# One demonstration of the minimum entropy theorem ruling the fluid flow field.

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**Research Article**

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One demonstration of the minimum entropy theorem ruling the fluid flow field.

It is known that the second theorem of the increase of the entropy S and the minimum entropy theorem of the several fields. [1)]

But there is no clear review that it shows the possibility of minimum entropy theorem , the general viscous fluid flow field is derived from the stationary of the entropy S. I intend to demonstrate the above in this article.

At first, one functional F is defined as the time-differential total external enegy Wt of the flow field by the space-dependant parameter(velocities,density etc) at a certain time t=t0 , using of continuity and general Navier-Stokes equation. And it is shown that the functional F on the Euler description is solved as that δF≡0, identically zero, by variational method. Therefore , the flow field satisfies the conditions of continuity and general Navier-Stokes equations.

On the second, the Wt of the flow field on an unit time is descrived by using the reversible energy Et (ex. kinetic energy, internal energy) and the entropy St (ex. viscous dissipation energy) as like Wt = Et + St.

Then if the Wt is assumed under the conditions at the a certain time and a period in appropriate size of field, the condition stationary δWt=0 means that Wt is expected to be a constant value. Here the theorem of minimum entropy production is well known, so the condition of stationary δSt=0 decides δEt=0 (=δWt- δSt) . Consequently the stationary δSt directly derives the solution of the viscous fluid flow, as mentioned in the top paragraph

As a result , it will help an understanding of the importance of the theorem of minimum entropy production and the simplification of the simulation of several fields.

Actual demonstration is shown as follows.

As I said above, a certain time and period energy function is defined like as below,

$$\frac{dW}{dt}=\left\{\Phi+\frac{d}{dt}\frac{1}{2}\rho(u^2+v^2+w^2)-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\}$$

The first term of the right-hand side,general dissipation function (viscous fuluid) $\Phi$ means

$$\Phi = \mu\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right\}$$

, μ is viscosity coefficient of fluid and defined as a constant value.

And the second term $\frac{d}{dt}\frac{1}{2}\rho(u^2+v^2+w^2)$ means the kinetic energy, $\rho$ is the fluid density and $u, v, w$ are velocities in $x, y, z$ axes.

The third term of the product of pressure $p$ and $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$ means the external energy, because $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$ equals $\frac{1}{\delta V}\frac{D\delta V}{Dt}$ ; the volume rate of change , which is derived from the continuity equation

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) + \left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right) = 0$$

and the definition of a small volume element $\delta V = \delta x\delta y\delta z$ and $\delta x, \delta y, \delta z \to 0$.

Total energy per an certain unit time; functional Wt is defined as below in the Euler coordinates

$$\mathrm{Wt} = \iiint \frac{dW}{dt}\, dxdydz$$

$$= \iiint \left\{\Phi + \frac{d}{dt}\frac{1}{2}\rho(u^2+v^2+w^2) - p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\} dxdydz$$

Then the intergrated function F is defined as described below,

$$\mathrm{F} = \Phi + \frac{\partial}{\partial t}\frac{1}{2}\rho(u^2+v^2+w^2) - p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$= \Phi + \frac{\partial \rho}{\partial t}\frac{1}{2}(u^2+v^2+w^2) + \rho\frac{\partial}{\partial t}\left(\frac{1}{2}(u^2+v^2+w^2)\right) - p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

Here, the continuity equation and the condition of the constant $\rho$ value in space $(\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = \frac{\partial \rho}{\partial z} = 0)$ derives $\frac{\partial \rho}{\partial t} = -\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$.

Then, the last equation becomes

$$\mathrm{F} = \Phi - \rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\frac{1}{2}(u^2+v^2+w^2) + \rho\left(u\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}+w\frac{\partial w}{\partial t}\right)$$

$$- p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

Here secondly, $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}$ are derived from general Navier-Stokes equation.

For example , in the x-axis of general Navier-Stokes equation ,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right)$$

So,

$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-w\frac{\partial u}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right)$$

Then $\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}$ are expressed as follows,

$$\frac{\partial v}{\partial t}=-u\frac{\partial v}{\partial x}-v\frac{\partial v}{\partial y}-w\frac{\partial v}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial^2 x}+\frac{\partial^2 v}{\partial^2 y}+\frac{\partial^2 v}{\partial^2 z}\right)$$

$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial x}-v\frac{\partial w}{\partial y}-w\frac{\partial w}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial^2 x}+\frac{\partial^2 w}{\partial^2 y}+\frac{\partial^2 w}{\partial^2 z}\right)$$

