Inner Product of Fuzzy Vectors

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Abstract

The inner product of vectors of non-normal fuzzy intervals will be studied in this paper by using the extension principle and the form of decomposition theorem. The membership functions of inner product will be different with respect to these two different methodologies. Since the non-normal fuzzy interval is more general than the normal fuzzy interval, the corresponding membership functions will become more complicated. Therefore, we shall establish their relationship including the equivalence and fuzziness based on the α-level sets. The potential application of inner product of fuzzy vectors is to study the fuzzy linear optimization problems.

Keywords: Canonical fuzzy intervals; Decomposition theorem; Extension principle; Fuzzy intervals; Non-normal fuzzy sets.

1 Introduction

A fuzzy interval in $\mathbb{R}$ is a fuzzy set in $\mathbb{R}$ such that its α-level sets are bounded and closed intervals. The purpose of this paper is to study the inner product of vectors of fuzzy intervals using two different methodologies called the extension principle and the form of decomposition theorem. Since the fuzzy linear optimization problems can be formulated as the form of inner product of fuzzy vectors, the results obtained in this paper can be useful for studying the fuzzy linear optimization problems.

There are two types of inner product will be studied in this paper. The first type of inner product of fuzzy vectors is directly based on the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ given by the following expression

$$\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + \cdots + x_n y_n,$$

where $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^n$. The extension principle and the form of decomposition theorem will directly apply to the (conventional) inner product $\mathbf{x} \bullet \mathbf{y}$ given above without considering the addition and multiplication of fuzzy intervals.

The second type of inner product of fuzzy vectors will be based on the addition and multiplication of fuzzy intervals by considering the following expression

$$\left(\tilde{a}^{(1)} \otimes \tilde{b}^{(1)}\right) \oplus \cdots \oplus \left(\tilde{a}^{(n)} \otimes \tilde{b}^{(n)}\right),$$

where $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are fuzzy intervals in $\mathbb{R}$ for $i = 1, \cdots, n$. The main issue of second type is the addition and multiplication of fuzzy intervals. In this paper, the addition and multiplication of fuzzy intervals will also be formulated based on the extension principle and the form of decomposition theorem. Therefore, the different combinations of using different addition and multiplication will

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generate many different second type of inner product of fuzzy vectors. Their relationship will be established. Moreover, the relationship between the first type and second type of inner product will also be studied.

The second type of inner product of vector of fuzzy intervals needs to consider the arithmetic operations $\oplus$ and $\otimes$. The original arithmetic operations are based on the minimum functions and maximum function. The general t-norms and s-norms instead of minimum functions and maximum functions, respectively, are used by referring to Bede and Stefanini [3], Dubois and Prade [4], Gebhardt [5], Gomes and Barros [6], Fullér and Keresztfalvi [7], Mesiar [8], Ralescu [9], Weber [10], Wu [11, 12, 13] and Yager [16]. More detailed properties regarding theses arithmetic operations $\oplus$ and $\otimes$ can refer to the monographs Dubois and Prade [1] and Klir and Yuan [2]. In this paper, we shall consider the general aggregation function rather than using t-norms and s-norms.

In Section 2, we shall present the basic properties of non-normal fuzzy sets. In Section 3, using the general aggregation functions, the inner product of vectors of fuzzy intervals will be studied. On the other hand, the form of decomposition theorem will be used to study the first type inner product based on three different families. The equivalence and comparison of fuzziness will also be studied. In Section 4, the second type of inner product of vectors of fuzzy intervals will be proposed by using the addition and multiplication of fuzzy intervals. The relationship between the first type and second type of inner product will also be studied. Based on the fuzziness, the suitable appropriation for using the first type or second type is also suggested.

2 Non-Normal Fuzzy Sets

Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\xi_{\tilde{A}}$. For $\alpha \in (0,1]$, the α-level set of $\tilde{A}$ is denoted and defined by

$$\tilde{A}_\alpha = \left\{x \in \mathbb{R} : \xi_{\tilde{A}}(x) \geq \alpha\right\}. \tag{1}$$

It is clear to see that if

$$\alpha \geq \sup_{x\in\mathbb{R}} \xi_{\tilde{A}}(x)$$

then $\tilde{A}_\alpha = \emptyset$. In this paper, we shall carefully avoid to be trapped in the empty α-level sets.

