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Bicomplex Version of Lebesgue's Dominated Convergence Theorem and Hyperbolic Invariant Measure

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Abstract

In this article we have studied bicomplex valued measurable functions on an arbitrary measurable space. We have established the bicomplex version of Lebesgue's dominated convergence theorem and some other results related to this theorem. Also we have proved the bicomplex version of Lebesgue-Radon-Nikodym theorem. Finally we have introduced the idea of hyperbolic version of invariant measure.

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1 Introduction

In 1882 Corrado Segre [10] introduced a new number system called bicomplex numbers. Unlike quaternions this number system is a commutative generalization of complex numbers by four reals. The book of G. B. Price [8] is a good resource of the analysis of bicomplex numbers. Many works have been done on bicomplex functional analysis. Few researchers have worked on bicomplex dynamics, bicomplex topological modules. In this article we have studied bicomplex valued measurable functions on an arbitrary measurable space. We have established the bicomplex version of Lebesgue's dominated convergence theorem and some other results related to this theorem. Also we have proved the bicomplex version of Lebesgue-Radon-Nikodym theorem. Finally we have introduced the idea of hyperbolic version of invariant measure. To prove the results in the first two subsections in our main results we have used the ideas of the book of W. Rudin [9] and for the results in the last subsection we have used [7] and [11].

2 Basis definitions

We denote the set of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$ respectively. We may think three imaginary numbers $\mathbf{i}_1, \mathbf{i}_2$ and $\mathbf{j}$ governed by the rules

$$\mathbf{i}_1^2 = -1, \mathbf{i}_2^2 = -1, \mathbf{j}^2 = 1$$

$$\begin{aligned} \mathbf{i}_1\mathbf{i}_2 &= \mathbf{i}_2\mathbf{i}_1 = \mathbf{j} \\ \mathbf{i}_1\mathbf{j} &= \mathbf{j}\mathbf{i}_1 = -\mathbf{i}_2 \\ \mathbf{i}_2\mathbf{j} &= \mathbf{j}\mathbf{i}_2 = -\mathbf{i}_1. \end{aligned}$$

Then we have two complex planes $\mathbb{C}(\mathbf{i}_1) = \{x + \mathbf{i}_1 y : x, y \in \mathbb{R}\}$ and $\mathbb{C}(\mathbf{i}_2) = \{x + \mathbf{i}_2 y : x, y \in \mathbb{R}\}$, both of which are identical to $\mathbb{C}$. Bicomplex numbers are defined as $\zeta = z_1 + \mathbf{i}_2 z_2$ for $z_1, z_2 \in \mathbb{C}(\mathbf{i}_1)$. The set of all bicomplex numbers is denoted by $\mathbb{T}$. In particular if $z_1 = x, z_2 = \mathbf{i}_1 y$ where $x, y \in \mathbb{R}$ we get $\zeta = x + \mathbf{j}y$ and these type of numbers are called hyperbolic numbers or duplex numbers. The set of all hyperbolic numbers is denoted by $\mathbb{D}$. For $(z_1 + \mathbf{i}_2 z_2), (w_1 + \mathbf{i}_2 w_2) \in \mathbb{T}$, the addition and multiplication are definde as

$$\begin{aligned} (z_1 + \mathbf{i}_2 z_2) + (w_1 + \mathbf{i}_2 w_2) &= (z_1 + w_1) + \mathbf{i}_2 (z_2 + w_2) \\ (z_1 + \mathbf{i}_2 z_2)(w_1 + \mathbf{i}_2 w_2) &= (z_1 w_1 - z_2 w_2) + \mathbf{i}_2 (z_1 w_2 + z_2 w_1). \end{aligned}$$

With these operations $\mathbb{T}$ forms a commutative ring with zero divisors. The elements $z_1 + \mathbf{i}_2 z_2 \in \mathbb{T}$ such that $z_1^2 + z_2^2 = 0$ are the zero divisors. The interesting property of a bicomplex number is its idempotent representation. Setting $\mathbf{e}_1 = \frac{1+\mathbf{j}}{2}$ and $\mathbf{e}_2 = \frac{1-\mathbf{j}}{2}$, we get

$$z_1 + \mathbf{i}_2 z_2 = (z_1 - \mathbf{i}_1 z_2)\,\mathbf{e}_1 + (z_1 + \mathbf{i}_1 z_2)\,\mathbf{e}_2.$$

Many calculations become easier for this representation.

Throughout this article we will consider $\mathfrak{M}$ to be a $\sigma-$algebra in a set X, unless stated otherwise.

2.1 Partial order on $\mathbb{D}$

The set of nonnegative hyperbolic numbers is

$$\mathbb{D}^+ = \{\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2 : \nu_1, \nu_2 \geq 0\}.$$

A hyperbolic number ζ is said to be (strictly) positive if $\zeta \in \mathbb{D}^+ \setminus \{0\}$ .The set of nonnegative hyperbolic numbers is also defined as

$$\mathbb{D}^+ = \left\{x + y\mathbf{k} : x^2 - y^2 \geq 0, x \geq 0\right\}.$$

On the realization of $\mathbb{D}^+$, M.E. Luna-Elizarraras et.al.[6] defined a partial order relation on $\mathbb{D}$. For two hyperbolic numbers ζ_1, ζ_2 the relation $\preceq_{\mathbb{D}}$ is defined as

$$\zeta_1 \preceq_{\mathbb{D}} \zeta_2 \text{ if and only if } \zeta_2 - \zeta_1 \in \mathbb{D}^+.$$

One can check that this relation is reflexive, transitive and antisymmetric. Therefore $\preceq_{\mathbb{D}}$ is a partial order relation on $\mathbb{D}$. This partial order relation $\preceq_{\mathbb{D}}$ on $\mathbb{D}$ is an extension of the total order relation $\leq$ on $\mathbb{R}$. We say $\zeta_1 \prec_{\mathbb{D}} \zeta_2$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ but $\zeta_1 \neq \zeta_2$. Also we say $\zeta_2 \succeq_{\mathbb{D}} \zeta_1$ if $\zeta_1 \preceq_{\mathbb{D}} \zeta_2$ and $\zeta_2 \succ_{\mathbb{D}} \zeta_1$ if $\zeta_1 \prec_{\mathbb{D}} \zeta_2$.