After the $\frac{\partial u}{\partial t},\frac{\partial v}{\partial t},\frac{\partial w}{\partial t}$ terms of function F is substituted by above equations and leads the stationary resolution using the the variational derivaitive as below,

$$[\mathrm{F}\,]_u=\frac{\partial \mathrm{F}}{\partial u}-\frac{\partial}{\partial x}\frac{\partial \mathrm{F}}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial \mathrm{F}}{\partial u_y}-\frac{\partial}{\partial z}\frac{\partial \mathrm{F}}{\partial u_z}+\frac{\partial^2}{\partial^2 x}\frac{\partial \mathrm{F}}{\partial u_{xx}}+\frac{\partial^2}{\partial^2 y}\frac{\partial \mathrm{F}}{\partial u_{yy}}+\frac{\partial^2}{\partial^2 z}\frac{\partial \mathrm{F}}{\partial u_{zz}}$$

$$[\mathrm{F}\,]_v=\frac{\partial \mathrm{F}}{\partial v}-\frac{\partial}{\partial x}\frac{\partial \mathrm{F}}{\partial v_x}-\frac{\partial}{\partial y}\frac{\partial \mathrm{F}}{\partial v_y}-\frac{\partial}{\partial z}\frac{\partial \mathrm{F}}{\partial v_z}+\frac{\partial^2}{\partial^2 x}\frac{\partial \mathrm{F}}{\partial v_{xx}}+\frac{\partial^2}{\partial^2 y}\frac{\partial \mathrm{F}}{\partial v_{yy}}+\frac{\partial^2}{\partial^2 z}\frac{\partial \mathrm{F}}{\partial v_{zz}}$$

$$[\mathrm{F}\,]_w=\frac{\partial \mathrm{F}}{\partial w}-\frac{\partial}{\partial x}\frac{\partial \mathrm{F}}{\partial w_x}-\frac{\partial}{\partial y}\frac{\partial \mathrm{F}}{\partial w_y}-\frac{\partial}{\partial z}\frac{\partial \mathrm{F}}{\partial w_z}+\frac{\partial^2}{\partial^2 x}\frac{\partial \mathrm{F}}{\partial w_{xx}}+\frac{\partial^2}{\partial^2 y}\frac{\partial \mathrm{F}}{\partial w_{yy}}+\frac{\partial^2}{\partial^2 z}\frac{\partial \mathrm{F}}{\partial w_{zz}}$$

$$[\mathrm{F}\,]_p=\frac{\partial \mathrm{F}}{\partial p}-\frac{\partial}{\partial x}\frac{\partial \mathrm{F}}{\partial p_x}-\frac{\partial}{\partial y}\frac{\partial \mathrm{F}}{\partial p_y}-\frac{\partial}{\partial z}\frac{\partial \mathrm{F}}{\partial p_z}$$

For example, actualy functional F substitute $[\mathrm{F}\,]_u$ then the first term of the right side F equation

$$[\Phi\,]_u=-2\ \mu\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right)$$

the second term,

$$\left[-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\frac{1}{2}(u^2+v^2+w^2)\right]_u$$

$$=-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)u+\ \rho\frac{\partial}{\partial x}\left\{\frac{1}{2}(u^2+v^2+w^2)\right\}$$

the third term,

$$\left[\rho\left(u\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}+w\frac{\partial w}{\partial t}\right)\right]_{u}$$

$$=-\frac{\partial p}{\partial x}+2\ \mu\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right)-\rho\frac{\partial}{\partial x}\left\{\frac{1}{2}\left(u^2+v^2+w^2\right)\right\}$$

the fourth term,

$$\left[-p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]_{u}=\frac{\partial p}{\partial x}$$

Therfore,

$$[\mathrm{F}\,]_{u}=-\,\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)u=0$$

Here, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ is defined as the continuity equation of the incompressible fluid,

As a result, the variational derivaitive equations are derived as

$[\mathrm{F}\,]_{u}\equiv 0, [\mathrm{F}\,]_{v}\equiv 0, [\mathrm{F}\,]_{w}\equiv 0, [\mathrm{F}\,]_{p}\equiv 0$ ; identically zero. They satisfies Euler equation, so that means the stationary function $\delta$Wt=0 of the flow field.

And at the same time, $\delta$Wt=0 is derived by the conditions of restricted continuity and general Navier-Stokes equations, so those are satisfied in this flow field.

On the second, the total enegy Wt of the flow field from outside on an unit time is descrived by using the reversible energy Et (ex. kinetic energy, internal energy) and the entropy St (ex. viscous dissipation energy) as Wt= St + Et.

By the variational method $\delta$Wt=0 as mentioned above, $\delta$Wt = $\delta$St+ $\delta$Et = 0.

Therefore the condition of stationary δSt=0 means δEt=0 (=δWt- δSt). So the appropriate function St is defined as mentioned above,

$$\mathrm{St}=\iiint \Phi\ dxdydz$$

, then it resolves the imcompressible viscous fluid flow field that satisfies general Navier-Stokes equation on the continuity equation. Consequently it shows the importance of the minimum entropy theorm and may help the reduction of the computational calculation load by saving the calculation of the equation of motion.

Reference

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