The *support* of fuzzy set $\tilde{A}$ is a crisp set defined by

$$\tilde{A}_{0+} = \{x \in \mathbb{R} : \xi_{\tilde{A}}(x) > 0\}.$$

When $\mathbb{R}$ is endowed with a topology, the 0-level set $\tilde{A}_0$ is defined to be the closure of the support of $\tilde{A}$, i.e., $\tilde{A}_0 = \mathrm{cl}(\tilde{A}_{0+})$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function defined on $\mathbb{R}$, and let S be a subset of $\mathbb{R}$. We say that the supremum $\sup_{x\in S} f(x)$ is attained if there exists $x^* \in S$ satisfying $f(x) \leq f(x^*)$ for all $x \in S$ with $x \neq x^*$. When the supremum $\sup_{x\in S} f(x)$ is attained, we see that

$$\sup_{x\in S} f(x) = \max_{x\in S} f(x).$$

The range of membership function $\xi_{\tilde{A}}$ is denoted by $\mathcal{R}(\xi_{\tilde{A}})$. We define

$$I_{\tilde{A}} = \begin{cases} [0,\alpha^*), & \text{if the supremum } \sup\mathcal{R}(\xi_{\tilde{A}}) \text{ is not attained} \\ [0,\alpha^*], & \text{if the supremum } \sup\mathcal{R}(\xi_{\tilde{A}}) \text{ is attained.} \end{cases} \tag{2}$$

Then $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$. It is clear to see $\mathcal{R}(\xi_{\tilde{A}}) \subseteq I_{\tilde{A}}$. By referring to Wu [13, 15], we also have

$$\tilde{A}_{0+} = \bigcup_{\{\alpha\in I_{\tilde{A}}:\alpha>0\}} \tilde{A}_\alpha = \bigcup_{\{\alpha\in\mathcal{R}(\xi_{\tilde{A}}):\alpha>0\}} \tilde{A}_\alpha. \tag{3}$$

The interval $I_{\tilde{A}}$ presented in (2) is called an *interval range* of $\tilde{A}$. In general, we see that $\mathcal{R}(\xi_{\tilde{A}}) \neq I_{\tilde{A}}$. The role of interval range $I_{\tilde{A}}$ can be used to check $\tilde{A}_\alpha \neq \emptyset$ for all $\alpha \in I_{\tilde{A}}$ and $\tilde{A}_\alpha = \emptyset$ for all $\alpha \notin I_{\tilde{A}}$. The range $\mathcal{R}(\xi_{\tilde{A}})$ is not helpful for identifying the α-level sets.

Recall that $\tilde{A}$ is called a *normal fuzzy set* in $\mathbb{R}$ when there exists $x \in \mathbb{R}$ satisfying $\xi_{\tilde{A}}(x) = 1$. In this case, the interval range of $\tilde{A}$ is given by $I_{\tilde{A}} = [0,1]$. However, the range $\mathcal{R}(\xi_{\tilde{A}})$ is not necessarily equal to the whole unit interval $[0,1]$ even though $\tilde{A}$ is normal.

The characteristic function χ_A of a crisp set A is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$$

The well-known decomposition theorem is based on the normal fuzzy sets in $\mathbb{R}$, which says that the membership function $\xi_{\tilde{A}}$ can be expressed as

$$\xi_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \sup_{\alpha \in (0,1]} \alpha \cdot \chi_{\tilde{A}_\alpha}(x),$$

If $\tilde{A}$ is not normal, we can similarly obtain the following form.

Theorem 2.1. (Wu [14])(**Decomposition Theorem**) *Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$. Then the membership function $\xi_{\tilde{A}}$ can be expressed as*

$$\begin{aligned} \xi_{\tilde{A}}(x) &= \sup_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in \mathcal{R}(\xi_{\tilde{A}})} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) \\ &= \sup_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x) = \max_{\alpha \in I_{\tilde{A}}} \alpha \cdot \chi_{\tilde{A}_\alpha}(x), \end{aligned}$$

where the supremum is attained and the interval range $I_{\tilde{A}}$ is given in (2).

Definition 2.2. We say that $\tilde{a}$ is a fuzzy interval in $\mathbb{R}$ when the following conditions are satisfied:

- The membership function $\xi_{\tilde{a}}$ is upper semi-continuous and quasi-concave on $\mathbb{R}$.
- The 0-level set $\tilde{a}_0$ is a closed and bounded subset of $\mathbb{R}$.