Definition 1 *For any hyperbolic number $\zeta = \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2$, the $\mathbb{D}-$modulus of ζ is defined by*

$$|\zeta|_{\mathbb{D}} = |\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2|_{\mathbb{D}} = |\nu_1|\,\mathbf{e}_1 + |\nu_2|\,\mathbf{e}_2 \in \mathbb{D}^+$$

where $|\nu_1|$ and $|\nu_2|$ are the usual modulus of real numbers.

Definition 2 *A subset A of $\mathbb{D}$ is said to be $\mathbb{D}-$bounded if there exists $M \in \mathbb{D}^+$ such that $|\zeta|_{\mathbb{D}} \preceq_{\mathbb{D}} M$ for any $\zeta \in A$.*

Set

$$\begin{aligned} A_1 &= \{x \in \mathbb{R} : \exists\, y \in \mathbb{R},\ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}, \\ A_2 &= \{y \in \mathbb{R} : \exists\, x \in \mathbb{R},\ x\mathbf{e}_1 + y\mathbf{e}_2 \in A\}. \end{aligned}$$

If A is $\mathbb{D}-$bounded then A_1 and A_2 are bounded subset of $\mathbb{R}$.

Definition 3 *For a $\mathbb{D}-$bounded subset A of $\mathbb{D}$, the **supremum** of A with respect to the $\mathbb{D}-$modulus is defined by*

$$\sup\nolimits_{\mathbb{D}} A = \sup A_1\mathbf{e}_1 + \sup A_2\mathbf{e}_2.$$

Definition 4 *A sequence of hyperbolic numbers* $\{\zeta_n\}_{n\geq 1}$ *is said to be* ***convergent to*** $\zeta \in \mathbb{D}$ *if for* $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ *there exists* $k \in \mathbb{N}$ *such that*

$$|\zeta_n - \zeta|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon.$$

Then we write

$$\lim_{n\to\infty} \zeta_n = \zeta.$$

Definition 5 *A sequence of hyperbolic numbers* $\{\zeta_n\}_{n\geq 1}$ *is said to be* $\mathbb{D}-$***Cauchy sequence*** $\zeta \in \mathbb{D}$ *if for* $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ $\exists$ $N \in \mathbb{N}$ *such that*

$$\left|\zeta_{N+m} - \zeta_N\right|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon$$

for all $m = 1, 2, 3, \ldots$ *.*

Note that a sequence of hyperbolic numbers $\{\zeta_n\}_{n\geq 1}$ is **convergent** if and only if it is a $\mathbb{D}-$**Cauchy sequence.**

Definition 6 *A hyperbolic series* $\sum_{n=1}^{\infty} \zeta_n$ *is* ***convergent*** *if and only if its partial sums is a* $\mathbb{D}-$*Cauchy sequence, i.e., for any* $\varepsilon \in \mathbb{D}^+ \setminus \{0\}$ $\exists$ $N \in \mathbb{N}$ *such that*

$$\left|\sum_{k=1}^{m} \zeta_{N+k}\right|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon$$

for any $m \in \mathbb{N}$.

Definition 7 *A hyperbolic series* $\sum_{n=1}^{\infty} \zeta_n$ *is* $\mathbb{D}-$***absolutely convergent*** *if the series* $\sum_{n=1}^{\infty} |\zeta_n|_{\mathbb{D}}$ *is convergent.*

Every $\mathbb{D}-$absolutely convergent series is convergent.

3 Main Results

In this section we have established our main results. We have arranged these in three subsections. In the first subsection we have proved the bicomplex version of Lebesgue's dominated convergence theorem. In the second subsection we have focussed on the bicomplex version of Lebesgue-Radon-Nikodym theorem and also we have established bicomplex version of Hahn decomposition theorem. Finally in the last subsection we have introduced the idea of hyperbolic version of invariant measure.

3.1 Bicomplex Version of Lebesgue's Dominated Convergence Theorem

Definition 8 *[2] Let X be a measurable space then the bicomplex valued function $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ is called $\mathbb{T}-$measurable on X if f_1 and f_2 are complex measurable functions on X. In particular if f_1 and f_2 are real measurable functions on X then $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ is called $\mathbb{D}-$measurable function on X.*

For a bicomplex measurable function $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ one can easily check that $|f| = |f_1|\,\mathbf{e}_1 + |f_2|\,\mathbf{e}_2$ is $\mathbb{D}-$measurable. Also for two $\mathbb{T}-$measurable functions f and g it is routine check up that $f + g$ and fg are also $\mathbb{T}-$measurable functions.

Theorem 1 *If f is a $\mathbb{T}-$measurable function on a measurable space X then there is a $\mathbb{T}-$measurable function $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$ such that $|\alpha_1| = 1, |\alpha_2| = 1$ and $f = \alpha\,|f|$.*

Proof. Let $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$.

Since f is a $\mathbb{T}-$measurable function on a measurable space X, f_1 and f_2 are complex measurable functions on X.

So there exist complex measurable functions α_1, α_2 such that $|\alpha_1| = 1, |\alpha_2| = 1$ and $f_1 = \alpha_1\,|f_1|\,, f_2 = \alpha_2\,|f_2|\,.$

Set $\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$. Obviously α is a $\mathbb{T}-$measurable function on X and the result follows.

Definition 9 *Let $\mathfrak{M}$ be a $\sigma-$algebra in a set X. A bicomplex function $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ defined on X is called a $\mathbb{T}-$measure on $\mathfrak{M}$ if μ_1,μ_2 are complex measures on $\mathfrak{M}$. In particular if μ_1,μ_2 are positive measures on $\mathfrak{M}$ i.e range of both μ_1,μ_2 are $[0, \infty]$ then μ is called a $\mathbb{D}-$measure on $\mathfrak{M}$ and if μ_1,μ_2 are real measures on $\mathfrak{M}$ i.e range of both μ_1,μ_2 are $[0, \infty)$ then μ is called a $\mathbb{D}^+-$measure on $\mathfrak{M}$.*

Definition 10 *Let $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ be a $\mathbb{D}-$measure on an arbitrary measurable space X. We say $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ to be bicomplex Lebesgue integrable function on X if*

$$\int_X |f_1|\,d\mu_1 \;<\; \infty,$$

$$\int_X |f_2|\,d\mu_2 \;<\; \infty$$

i.e., $f_i \in L^1(\mu_i)$, the set of all complex Lebesgue integrable functions with respect to μ_i for $i = 1, 2$.

