \end{pmatrix}$	$\dim\mathrm{Der}_{\bar{0}}(g)=2$ $\dim\mathrm{Der}(g)=4$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Not Soluble
S_k $(k\neq 1)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (k-1)c & c & a & 0 \\ e & (e-c)/(k-1) & b & a \end{pmatrix}$	$\dim\mathrm{Der}_{\bar{0}}(g)=2$ $\dim\mathrm{Der}(g)=4$	$\mathrm{Der}_{\bar{0}}(g)$: Abelian $\mathrm{Der}(g)$: Soluble

Table 4: Automorphisms about 4-dimensional ω-Lie superalgebras

g	Relations in g	Elements in $\mathrm{Aut}(g)$	properties of $\mathrm{Aut}(g)$
$P_{1,k}$	$[x_1,x_2]=x_2, [x_2,x_3]=x_3$ $[x_1,y]=ky, \omega(x_1,x_2)=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & -a & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}$, $b\neq 0, c\neq 0$	Soluble
N_0	$[x_1,x_3]=x_2, [x_2,x_3]=x_3$ $[x_2,y]=y, \omega(x_1,x_3)=1$	$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$, $a\neq 0, b\neq 0$	Abelian
N_k $(k\neq 0)$	$[x_1,x_3]=x_2, [x_2,x_3]=x_3$ $[x_1,y]=ky, [x_2,y]=y$ $\omega(x_1,x_3)=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$, $a\neq 0$	Abelian
M_k	$[x_1,x_2]=x_1, [x_3,y]=-y,$ $[x_1,x_3]=x_1+x_2,$ $[x_2,x_3]=kx_1+x_3,$ $\omega(x_2,x_3)=-1$	$\begin{pmatrix} 1 & a & (a^2+a)/2 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$, $b\neq 0$	Abelian
Q	$[x_1,x_2]=x_2, [x_1,y]=2y,$ $[x_1,x_3]=x_2+x_3,$ $[x_2,x_3]=x_1, \omega(x_2,x_3)=2$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$, $b\neq 0$	Abelian
$P_{2,1}$	$[x_1,x_2]=x_2, [x_1,x_3]=x_3,$ $[x_2,x_3]=x_1, [x_1,y]=2y,$ $\omega(x_2,x_3)=2,$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & e & f & 0 \\ 0 & 0 & 0 & h \end{pmatrix}$, $\begin{vmatrix} a & b \\ e & f \end{vmatrix}=1$	Not soluble
$P_{2,k}$ $(k\neq 1)$	$[x_1,x_2]=x_2, [x_2,x_3]=x_1,$ $[x_1,x_3]=kx_3,$ $[x_1,y]=(1+k)y,$ $\omega(x_2,x_3)=1+k,$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$, $a\neq 0, b\neq 0$	Abelian
T_k	$[x_1,y_1]=(k-1)y_1,$ $[x_2,y_1]=y_1+y_2, [x_1,x_2]=x_2,$ $[x_1,y_2]=ky_2, [x_2,y_2]=y_2,$ $\omega(x_1,x_2)=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}$, $a\neq 0$	Abelian
$R_{k,k}$	$[x_1,y_1]=ky_1, [x_2,y_1]=y_1,$ $[x_1,x_2]=x_2, [x_1,y_2]=ky_2,$ $[x_2,y_2]=y_2, \omega(x_1,x_2)=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & e & f \end{pmatrix}$, $\begin{vmatrix} a & b \\ e & f \end{vmatrix}\neq 0$	Not soluble
$R_{t,k}$ $(t\neq k)$	$[x_1,y_1]=ty_1, [x_2,y_1]=y_1,$ $[x_1,x_2]=x_2, [x_1,y_2]=ky_2,$ $[x_2,y_2]=y_2, \omega(x_1,x_2)=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$, $a\neq 0, b\neq 0$	Abelian

S_k	$[x_1, y_1] = ky_1 + y_2$, $[x_2, y_1] = y_1, [x_1, x_2] = x_2$, $[x_1, y_2] = ky_2, [x_2, y_2] = y_2$, $\omega(x_1, x_2) = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & b & a \end{pmatrix}$, $a \neq 0$	Abelian

Table 5: Jordan standard forms about elements in $\mathrm{Der}(g)$

g	Jordan standard form about elements in $\mathrm{Der}(g)$
$P_{1,1}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, b, c \in \mathbb{C}.$
$P_{1,k}\ (k \neq 1)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, b, c \in \mathbb{C}.$
N_0	$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 1 & -a \end{pmatrix}; \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & c \end{pmatrix}, a, c \in \mathbb{C}.$
$N_k\ (k \neq 0)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, a \in \mathbb{C}.$
M_k	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, b \in \mathbb{C}.$
Q	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, b \in \mathbb{C}.$
$P_{2,1}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, a, b \in \mathbb{C}.$
$P_{2,k}\ (k \neq 1)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, a, b \in \mathbb{C}.$
T_k	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, a \in \mathbb{C}.$
R_{kk}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 1 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}.$
$R_{tk}\ (t \neq k)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, a, b \in \mathbb{C}.$
S_1	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$
$S_k\ (k \neq 2)$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}.$