It is well-known that the α-level sets of fuzzy interval $\tilde{a}$ are all closed and bounded intervals denoted by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ for $\alpha \in [0,1]$. When the fuzzy interval $\tilde{a}$ is normal and the 1-level set $\tilde{a}_1$ is a singleton set $\{a\}$ for some $a \in \mathbb{R}$, the fuzzy interval $\tilde{a}$ is then called a *fuzzy number* with *core value* a.

3 The First Type of Inner Product

Recall that the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\mathbf{x} \bullet \mathbf{y} = x_1 y_1 + \cdots + x_n y_n,$$

where $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$ be two vectors in $\mathbb{R}^n$.

Given any fuzzy intervals $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ in $\mathbb{R}$, we define

$$\alpha^*_i = \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) = \sup_{x \in \mathbb{R}} \xi_{\tilde{a}^{(i)}}(x) \text{ and } \beta^*_i = \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) = \sup_{x \in \mathbb{R}} \xi_{\tilde{b}^{(i)}}(x) \tag{4}$$

for $i = 1, \cdots, n$. The interval ranges $I_{\tilde{a}^{(i)}}$ of $\tilde{a}^{(i)}$ and $I_{\tilde{b}^{(i)}}$ of $\tilde{b}^{(i)}$ can be realized from (2). More precisely, we have

$$I_{\tilde{a}^{(i)}} = \begin{cases} [0, \alpha^*_i), & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ is not attained} \\ [0, \alpha^*_i], & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ is attained} \end{cases} \tag{5}$$

and

$$I_{\tilde{b}^{(i)}} = \begin{cases} [0, \beta_i^*), & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \text{ is not attained} \\ [0, \beta_i^*], & \text{if the supremum } \sup \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \text{ is attained} \end{cases} \tag{6}$$

Let

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}. \tag{7}$$

By referring to (5) and (6), we see that $I^* \neq \emptyset$. Therefore, for each $\alpha \in I^*$, the α-level sets of $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are nonempty and denoted by

$$\tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \text{ and } \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right].$$

For convenience, we write

$$\left(\tilde{a}_{1\alpha}^L, \tilde{a}_{2\alpha}^L, \cdots, \tilde{a}_{n\alpha}^L\right) = \tilde{\mathbf{a}}_\alpha^L \in \mathbb{R}^n \text{ and } \left(\tilde{a}_{1\alpha}^U, \tilde{a}_{2\alpha}^U, \cdots, \tilde{a}_{n\alpha}^U\right) = \tilde{\mathbf{a}}_\alpha^U \in \mathbb{R}^n. \tag{8}$$

We also write

$$\tilde{\mathbf{a}}_\alpha = \tilde{a}_\alpha^{(1)} \times \cdots \times \tilde{a}_\alpha^{(n)} = \left[\tilde{a}_{1\alpha}^L, \tilde{a}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{a}_{n\alpha}^L, \tilde{a}_{n\alpha}^U\right] \tag{9}$$

and

$$\tilde{\mathbf{b}}_\alpha = \tilde{b}_\alpha^{(1)} \times \cdots \times \tilde{b}_\alpha^{(n)} = \left[\tilde{b}_{1\alpha}^L, \tilde{b}_{1\alpha}^U\right] \times \cdots \times \left[\tilde{b}_{n\alpha}^L, \tilde{b}_{n\alpha}^U\right]. \tag{10}$$

Let $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ be two vectors of fuzzy intervals in $\mathbb{R}$ given by

$$\tilde{\mathbf{a}} = \left(\tilde{a}^{(1)}, \tilde{a}^{(2)}, \cdots, \tilde{a}^{(n)}\right) \text{ and } \tilde{\mathbf{b}} = \left(\tilde{b}^{(1)}, \tilde{b}^{(2)}, \cdots, \tilde{b}^{(n)}\right). \tag{11}$$

We shall study the inner product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ using the extension principle, and the inner product $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$ using the form of decomposition theorem.

3.1 Using the Extension Principle

We are going to use the extension principle to define the membership function of inner product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$. Given two vectors $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ of fuzzy intervals in (11) and an aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ defined on $[0,1]^{2n}$, the membership function of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}(z) = \sup_{\{(\mathbf{x},\mathbf{y}): z = \mathbf{x} \bullet \mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \tag{12}$$

for each $z \in \mathbb{R}$. If the aggregation function $\mathfrak{A}$ is taken to be the minimum function, it recovers the conventional form of extension principle.