In that case we write

$$\int_X f d\mu = \left(\int_X f_1 d\mu_1\right) \mathbf{e}_1 + \left(\int_X f_2 d\mu_2\right) \mathbf{e}_2.$$

We define $L^1_{\mathbb{T}}(\mu)$ to be the collection of all bicomplex Lebesgue integrable functions on X.

Theorem 2 *Let $f, g \in L^1_{\mathbb{T}}(\mu)$ and $\alpha, \beta \in \mathbb{C}$. Then $\alpha f + \beta g \in L^1_{\mathbb{T}}(\mu)$, and*

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

Proof. Let $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2, g = g_1\mathbf{e}_1 + g_2\mathbf{e}_2, \mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2, \alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2, \beta = \beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2$.

Then

$$\alpha f + \beta g = (\alpha_1 f_1 + \beta_1 g_1)\, \mathbf{e}_1 + (\alpha_2 f_2 + \beta_2 g_2)\, \mathbf{e}_2.$$

Since $f, g \in L^1_{\mathbb{T}}(\mu)$ we have $\alpha_1 f_1 + \beta_1 g_1 \in L^1(\mu_1)$ and $\alpha_2 f_2 + \beta_2 g_2 \in L^1(\mu_2)$ and therefore $\alpha f + \beta g \in L^1_{\mathbb{T}}(\mu)$.

The last part follows from the facts

$$\begin{aligned} \int_X (\alpha_1 f_1 + \beta_1 g_1)\, d\mu_1 &= \alpha_1 \int_X f_1 d\mu_1 + \beta_1 \int_X g_1 d\mu_1, \\ \int_X (\alpha_2 f_2 + \beta_2 g_2)\, d\mu_2 &= \alpha_2 \int_X f_2 d\mu_2 + \beta_2 \int_X g_2 d\mu_2, \end{aligned}$$

and

$$\int_X (\alpha f + \beta g)\, d\mu = \left(\int_X (\alpha_1 f_1 + \beta_1 g_1)\, d\mu_1\right) \mathbf{e}_1 + \left(\int_X (\alpha_2 f_2 + \beta_2 g_2)\, d\mu_2\right) \mathbf{e}_2.$$

Theorem 3 *If $f \in L^1_{\mathbb{T}}(\mu)$, then*

$$\left|\int_X f d\mu\right|_{\mathbb{D}} \preceq_{\mathbb{D}} \int_X |f|_{\mathbb{D}}\, d\mu.$$

Proof. Let $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2, \mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$.

Since $f \in L^1_{\mathbb{T}}(\mu)$, we have $f_1 \in L^1(\mu_1)$ and $f_2 \in L^1(\mu_2)$.

Therefore,

$$\begin{aligned}\left|\int_X f d\mu\right|_{\mathbb{D}} &= \left|\left(\int_X f_1 d\mu_1\right)\mathbf{e}_1 + \left(\int_X f_2 d\mu_2\right)\mathbf{e}_2\right|_{\mathbb{D}} \\ &= \left|\int_X f_1 d\mu_1\right|\mathbf{e}_1 + \left|\int_X f_2 d\mu_2\right|\mathbf{e}_2 \preceq_{\mathbb{D}} \int_X |f_1|\, d\mu_1\mathbf{e}_1 + \int_X |f_2|\, d\mu_2\mathbf{e}_2 = \int_X |f|_{\mathbb{D}}\, d\mu.\end{aligned}$$

Theorem 4 (Lebesgue's Dominated Convergence Theorem) *Let* $\{f_n = f_{n1}\mathbf{e}_1 + f_{n2}\mathbf{e}_2\}$ *be a sequence of* $\mathbb{T}-$*measurable functions on* X *such that*

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists for all $x \in X$. *If there exists* $g = g_1\mathbf{e}_1 + g_2\mathbf{e}_2 \in L^1_{\mathbb{T}}(\mu)$ *such that*

$$|f_{ni}(x)| \leq g_i(x)$$

for all $n = 1, 2, 3, \ldots$; $i = 1, 2$; $x \in X$, *then* $f \in L^1_{\mathbb{T}}(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n(x) - f(x)|\, d\mu = 0,$$

and

$$\lim_{n\to\infty} \int_X f_n(x)\, d\mu = f(x).$$

Proof. Since $\{f_n\}$ is a sequence of $\mathbb{T}-$measurable functions on X, both $\{f_{n1}\}$ and $\{f_{n2}\}$ are sequences of complex measurable functions on X.

Let $f = f_1\mathbf{e}_1 + f_2\mathbf{e}_2, \mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$.

Thus

$$\lim_{n\to\infty} f_{ni}(x) = f_i(x)$$

for $i = 1, 2$.

Now since

$$|f_{ni}(x)| \leq g_i(x)$$

for all $n = 1, 2, 3, \ldots$; $i = 1, 2$; $x \in X$, we get $f_i \in L^1(\mu_i)$ for $i = 1, 2$ and therefore $f \in L^1_{\mathbb{T}}(\mu)$.

Also,

$$\lim_{n\to\infty}\int_X |f_{ni}(x) - f_i(x)|\, d\mu_i = 0,$$

and

$$\lim_{n\to\infty}\int_X f_{n_i}(x)\, d\mu_i = f_i(x)$$

for $i = 1, 2$. Hence

$$\begin{aligned} & \lim_{n\to\infty}\int_X |f_n(x) - f(x)|_{\mathbb{D}}\, d\mu \\ = & \lim_{n\to\infty}\left(\int_X |f_{n1}(x) - f_1(x)|\, d\mu_1\right)\mathbf{e}_1 + \lim_{n\to\infty}\left(\int_X |f_{n2}(x) - f_2(x)|\, d\mu_2\right)\mathbf{e}_2 \\ = & \ 0 \end{aligned}$$

and

$$\begin{aligned} & \lim_{n\to\infty}\int_X f_n(x)\, d\mu \\ = & \lim_{n\to\infty}\left(\int_X f_{n1}(x)\, d\mu_1\right)\mathbf{e}_1 + \lim_{n\to\infty}\left(\int_X f_{n2}(x)\, d\mu_2\right)\mathbf{e}_2 \\ = & \ f_1(x)\,\mathbf{e}_1 + f_2(x)\,\mathbf{e}_2 \\ = & \ f(x). \end{aligned}$$

3.2 Bicomplex Version of Lebesgue-Radon-Nikodym Theorem

Let $\mathfrak{M}$ be a measure space and $E \in \mathfrak{M}$. Let $P = \{E_k\}$ be a partition of E. Then for all $E \in \mathfrak{M}$ the $\mathbb{T}-$measure $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ on $\mathfrak{M}$ satisfies

$$\mu(E) = \sum_{k=1}^{\infty} \mu(E_k)$$

for every partition $\{E_k\}$ of E.