Table 6: Jordan standard forms about elements in $\mathrm{Aut}(g)$

g	Jordan standard form about elements in $\mathrm{Aut}(g)$

$P_{1,k}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&b&0\\0&0&0&c\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&c\end{pmatrix}, 0\neq b,c\in\mathbb{C}.$
N_0	$\begin{pmatrix}a&0&0&0\\0&1&0&0\\0&0&1/a&0\\0&0&0&b\end{pmatrix}, 0\neq a,b\in\mathbb{C}.$
$N_k(k\neq 0)$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&a\end{pmatrix}, 0\neq a\in\mathbb{C}.$
M_k	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&b\end{pmatrix};\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&0&b\end{pmatrix}, 0\neq b\in\mathbb{C}.$
Q	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&b\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&b\end{pmatrix}, 0\neq b\in\mathbb{C}.$
$P_{2,1}$	$\begin{pmatrix}1&0&0&0\\0&a&0&0\\0&0&1/a&0\\0&0&0&h\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&h\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&1&-1&0\\0&0&0&h\end{pmatrix}, 0\neq a,h\in\mathbb{C}.$
$P_{2,k}\ (k\neq 1)$	$\begin{pmatrix}1&0&0&0\\0&a&0&0\\0&0&1/a&0\\0&0&0&b\end{pmatrix}, 0\neq a,b\in\mathbb{C}.$
T_k	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&a\end{pmatrix}, 0\neq a\in\mathbb{C}.$
R_{kk}	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&f\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&1&a\end{pmatrix}, 0\neq a,f\in\mathbb{C}.$
$R_{tk}\ (t\neq k)$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&b\end{pmatrix}, 0\neq a,b\in\mathbb{C}.$
S_k	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&0&a\end{pmatrix};\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&a&0\\0&0&1&a\end{pmatrix}, 0\neq a\in\mathbb{C}.$

8. Acknowledgements

The authors would like to thank the referee for valuable comments and suggestions on this article. Supported by NNSF of China (Nos. 11771069 and 12071405).

References

[1] Bobienski, M., Nurowski, P.:Irreducible SO(3) geometry in dimension five. J. Reine Angew. Math., **605**: 51–93 (2007). `https://doi.org/10.1515/CRELLE.2007.027`

[2] Chen Y., Liu C. and Zhang R.: Classification of three dimensional complex ω-Lie algebras. Port. Math., **71**, 97–108 (2014). `https://www.ems-ph.org/journals/show_abstract.php?issn=0032-5155&vol=71&iss=2&rank=2`

[3] Chen Y., Zhang R.: Simple ω-Lie algebras and 4-dimensional ω-Lie algebras over $\mathbb{C}$ Bull. Malays. Math. Sci. Soc., **40**(3), 1377–1390 (2017). https://link.springer.com/article/10.1007/s40840-015-0120-6

[4] Chen Y., Zhang Z., Zhang R. and Zhuang R.: Derivations, Automorphisms, and Representations of Complex ω-Lie algebras Comm. Algebra, **46**(2), 708–726 (2018). https://www.tandfonline.com/doi/abs/10.1080/00927872.2017.1327062?journalCode=lagb20

[5] Hall B.: Lie groups, Lie algebras, and representations: an elementary introduction. GTM222. Springer-Verlag (2000). https://www.springer.com/us/book/9783319134666#aboutBook

[6] Nurowski P.: Deforming a Lie algebra by means of a 2-form J. Geom. Phys., **57**, 1325–1329 (2007). https://www.sciencedirect.com/science/article/pii/S039304400600146X?via%3Dihub

[7] Nurowski P.: Distinguished dimensions for special Riemannian geometries. J. Geom. Phys., **58**, 1148–1170 (2008). https://mathscinet.ams.org/mathscinet-getitem?mr=2451275

[8] Sagle A., Walde R.: Introduction to Lie groups and Lie algebras. Academic Press (1973). https://www.researchgate.net/publication/44501120_Introduction_to_Lie_groups_and_Lie_algebras_Arthur_A_Sagle_Ralph_E_Walde

[9] Scheunert M.: The Theory of Lie Superalgebras: An Introduction. Springer, New York(1979). https://link.springer.com/book/10.1007%2FBFb0070929

[10] Wang Z.: Classifications of low-dimension Lie superalgebras Dissertation, East China Normal University (2008)(in Chinese). https://d.wanfangdata.com.cn/thesis/Y895657

[11] Zhou J., Chen L., Ma Y. and Sun B.: On ω-Lie superalgebras J. Algebra Appl., **17**(10), 18pp (2018). https://doi.org/10.1142/S0219498818502122

[12] Zusmanovich P.: ω-Lie algebras J. Geom. Phys., **60**, 1028–1044 (2010). https://www.sciencedirect.com/science/article/pii/S0393044010000604?via%3Dihub

Jia zhou
Department of Mathematics, Shanghai Normal University
100, Guilin Rd.
Xuhui District, Shanghai, China
College of Information Technology, Jilin Agricultural University
2888, Xincheng Str.
Nanguan District, Changchun, China
e-mail: 1316023705@qq.com

Liangyun Chen
School of Mathematics and Statistics, Northeast Normal University
5268, Renmin Str.
Changchun, China
e-mail: chenly640@nenu.edu.cn