In order to obtain the nonempty α-level sets of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, we need to consider the interval range $I_\circledast^{(EP)}$ of inner product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, which says that

$$I_\circledast^{(EP)} = \begin{cases} [0, \alpha^*], & \text{if the supremum } \alpha^* = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0, \alpha^*), & \text{otherwise.} \end{cases} \tag{13}$$

We also have

$$\begin{aligned} \alpha^* &= \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}\right) = \sup_{\mathbf{z} \in \mathbb{R}^m} \xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}(\mathbf{z}) \\ &= \sup_{\mathbf{z} \in \mathbb{R}^m} \sup_{\{(\mathbf{x},\mathbf{y}): \mathbf{z} = \mathbf{x} \bullet \mathbf{y}\}} \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \\ &= \sup_{(\alpha_1, \cdots, \alpha_{2n}) \in \mathcal{R}_1 \times \cdots \times \mathcal{R}_{2n}} \mathfrak{A}\left(\alpha_1, \cdots, \alpha_{2n}\right). \end{aligned}$$

By referring to (2), we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I_\circledast^{(EP)} \text{ and } \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I_\circledast^{(EP)}.$$

Proposition 3.1. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is taken by*

$$\begin{aligned}
&\mathfrak{A}\left(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\right) \\
&= \begin{cases} \min\left\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\right\}, & \text{if } \alpha_i \in \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ and } \beta_i \in \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \\ & \quad \text{for } i = 1, \cdots, n \\ \text{any expression}, & \text{otherwise}, \end{cases}
\end{aligned} \tag{14}$$

Then $I_{\circledast}^{(EP)} = I^*$ and

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}}\right) = \alpha^* = \min\left\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\right\}.$$

Proof. The similar proof can refer to Wu [15]. ■

For each $\alpha \in I_{\circledast}^{(EP)}$ with $\alpha > 0$, by applying the results in Wu [13] to the inner product $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, the α-level set $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_\alpha$ of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ for $\alpha \in I_{\circledast}^{(EP)}$ is given by

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} \bullet \mathbf{y} : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\} \\
&= \left\{x_1 y_1 + \cdots + x_n y_n : \mathfrak{A}\left(\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right) \geq \alpha\right\}.
\end{aligned} \tag{15}$$

Also, the 0-level set is given by

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_0 = \tilde{\mathbf{a}}_0 \bullet \tilde{\mathbf{b}}_0 = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_0 \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_0\right\}.$$

On the other hand, the α-level sets $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_\alpha$ are closed and bounded subsets of $\mathbb{R}^m$ for $\alpha \in I_{\circledast}^{(EP)}$.

Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is taken by the form of (14). For each $\alpha \in I_{\circledast}^{(EP)}$ with $\alpha > 0$, using (15), we have

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \left\{\mathbf{x} \bullet \mathbf{y} : \min\left\{\xi_{\tilde{a}^{(1)}}(x_1), \cdots, \xi_{\tilde{a}^{(n)}}(x_n), \xi_{\tilde{b}^{(1)}}(y_1), \cdots, \xi_{\tilde{b}^{(n)}}(y_n)\right\} \geq \alpha\right\} \\
&= \left\{\mathbf{x} \bullet \mathbf{y} : \xi_{\tilde{a}^{(i)}}(x_i) \geq \alpha \text{ and } \xi_{\tilde{b}^{(i)}}(y_i) \geq \alpha \text{ for each } i = 1, \cdots, n\right\} \\
&= \Big\{x_1 y_1 + \cdots + x_n y_n : x_i \in \tilde{a}_\alpha^{(i)} \equiv \left[\tilde{a}_{i\alpha}^L, \tilde{a}_{i\alpha}^U\right] \\
&\qquad \text{and } y_i \in \tilde{b}_\alpha^{(i)} \equiv \left[\tilde{b}_{i\alpha}^L, \tilde{b}_{i\alpha}^U\right] \text{ for each } i = 1, \cdots, n\Big\} \\
&= \left[\min_{(\mathbf{x}, \mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \left(x_1 y_1 + \cdots + x_n y_n\right), \max_{(\mathbf{x}, \mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \left(x_1 y_1 + \cdots + x_n y_n\right)\right],
\end{aligned} \tag{16}$$

where $\tilde{\mathbf{a}}_\alpha$ and $\tilde{\mathbf{b}}_\alpha$ are given in (9) and (10).