Let $\lambda = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2$ be a $\mathbb{D}-$measure on $\mathfrak{M}$. We say that λ dominates μ on $\mathfrak{M}$ if $|\mu_i(E)| \leq \lambda_i(E)$ for all $E \in \mathfrak{M}$ and for $i = 1, 2$. The $\mathbb{D}-$modulus of μ, denoted by $|\mu|_{\mathbb{D}}$, is defined on $\mathfrak{M}$ by

$$|\mu|_{\mathbb{D}}(E) = \sup_{P} \sum_{k=1}^{\infty} |\mu(E_k)|_{\mathbb{D}}$$

for all $E \in \mathfrak{M}$.

Theorem 5 *For a $\mathbb{T}-$measure μ on $\mathfrak{M}$, $|\mu|_{\mathbb{D}}$ is a $\mathbb{D}-$measure on $\mathfrak{M}$.*

Proof. Let $\mu = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2$.

Since μ_1,μ_2 being complex measures on $\mathfrak{M}$, $|\mu_1|$ and $|\mu_2|$ are positive measures on $\mathfrak{M}$.

Hence $|\mu|_{\mathbb{D}} = |\mu_1| \mathbf{e}_1 + |\mu_2| \mathbf{e}_2$ is a $\mathbb{D}-$measure on $\mathfrak{M}$.

Theorem 6 *For a $\mathbb{T}-$measure μ on X,*

$$|\mu|_{\mathbb{D}}(X) \prec_{\mathbb{D}} \infty_{\mathbb{D}}.$$

Proof. Let $\mu = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2$.

Since μ_1,μ_2 being complex measures on $\mathfrak{M}$,

$$|\mu_i|(X) < \infty$$

for $i = 1, 2$.

Hence

$$|\mu|_{\mathbb{D}}(X) = |\mu_1|(X) \mathbf{e}_1 + |\mu_2|(X) \mathbf{e}_2 \prec_{\mathbb{D}} \infty_{\mathbb{D}}.$$

Let μ, λ be two $\mathbb{T}-$measures on $\mathfrak{M}$ and $c \in \mathbb{D}$. For all $E \in \mathfrak{M}$ define

$$\begin{aligned} (\mu + \lambda)(E) &= \mu(E) + \lambda(E), \\ (c\mu)(E) &= c\mu(E). \end{aligned}$$

One can easily check that $\mu + \lambda$ and $c\mu$ are also $\mathbb{T}-$measures on $\mathfrak{M}$. The collection of all $\mathbb{T}-$measures on $\mathfrak{M}$ forms a module space over $\mathbb{D}$.

Slight modifying the definition of hyperbolic valued signed measure from [3], we now define it to be a $\mathbb{T}-$measure on $\mathfrak{M}$ having range in $\mathbb{D}^+ \cup \mathbb{D}^-$. Let μ be a signed $\mathbb{D}-$measure on $\mathfrak{M}$. Then both $\mu^+ = \frac{1}{2}(|\mu|_{\mathbb{D}} + \mu)$ and $\mu^- = \frac{1}{2}(|\mu|_{\mathbb{D}} - \mu)$ are $\mathbb{D}-$measures on $\mathfrak{M}$. Obviously μ^+ and μ^- are $\mathbb{D}-$bounded. Also the Jordan decomposition of a signed $\mathbb{D}-$measure is given by

$$\begin{aligned} \mu &= \mu^+ - \mu^-, \\ |\mu|_{\mathbb{D}} &= \mu^+ + \mu^-. \end{aligned}$$

Definition 11 *Let $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ be a $\mathbb{D}-$measure and $\lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ be a $\mathbb{T}-$measure on $\mathfrak{M}$. Then λ is said to be absolutely $\mathbb{T}-$continuous with respect to μ if λ_i is absolutely continuous with respect to μ_i for $i = 1, 2$. We denote this by $\lambda \ll_{\mathbb{T}} \mu$.*

If for $A \in \mathfrak{M}$, λ_i is concentrated on A for $i = 1, 2$, then λ is said to be $\mathbb{T}-$concentrated on A.

Two $\mathbb{T}-$measures $\lambda' = \lambda_1'\mathbf{e}_1 + \lambda_2'\mathbf{e}_2, \lambda'' = \lambda_1''\mathbf{e}_1 + \lambda_2''\mathbf{e}_2$ on $\mathfrak{M}$ are called mutually $\mathbb{T}-$singular if λ_i' and λ_i'' are mutually singular for $i = 1, 2$. We denote this by $\lambda' \perp_{\mathbb{T}} \lambda''$.

Theorem 7 *Let λ, λ' and λ'' be $\mathbb{T}-$measures on $\mathfrak{M}$. Also let μ be a $\mathbb{D}-$measure on $\mathfrak{M}$. Then the following hold:*

a) If λ is $\mathbb{T}-$concentrated on A, then $|\lambda|_{\mathbb{D}}$ is also so.

b) If $\lambda' \perp_{\mathbb{T}} \lambda''$, then $|\lambda'|_{\mathbb{D}} \perp_{\mathbb{T}} |\lambda''|_{\mathbb{D}}$.

c) If $\lambda' \perp_{\mathbb{T}} \mu$, $\lambda'' \perp_{\mathbb{T}} \mu$, then $\lambda' + \lambda'' \perp_{\mathbb{T}} \mu$.

d) If $\lambda' \ll_{\mathbb{T}} \mu$, $\lambda'' \ll_{\mathbb{T}} \mu$, then $\lambda' + \lambda'' \ll_{\mathbb{T}} \mu$.

e) If $\lambda \ll_{\mathbb{T}} \mu$, then $|\lambda|_{\mathbb{D}} \ll_{\mathbb{T}} \mu$.

f) If $\lambda' \ll_{\mathbb{T}} \mu$, $\lambda'' \perp_{\mathbb{T}} \mu$, then $\lambda' \perp_{\mathbb{T}} \lambda''$.

g) If $\lambda \ll_{\mathbb{T}} \mu$ and $\lambda \perp_{\mathbb{T}} \mu$ then $\lambda = 0$.