In order to simplify the mathematical expression of 0-level set $(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}})_0$, we introduce the concept of canonical fuzzy interval.

Definition 3.2. We say that $\tilde{a}$ is a *canonical fuzzy interval* when the following conditions are satisfied.

- $\tilde{a}$ is a fuzzy interval.
- The functions $l(\alpha) = \tilde{a}_\alpha^L$ and $u(\alpha) = \tilde{a}_\alpha^U$ are continuous on $I_{\tilde{a}}$, where $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$ for $\alpha \in I_{\tilde{a}}$.

Now, we assume that $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are taken to be the canonical fuzzy intervals for $i = 1, \cdots, n$. For the 0-level set, from (16) and (3), using the nestedness and the continuities regarding the canonical fuzzy intervals, we can show that

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_0 = \text{cl}\left(\bigcup_{\{\alpha \in I_{\circledast}^{(EP)}:\alpha>0\}} \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha\right)$$
$$= \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_0,\tilde{\mathbf{b}}_0)} (x_1y_1 + \cdots + x_ny_n), \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_0,\tilde{\mathbf{b}}_0)} (x_1y_1 + \cdots + x_ny_n)\right].$$

In order to simplify the expression, we consider the nonnegativity of fuzzy sets.

Definition 3.3. Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\xi_{\tilde{A}}$. We say that $\tilde{A}$ is nonnegative when $\xi_{\tilde{A}}(x) = 0$ for each $x < 0$.

We see that a fuzzy interval $\tilde{a}$ is nonnegative if and only if $\tilde{a}_\alpha^L \geq 0$ for each $\alpha \in I_{\tilde{a}}$. Now, we assume that $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be nonnegative canonical fuzzy intervals. Then

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{a}_{1\alpha}^L\tilde{b}_{1\alpha}^L + \cdots + \tilde{a}_{n\alpha}^L\tilde{b}_{n\alpha}^L, \tilde{a}_{1\alpha}^U\tilde{b}_{1\alpha}^U + \cdots + \tilde{a}_{n\alpha}^U\tilde{b}_{n\alpha}^U\right].$$

The above results are summarized in the following theorem.

Theorem 3.4. *Let $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ be canonical fuzzy intervals. Suppose that the aggregation function $\mathfrak{A} : [0,1]^{2n} \to [0,1]$ is taken by*

$$\mathfrak{A}(\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n) = \begin{cases} \min\{\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n\}, & \text{if } \alpha_i \in \mathcal{R}(\xi_{\tilde{a}^{(i)}}) \text{ and } \beta_i \in \mathcal{R}(\xi_{\tilde{b}^{(i)}}) \\ & \quad \text{for } i = 1, \cdots, n \\ \text{any expression}, & \text{otherwise}, \end{cases}$$

Then $I_{\circledast}^{(EP)} = I^*$ and the α-level sets of $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ are given by

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}, \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y}\right]$$

for $\alpha \in I^*$, where $\tilde{\mathbf{a}}_\alpha$ and $\tilde{\mathbf{b}}_\alpha$ are given in (9) *and* (10)*. Suppose that $\tilde{a}^{(1)}, \cdots, \tilde{a}^{(n)}$ and $\tilde{b}^{(1)}, \cdots, \tilde{b}^{(n)}$ are taken to be nonnegative canonical fuzzy intervals. Then*

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}_\alpha^L \bullet \tilde{\mathbf{b}}_\alpha^L, \tilde{\mathbf{a}}_\alpha^U \bullet \tilde{\mathbf{b}}_\alpha^U\right],$$

where $\tilde{\mathbf{a}}_\alpha^L$, $\tilde{\mathbf{a}}_\alpha^U$, $\tilde{\mathbf{b}}_\alpha^L$ and $\tilde{\mathbf{b}}_\alpha^U$ are given in (8).

3.2 Using the Form of Decomposition Theorem

Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be fuzzy intervals for $i = 1, \cdots, n$. Now, we are going to use the form of decomposition theorem to define three different inner product by considering three different families.

- We consider the family $\{M_\alpha^\bullet : \alpha \in I^* \text{ with } \alpha > 0\}$ by taking

$$M_\alpha^\bullet = \tilde{\mathbf{a}}_\alpha \bullet \tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x} \bullet \mathbf{y} : \mathbf{x} \in \tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y} \in \tilde{\mathbf{b}}_\alpha\right\} \tag{17}$$

  to define the inner product $\tilde{\mathbf{a}} \circledast_{DT}^\diamond \tilde{\mathbf{b}}$, where $\tilde{\mathbf{a}}_\alpha$ and $\tilde{\mathbf{b}}_\alpha$ are given in (9) and (10).

- Let M_β be bounded closed intervals given by

$$M_\beta = \left[\min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right],$$

  where $\tilde{\mathbf{a}}^L_\alpha$, $\tilde{\mathbf{a}}^U_\alpha$, $\tilde{\mathbf{b}}^L_\alpha$ and $\tilde{\mathbf{b}}^U_\alpha$ are given in (8). We consider the family $\{M^\bullet_\alpha : \alpha \in I^* \text{ with } \alpha > 0\}$ by taking

$$M^\bullet_\alpha = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta, \tag{18}$$

  to define the inner product $\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}$.

- We consider the family $\{M^\bullet_\alpha : \alpha \in I^* \text{ with } \alpha > 0\}$ by directly taking

$$M^\bullet_\alpha = \left[\min\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}, \max\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}\right] \tag{19}$$

  to define the inner product $\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}$.

Using the form of decomposition theorem, for $\circledast_{DT} \in \{\circledast^\diamond_{DT}, \circledast^\star_{DT}, \circledast^\dagger_{DT}\}$, the membership function of $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M^\bullet_\alpha}(\mathbf{z}), \tag{20}$$

where $M^\bullet_\alpha$ corresponds to the above three cases (17), (18) and (19). We also have

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}}\right) = \sup_{z \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}}(z) = \sup_{z \in \mathbb{R}} \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M^\bullet_\alpha}(z) = \sup I^* \equiv \alpha^\bullet. \tag{21}$$

In order to consider the nonempty α-level sets. The interval ranges of $\xi_{\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}}$, $\xi_{\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}}$ and $\xi_{\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}}$ are denoted by $I^{(\diamond DT)}_\circledast$, $I^{(\star DT)}_\circledast$ and $I^{(\dagger DT)}_\circledast$, respectively. More precisely, by referring to (2), we have

$$I^{(\diamond DT)}_\circledast = \begin{cases} [0, \alpha^\bullet], & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0, \alpha^\bullet), & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}}\right) \text{ is not attained.} \end{cases} \tag{22}$$

and

$$I^{(\star DT)}_\circledast = \begin{cases} [0, \alpha^\bullet], & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0, \alpha^\bullet), & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}}\right) \text{ is not attained.} \end{cases} \tag{23}$$

and

$$I^{(\dagger DT)}_\circledast = \begin{cases} [0, \alpha^\bullet], & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0, \alpha^\bullet), & \text{if the supremum } \alpha^\bullet = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}}\right) \text{ is not attained.} \end{cases} \tag{24}$$

Therefore, the nonempty α-level sets can be realized below:

$$\left(\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I^{(\diamond DT)}_\circledast \text{ and } \left(\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I^{(\diamond DT)}_\circledast$$

and

$$\left(\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I^{(\star DT)}_\circledast \text{ and } \left(\tilde{\mathbf{a}} \circledast^\star_{DT} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I^{(\star DT)}_\circledast$$

and

$$\left(\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}\right)_\alpha \neq \emptyset \text{ for } \alpha \in I^{(\dagger DT)}_\circledast \text{ and } \left(\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}\right)_\alpha = \emptyset \text{ for } \alpha \notin I^{(\dagger DT)}_\circledast.$$

Then, we have the following useful results regarding the interval ranges.

Proposition 3.5. *Let $\tilde{a}^{(1)},\cdots,\tilde{a}^{(n)}$ and $\tilde{b}^{(1)},\cdots,\tilde{b}^{(n)}$ be fuzzy intervals. Suppose that the following supremum*

$$\sup I^* = \sup\left(I_{\tilde{a}^{(1)}}\cap\cdots\cap I_{\tilde{a}^{(n)}}\cap I_{\tilde{b}^{(1)}}\cap\cdots\cap I_{\tilde{b}^{(n)}}\right).$$

is attained. Then, the interval ranges are all identical given by

$$I_{\circledast}^{(\diamond DT)} = I_{\circledast}^{(\star DT)} = I_{\circledast}^{(\dagger DT)} = I^* = [0,\alpha^\bullet].$$

Proof. The similar proof can refer to Wu [15]. ■

In the sequel, we shall separately study the three different families $\{M_\alpha^\bullet : \alpha\in I^* \text{ for } \alpha>0\}$ given in (17), (18) and (19).