Proof. Let $\lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$, $\lambda' = \lambda_1'\mathbf{e}_1 + \lambda_2'\mathbf{e}_2$, $\lambda'' = \lambda_1''\mathbf{e}_1 + \lambda_2''\mathbf{e}_2$ and $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$.

a) λ is $\mathbb{T}-$concentrated on A implies λ_i is concentrated on A for $i = 1, 2$.

Thus $|\lambda_i|$ is concentrated on A for $i = 1, 2$.

Therefore $|\lambda|_{\mathbb{D}} = |\lambda_1|\,\mathbf{e}_1 + |\lambda_2|\,\mathbf{e}_2$ is $\mathbb{T}-$concentrated on A.

b) $\lambda' \perp_{\mathbb{T}} \lambda''$ implies λ_i' and λ_i'' are mutually singular for $i = 1, 2$.

Thus $|\lambda_i'|$ and $|\lambda_i''|$ are mutually singular for $i = 1, 2$.

Therefore $|\lambda'|_{\mathbb{D}} = |\lambda_1'|\,\mathbf{e}_1 + |\lambda_2'|\,\mathbf{e}_2$ and $|\lambda''|_{\mathbb{D}} = |\lambda_1''|\,\mathbf{e}_1 + |\lambda_2''|\,\mathbf{e}_2$ are mutually $\mathbb{T}-$singular.

c) $\lambda' \perp_{\mathbb{T}} \mu$, $\lambda'' \perp_{\mathbb{T}} \mu$ implies λ_i' and μ_i are mutually singular for $i = 1, 2$ and λ_i'' and μ_i are mutually singular for $i = 1, 2$.

Thus $\lambda_i' + \lambda_i''$ and μ_i are mutually singular for $i = 1, 2$.

Therefore $\lambda' + \lambda'' = (\lambda_1' + \lambda_1'')\,\mathbf{e}_1 + (\lambda_2' + \lambda_2'')\,\mathbf{e}_2$ and μ are mutually $\mathbb{T}-$singular.

d) $\lambda' \ll_{\mathbb{T}} \mu$, $\lambda'' \ll_{\mathbb{T}} \mu$ implies λ_i' is absolutely continuous with respect to μ_i for $i = 1, 2$ and λ_i'' is absolutely continuous with respect to μ_i for $i = 1, 2$.

Thus $\lambda_i' + \lambda_i''$ is absolutely continuous with respect to μ_i for $i = 1, 2$.

Therefore $\lambda' + \lambda'' = (\lambda_1' + \lambda_1'')\,\mathbf{e}_1 + (\lambda_2' + \lambda_2'')\,\mathbf{e}_2$ is absolutely $\mathbb{T}-$continuous with respect to μ.

e) $\lambda \ll_{\mathbb{T}} \mu$ implies λ_i is absolutely continuous with respect to μ_i for $i = 1, 2$.

Thus $|\lambda_i|$ is absolutely continuous with respect to μ_i for $i = 1, 2$.

Therefore $|\lambda|_{\mathbb{D}} = |\lambda_1| \mathbf{e}_1 + |\lambda_2| \mathbf{e}_2$ is absolutely $\mathbb{T}-$continuous with respect to μ.

f) $\lambda' \ll_{\mathbb{T}} \mu$ implies λ'_i is absolutely continuous with respect to μ_i for $i = 1, 2$ and $\lambda'' \perp_{\mathbb{T}} \mu$ implies λ''_i and μ_i are mutually singular for $i = 1, 2$.

Thus λ'_i and λ''_i are mutually singular for $i = 1, 2$.

Therefore $\lambda' = \lambda'_1\mathbf{e}_1 + \lambda'_2\mathbf{e}_2$ and $\lambda'' = \lambda''_1\mathbf{e}_1 + \lambda''_2\mathbf{e}_2$ are mutually $\mathbb{T}-$singular.

g) $\lambda \ll_{\mathbb{T}} \mu$ implies λ_i is absolutely continuous with respect to μ_i for $i = 1, 2$ and $\lambda \perp_{\mathbb{T}} \mu$ implies λ_i and μ_i are mutually singular for $i = 1, 2$.

Thus $\lambda_i = 0$ for $i = 1, 2$.

Therefore $\lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 = 0$.

Theorem 8 (Lebesgue-Radon-Nikodym Theorem) *Let μ be a $\sigma-$finite $\mathbb{D}-$measure on $\mathfrak{M}$, and let λ be $\mathbb{T}-$measure on $\mathfrak{M}$.*

a) There is a unique pair of $\mathbb{T}-$measures λ', λ'' on $\mathfrak{M}$ such that

$$\lambda = \lambda' + \lambda''$$

where $\lambda' \ll_{\mathbb{T}} \mu, \lambda'' \perp_{\mathbb{T}} \mu$. If λ is $\mathbb{D}-$finite measure on $\mathfrak{M}$ then λ', λ'' are also so.

b) For all $E \in \mathfrak{M}$ there is a unique $h \in L^1_{\mathbb{T}}(\mu)$ such that

$$\lambda'(E) = \int_E h d\mu.$$

Proof. Let $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2, \lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$.

Since μ is a $\sigma-$finite $\mathbb{D}-$measure on $\mathfrak{M}$, μ_1 and μ_2 are positive $\sigma-$finite measures on $\mathfrak{M}$. Also λ_1,λ_2 are complex measures on $\mathfrak{M}$.

a) Then for each $i = 1, 2$ there is a unique pair of complex measures λ'_i, λ''_i on $\mathfrak{M}$ such that

$$\lambda_i = \lambda'_i + \lambda''_i$$

where λ'_i is absolutely continuous with respect to μ_i and λ''_i, μ_i are mutually singular. If λ is positive and finite measure on $\mathfrak{M}$ then λ'_i, λ''_i are also so.