3.2.1 The Inner Product $\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}$

We shall study the inner product $\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}$ considering the family given in (17). Since $\tilde{a}_\alpha^{(i)}\neq\emptyset$ and $\tilde{b}_\alpha^{(i)}\neq\emptyset$ for $\alpha\in I^*$ and $i=1,\cdots,n$, given any $\alpha\in I^*$ with $\alpha>0$, we have

$$M_\alpha^\bullet = \tilde{\mathbf{a}}_\alpha\bullet\tilde{\mathbf{b}}_\alpha = \left\{\mathbf{x}\bullet\mathbf{y} : \mathbf{x}\in\tilde{\mathbf{a}}_\alpha \text{ and } \mathbf{y}\in\tilde{\mathbf{b}}_\alpha\right\} = \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)}\mathbf{x}\bullet\mathbf{y},\ \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)}\mathbf{x}\bullet\mathbf{y}\right].$$

According to the form of decomposition theorem, the membership function of $\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}}(z) = \sup_{\{\alpha\in I^*:\alpha>0\}}\alpha\cdot\chi_{M_\alpha^\bullet}(z). \tag{25}$$

We have the following interesting results.

Theorem 3.6. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i=1,\cdots,n$, and let*

$$I^* = I_{\tilde{a}^{(1)}}\cap\cdots\cap I_{\tilde{a}^{(n)}}\cap I_{\tilde{b}^{(1)}}\cap\cdots\cap I_{\tilde{b}^{(n)}}.$$

The family $\{M_\alpha^\bullet : \alpha\in I^* \text{ for } \alpha>0\}$ is taken by $M_\alpha^\bullet = \tilde{\mathbf{a}}_\alpha\bullet\tilde{\mathbf{b}}_\alpha$. Suppose that the supremum $\sup I^*$ is attained. Then $I_{\circledast}^{(\diamond DT)} = I^*$, and, for $\alpha\in I^*$, we have

$$\left(\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}\right)_\alpha = \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)}\mathbf{x}\bullet\mathbf{y},\ \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)}\mathbf{x}\bullet\mathbf{y}\right].$$

When $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are taken to be nonnegative canonical fuzzy intervals for $i=1,\cdots,n$, we simply have

$$\left(\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}_\alpha^L\bullet\tilde{\mathbf{b}}_\alpha^L, \tilde{\mathbf{a}}_\alpha^U\bullet\tilde{\mathbf{b}}_\alpha^U\right].$$

Proof. It is clear to see that $\{M_\alpha^\bullet : \alpha\in I^* \text{ for } \alpha>0\}$ is a nested family in the sense of $M_\alpha^\bullet\subseteq M_\beta^\bullet$ for $\beta<\alpha$. Using the continuities regarding the canonical fuzzy intervals, we see that the family $\{M_\alpha^\bullet : \alpha\in I^* \text{ for } \alpha>0\}$ will continuously shrink when α increases on I^*. Therefore, for $\alpha\in I^*$ with $\alpha>0$, we have

$$M_\alpha^\bullet = \bigcap_{s=1}^{\infty} M_{\alpha_k}^\bullet \tag{26}$$

for $0<\alpha_k\uparrow\alpha$ with $\alpha_k\in I^*$ for all k.

The equality $I_{\circledast}^{(\diamond DT)} = I^*$ can be realized from Proposition 3.5. Next, we are going to show that $M_\alpha^\bullet = (\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}})_\alpha$ for $\alpha\in I^*$. For $\alpha\in I^*$ with $\alpha>0$ and any $\mathbf{z}\in M_\alpha^\bullet$, the expression (25) says that $\xi_{\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}}(\mathbf{z})\geq\alpha$, which implies $\mathbf{z}\in(\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}})_\alpha$ and proves the inclusion $M_\alpha^\bullet\subseteq(\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}})_\alpha$. On the other hand, given any $\mathbf{z}\in(\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}})_\alpha$, it means that $\xi_{\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}}(\mathbf{z})\geq\alpha$. Let $\hat{\alpha} = \xi_{\tilde{\mathbf{a}}\circledast_{DT}^{\diamond}\tilde{\mathbf{b}}}(\mathbf{z})$. We consider the following cases.