Hence the result follows from these facts.

b) For each $i = 1, 2$ and for all $E \in \mathfrak{M}$ there is a unique $h_i \in L^1(\mu_i)$ such that

$$\lambda'_i(E) = \int_E h_i d\mu_i.$$

Therefore for all $E \in \mathfrak{M}$ there is a unique $h = h_1\mathbf{e}_1 + h_2\mathbf{e}_2 \in L^1_{\mathbb{T}}(\mu)$ such that

$$\begin{aligned}
\lambda'(E) &= \lambda'_1(E)\,\mathbf{e}_1 + \lambda'_2(E)\,\mathbf{e}_2 \\
&= \left(\int_E h_1 d\mu_1\right)\mathbf{e}_1 + \left(\int_E h_2 d\mu_2\right)\mathbf{e}_2 \\
&= \int_E h d\mu.
\end{aligned}$$

Theorem 9 *Let λ be $\mathbb{T}-$measure on $\mathfrak{M}$ and μ be $\mathbb{D}-$measure on $\mathfrak{M}$. Then the following are equivalent:*

a) $\lambda \ll_{\mathbb{T}} \mu$.

b) For every $\varepsilon \in \mathbb{D}^+/\{0\}$ there exists $\delta \in \mathbb{D}^+/\{0\}$ such that $|\lambda(E)|_{\mathbb{D}} \prec_{\mathbb{D}} \varepsilon$ for all $E \in \mathfrak{M}$ with $|\mu(E)|_{\mathbb{D}} \prec_{\mathbb{D}} \delta$.

Proof. Let $\lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2, \mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ where λ_1,λ_2 are complex measures and μ_1,μ_2 are positive measures on $\mathfrak{M}$.

Then for $i = 1, 2$ the following two statements are equivalent

i) λ_i is absolutely continuous with respect to μ_i.

ii) For every $\varepsilon_i > 0$ there exists $\delta_i > 0$ such that $|\lambda_i(E)| < \varepsilon_i$ for all $E \in \mathfrak{M}$ with $\mu_i(E) < \delta_i$.

If for $i = 1, 2$, λ_i is absolutely continuous with respect to μ_i we get $\lambda \ll_{\mathbb{T}} \mu$.

Also if for every $\varepsilon_i > 0$ there exists $\delta_i > 0$ such that $|\lambda_i(E)| < \varepsilon_i$ for all $E \in \mathfrak{M}$ with $\mu_i(E) < \delta_i$ we can say for every $\varepsilon = \varepsilon_1\mathbf{e}_1 + \varepsilon_2\mathbf{e}_2 \in \mathbb{D}^+/\{0\}$ there exists $\delta = \delta_1\mathbf{e}_1 + \delta_2\mathbf{e}_2 \in \mathbb{D}^+/\{0\}$ such that $|\lambda(E)|_{\mathbb{D}} = |\lambda_1(E)|\,\mathbf{e}_1 + |\lambda_2(E)|\,\mathbf{e}_2 \prec_{\mathbb{D}} \varepsilon$ for all $E \in \mathfrak{M}$ with $|\mu(E)|_{\mathbb{D}} = |\mu_1(E)|\,\mathbf{e}_1 + |\mu_2(E)|\,\mathbf{e}_2 \prec_{\mathbb{D}} \delta$.

Theorem 10 *Let $\mathfrak{M}$ be a $\sigma-$algebra on X. Let μ be $\mathbb{T}-$measure on $\mathfrak{M}$. Then there exists a $\mathbb{T}-$measurable function h such that $|h(x)|_{\mathbb{D}} = \mathbf{e}_1 + \mathbf{e}_2$ for all $x \in X$ and such that*

$$d\mu = hd\,|\mu|_{\mathbb{D}}\,.$$

Proof. Let $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$.

Since for $i = 1, 2$, μ_i is a complex measure on the $\sigma-$algebra $\mathfrak{M}$ in X, there exists measurable functions h_i such that $|h_i(x)| = 1$ for all $x \in X$ and such that

$$d\mu_i = h_i d\,|\mu_i|\,.$$

Setting $h = h_1\mathbf{e}_1 + h_2\mathbf{e}_2$,we get

$$\begin{aligned} d\mu &= d\mu_1\mathbf{e}_1 + d\mu_2\mathbf{e}_2 \\ &= h_1 d\left|\mu_1\right|\mathbf{e}_1 + h_2 d\left|\mu_2\right|\mathbf{e}_2 \\ &= hd\left|\mu\right|_{\mathbb{D}}. \end{aligned}$$

Theorem 11 *Let μ be a $\mathbb{D}-$measure on $\mathfrak{M}$, $g \in L^1_{\mathbb{T}}(\mu)$, and for all $E \in \mathfrak{M}$*

$$\lambda(E) = \int_E g d\mu.$$

Then for all $E \in \mathfrak{M}$,

$$\left|\lambda\right|_{\mathbb{D}}(E) = \int_E \left|g\right|_{\mathbb{D}} d\mu.$$

Proof. *Let $\mu = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$, $\lambda = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2$ and $g = g_1\mathbf{e}_1 + g_2\mathbf{e}_2$. Then μ_1, μ_2 are positive measures on $\mathfrak{M}$.*

Since $g \in L^1_{\mathbb{T}}(\mu)$, then $g_1 \in L^1(\mu_1)$ and $g_2 \in L^1(\mu_2)$.

Also $\lambda(E) = \int_E g d\mu \Rightarrow \lambda_1(E) = \int_E g_1 d\mu_1$ and $\lambda_2(E) = \int_E g_2 d\mu_2$.

Then by Consequences of the Radon-Nikodym Theorem [9], we have

$$\left|\lambda_1\right|(E) = \int_E \left|g_1\right| d\mu_1 \text{ and } \left|\lambda_2\right|(E) = \int_E \left|g_2\right| d\mu_2.$$

Therefore

$$\begin{aligned} \left|\lambda\right|_{\mathbb{D}}(E) &= \left|\lambda_1\right|(E)\mathbf{e}_1 + \left|\lambda_1\right|(E)\mathbf{e}_2 \\ &= \left(\int_E \left|g_1\right| d\mu_1\right)\mathbf{e}_1 + \left(\int_E \left|g_2\right| d\mu_2\right)\mathbf{e}_2 \\ &= \int_E (\left|g_1\right|\mathbf{e}_1 + \left|g_1\right|\mathbf{e}_2) d(\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2) \\ &= \int_E \left|g\right|_{\mathbb{D}} d\mu. \end{aligned}$$

Theorem 12 (Hahn Decomposition Theorem) *Let $\mathfrak{M}$ be a $\sigma-$algebra on X. Let μ be a $\mathbb{D}^{+}-$measure on $\mathfrak{M}$. Then there exists a partition $\{A, B, C, D\} \subset \mathfrak{M}$ of X such that for all $E \in \mathfrak{M}$,*

$$\mu^{+}(E) = \mu(E \cap A) + \mathbf{e}_1 |\mu(E \cap C)|_{\mathbb{D}} + \mathbf{e}_2 |\mu(E \cap D)|_{\mathbb{D}},$$

$$\mu^{-}(E) = -\mu(E \cap B) - \mu(E \cap C) - \mu(E \cap D) + \mathbf{e}_1 |\mu(E \cap C)|_{\mathbb{D}} + \mathbf{e}_2 |\mu(E \cap D)|_{\mathbb{D}}. \tag{1}$$

Proof. *By Theorem 10, $d\mu = hd|\mu|_{\mathbb{D}}$, where $|h(x)|_{\mathbb{D}} = \mathbf{e}_1 + \mathbf{e}_2$ for all $x \in X$. Since μ is hyperbolic, it follows that h is hyperbolic, hence $h(x) = \pm\mathbf{e}_1 \pm \mathbf{e}_2$ for all $x \in X$. Put*

$$A = \{x : h(x) = \mathbf{e}_1 + \mathbf{e}_2\},$$
$$B = \{x : h(x) = -\mathbf{e}_1 - \mathbf{e}_2\},$$
$$C = \{x : h(x) = \mathbf{e}_1 - \mathbf{e}_2\},$$
$$D = \{x : h(x) = -\mathbf{e}_1 + \mathbf{e}_2\}.$$

Since $\mu^{+} = \frac{1}{2}(|\mu|_{\mathbb{D}} + \mu)$, and since

$$\frac{1}{2}(1+h) = \begin{cases} h & \text{on } A, \\ 0 & \text{on } B, \\ \mathbf{e}_1 & \text{on } C, \\ \mathbf{e}_2 & \text{on } D, \end{cases}$$

we have, for any $E \in \mathfrak{M}$,

$$\begin{aligned} \mu^{+}(E) &= \frac{1}{2}\int_{E}(1+h)d|\mu|_{\mathbb{D}} \\ &= \int_{E\cap A} hd|\mu|_{\mathbb{D}} + \int_{E\cap C} \mathbf{e}_1 d|\mu|_{\mathbb{D}} + \int_{E\cap D} \mathbf{e}_2 d|\mu|_{\mathbb{D}} \\ &= \mu(E \cap A) + \mathbf{e}_1 |\mu(E \cap C)|_{\mathbb{D}} + \mathbf{e}_2 |\mu(E \cap D)|_{\mathbb{D}}. \end{aligned}$$

Since $\mu(E) = \mu(E \cap A) + \mu(E \cap B) + \mu(E \cap C) + \mu(E \cap D)$ and since $\mu = \mu^{+} - \mu^{-}$, (1) follows.

3.3 Hyperbolic Invariant Measure

Definition 12 *Let $\mathfrak{B}$ be a Borel $\sigma-$algebra on a metric space X. A $\mathbb{D}-$measure μ on X is $\mathbb{D}-$finite if $\mu(X) \prec_{\mathbb{D}} \infty_{\mathbb{D}}$ and μ is called a Borel $\mathbb{D}-$probability measure if $\mu(X) = \mathbf{e}_1 + \mathbf{e}_2$ or $\mathbf{e}_1$ or $\mathbf{e}_2$.*

For any $\mathbb{D}-$finite, nonzero, $\mathbb{D}-$measure $\widehat{\mu}$ on X we may define a Borel $\mathbb{D}-$probability measure as

$$\mu(A) = \frac{\widehat{\mu}(A)}{\widehat{\mu}(X)}$$

for all $A \in \mathfrak{B}$.

Throughout this section we will need a set X equipped with a Borel $\sigma-$algebra $\mathfrak{B}$ and a measurable function $f : X \to X$. We denote by $\mathcal{M}_{\mathbb{D}}$, the space of all Borel $\mathbb{D}-$probability measures on X.

Definition 13 *A Borel $\mathbb{D}-$probability measure $\mu = \mu_1 \mathbf{e}_1 + \mu_2 \mathbf{e}_2$ on X is said to be $\mathbb{D}-$invariant with respect to a measurable function $f : X \to X$ if μ_1 and μ_2 are invariant i.e., $\mu_1(f^{-1}(A)) = \mu_1(A)$ and $\mu_2(f^{-1}(A)) = \mu_2(A)$ for all $A \in \mathfrak{B}$.*

Definition 14 (Push-forward of measures) *Let $f_* : \mathcal{M}_{\mathbb{D}} \longrightarrow \mathcal{M}_{\mathbb{D}}$ be the map from the space of $\mathbb{D}-$probability measures to itself, defined by*

$$f_*\mu(A) := \mu\left(f^{-1}(A)\right) = \mu_1\left(f^{-1}(A)\right)\mathbf{e}_1 + \mu_2\left(f^{-1}(A)\right)\mathbf{e}_2 = f_*\mu_1(A)\mathbf{e}_1 + f_*\mu_2(A)\mathbf{e}_2.$$

We call $f_*\mu$ the push forward of μ by f.

The mapping is well defined and μ is invarient iff $f_*\mu = \mu$. For any $\mu \in \mathcal{M}_{\mathbb{D}}$ and any $i \geq 1$ we also let

$$f_*^i\mu(A) := \mu\left(f^{-i}(A)\right).$$

We now prove some properties of the map f_*.

Lemma 1 *For all $\varphi \in L^1_{\mathbb{T}}(\mu)$ we have $\int \varphi d(f_*\mu) = \int \varphi \circ f d\mu$.*
Proof. *First let $\varphi = \chi_A$ be the characteristic function of some $A \subseteq X$. Then*

$$\begin{aligned}
\int \chi_A d(f_*\mu) &= f_*\mu(A) \\
&= \mu\left(f^{-1}(A)\right) \\
&= \mu_1\left(f^{-1}(A)\right)\mathbf{e}_1 + \mu_2\left(f^{-1}(A)\right)\mathbf{e}_2 \\
&= \left(\int \chi_{f^{-1}(A)} d\mu_1\right)\mathbf{e}_1 + \left(\int \chi_{f^{-1}(A)} d\mu_2\right)\mathbf{e}_2 \\
&= \int (\chi_A \circ f) d\mu.
\end{aligned}$$

So, the statement is true for characteristic functions and thus for simple functions. Hence the statement is true for general integrable functions by standard approximation arguments.

Corollary 1 *$f_* : \mathcal{M}_{\mathbb{D}} \longrightarrow \mathcal{M}_{\mathbb{D}}$ is continuous.*
Proof. *Consider a sequence $\mu_n \longrightarrow \mu$ in $\mathcal{M}_{\mathbb{D}}$. Then, by Lemma 1, for any continuous function $\varphi : X \longrightarrow \mathbb{D}$ we have*

$$\int \varphi d(f_*\mu_n) = \int \varphi \circ f d\mu_n \longrightarrow \int \varphi \circ f d\mu = \int \varphi d(f_*\mu)$$

which implies $f_*\mu_n \longrightarrow f_*\mu$. Hence f_* is continuous.

Corollary 2 *μ is $\mathbb{D}-$invariant with respect to a measurable function $f : X \longrightarrow X$ if and only if $\int \varphi \circ f d\mu = \int \varphi d\mu$ for all continuous function $\varphi : X \longrightarrow \mathbb{D}$.*
Proof. *$\varphi = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2$ is continuous $\Longleftrightarrow \varphi_1$ and φ_2 are continuous.*

Suppose first that μ is $\mathbb{D}-$invariant. Then $f_*\mu = \mu$. Now using Lemma 1, we have

$$\int \varphi \circ f d\mu = \int \varphi df_*\mu = \int \varphi d\mu.$$

For the converse, we have

$$\int \varphi d\mu = \int \varphi \circ f d\mu = \int \varphi d(f_*\mu).$$

which implies

$$\left(\int \varphi_1 d\mu_1\right)\mathbf{e}_1 + \left(\int \varphi_2 d\mu_2\right)\mathbf{e}_2 = \left(\int \varphi_1 df_*\mu_1\right)\mathbf{e}_1 + \left(\int \varphi_2 df_*\mu_2\right)\mathbf{e}_2.$$

So,

$$\int \varphi_1 d\mu_1 = \int \varphi_1 df_*\mu_1$$

and

$$\int \varphi_2 d\mu_2 = \int \varphi_2 df_*\mu_2$$

for every continuous function $\varphi_1, \varphi_2 : X \longrightarrow \mathbb{R}$.

By the Riesz Representation Theorem, measures correspond to linear functionals and therefore this can be restated as saying that

$$\mu_1(\varphi_1) = f_*\mu_1(\varphi_1)$$

and

$$\mu_2(\varphi_2) = f_*\mu_2(\varphi_2)$$

for all continuous function $\varphi_1, \varphi_2 : X \longrightarrow \mathbb{R}$.

Hence

$$\mu_1 = f_*\mu_1 \text{ and } \mu_2 = f_*\mu_2$$

and so

$$f_*\mu = \mu.$$

Therefore μ is $\mathbb{D}-$invariant.

We now prove a general result which gives conditions to guarantee that atleast some $\mathbb{D}-$invariant $\mathbb{D}-$probability measure exists.

Let

$$\mathcal{M}_{\mathbb{D}}(f) = \{\mu \in \mathcal{M}_{\mathbb{D}}\text{: } \mu \text{ is } \mathbb{D}-\text{invariant with respect to } f\}.$$

Theorem 13 *Suppose M is compact metric space and $f : M \longrightarrow M$ is continuous. Then $\mathcal{M}_{\mathbb{D}}(f)$ is non-empty, convex, compact.*

Proof. *Krylov-Boguliobov Theorem [7] states that if M is compact metric space and $f : M \longrightarrow M$ is continuous, then the set $\{\mu^* \in \mathcal{M}^*$: μ^* is $f-$invariant$\}$ is non-empty, where μ^* is probability measure and $\mathcal{M}^*$ is the space of probability measures.*

It is clear that

$$\{\mu^* \in \mathcal{M}^* : \mu^* \textit{ is } f- \textit{ invariant}\} \subseteq \mathcal{M}_{\mathbb{D}}(f).$$

Hence $\mathcal{M}_{\mathbb{D}}(f)$ is non-empty.

Now let $\mu^1, \mu^2 \in \mathcal{M}_{\mathbb{D}}(f)$. Then $\mu^1(f^{-1}(A)) = \mu^1(A)$ and $\mu^2(f^{-1}(A)) = \mu^2(A)$, for all $A \in \mathfrak{B}$.

Now for $t \in [0,1]$, let $\mu = t\mu^1 + (1-t)\mu^2$.

Then

$$\begin{aligned} \mu(f^{-1}(A)) &= t\mu^1(f^{-1}(A)) + (1-t)\mu^2(f^{-1}(A)) \\ &= t\mu^1(A) + (1-t)\mu^2(A) \\ &= \mu(A). \end{aligned}$$

Hence $\mathcal{M}_{\mathbb{D}}(f)$ is convex.

To show compactness, suppose that μ_n is a sequence in $\mathcal{M}_{\mathbb{D}}(f)$ converging to some $\mu \in \mathcal{M}_{\mathbb{D}}$. Then by Lemma 1 we have, for any continuous function φ, that $\int \varphi \circ f d\mu = \lim_{n\to\infty} \int \varphi \circ f d\mu_n = \lim_{n\to\infty} \int \varphi d\mu_n = \int \varphi d\mu$. Therefore by Corollary 2, μ is $\mathbb{D}-$invariant and so $\mu \in \mathcal{M}_{\mathbb{D}}(f)$.

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