- Assume that $\hat{\alpha} > \alpha$. Let $\epsilon = \hat{\alpha} - \alpha > 0$. By referring to (25), the concept of supremum says that there exists $\alpha_0 \in I^*$ satisfying $\mathbf{z} \in M^{\bullet}_{\alpha_0}$ and $\hat{\alpha} - \epsilon < \alpha_0$, which says that $\alpha < \alpha_0$. Therefore, we obtain $\mathbf{z} \in M^{\bullet}_{\alpha}$, since $M^{\bullet}_{\alpha_0} \subseteq M^{\bullet}_{\alpha}$ by the nestedness.
- Assume that $\hat{\alpha} = \alpha$. Since I^* is an interval with left end-point 0, for any $\alpha \in I^*$ with $\alpha > 0$, there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ in I^* satisfying $0 < \alpha_k \uparrow \alpha$ with $\alpha_k \in I^*$ for all k. Let $\epsilon_k = \alpha - \alpha_k > 0$. By referring to (25), the concept of supremum says that there exists $\alpha_0 \in I^*$ satisfying $\mathbf{z} \in M^{\bullet}_{\alpha_0}$ and $\hat{\alpha} - \epsilon_k = \alpha - \epsilon_k < \alpha_0$, which implies $\alpha_0 > \alpha_k \in I^*$. The nestedness also says that $\mathbf{z} \in M^{\bullet}_{\alpha_k}$ for all k, i.e., $\mathbf{z} \in \bigcap_{k=1}^{\infty} M^{\bullet}_{\alpha_k}$. From (26), we obtain $\mathbf{z} \in M^{\bullet}_{\alpha}$.

The above two cases conclude that $(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha \subseteq M^{\bullet}_{\alpha}$. Therefore, we obtain $M^{\bullet}_{\alpha} = (\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}})_\alpha$ for $\alpha \in I^*$ with $\alpha > 0$.

For the 0-level set, we also have

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_0 &= \mathrm{cl}\left(\bigcup_{\{\alpha \in I^{(\diamond DT)}_{\circledast} : \alpha > 0\}} \left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha\right) \text{ (referring to (3))} \\
&= \mathrm{cl}\left(\bigcup_{\{\alpha \in I^* : \alpha > 0\}} M^{\bullet}_{\alpha}\right) \text{ (since } I^{(\diamond DT)}_{\circledast} = I^*\text{)} \\
&= M^{\bullet}_0 \text{ (using the nestedness and continuities in Definition 3.2)}
\end{aligned}$$

This completes the proof. ■

3.2.2 The Inner Product $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$

We shall study the inner product $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ considering the family given in (18). According to the form of decomposition theorem, the membership function of inner product $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}}(z) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M^{\bullet}_{\alpha}}(z). \tag{27}$$

We have the following interesting results.

Theorem 3.7. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i = 1, \cdots, n$, and let*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

The family $\{M^{\bullet}_{\alpha} : \alpha \in I^*$ for $\alpha > 0\}$ is taken by

$$M^{\bullet}_{\alpha} = \left(\bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta\right),$$

where M_β is a bounded closed intervals given by

$$M_\beta = \left[\min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right].$$

Suppose that the supremum $\sup I^*$ is attained. Then $I^{(\star DT)}_{\circledast} = I^*$, and, for $\alpha \in I^*$, we have

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= M^{\bullet}_{\alpha} \\
&= \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta \in I^* : \beta \geq \alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right].
\end{aligned} \tag{28}$$

When $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are taken to be nonnegative canonical fuzzy intervals for $i = 1, \cdots, n$, we simply have

$$\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_{\alpha} = \left[\tilde{\mathbf{a}}^L_{\alpha} \bullet \tilde{\mathbf{b}}^L_{\alpha}, \tilde{\mathbf{a}}^U_{\alpha} \bullet \tilde{\mathbf{b}}^U_{\alpha}\right].$$

Proof. The equality $I^{(\star DT)}_{\circledast} = I^*$ can be realized from Proposition 3.5. Next, we are going to show that $M^{\bullet}_{\alpha} = (\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}})_{\alpha}$ for $\alpha \in I^*$. By using (27) and the proof of Theorem 3.6, we can similarly obtain the inclusion $M^{\bullet}_{\alpha} \subseteq (\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}})_{\alpha}$.

On the other hand, we can see that $\{M^{\bullet}_{\alpha} : \alpha \in I^* \text{ with } \alpha > 0\}$ is a nested family in the sense of $M^{\bullet}_{\alpha} \subseteq M^{\bullet}_{\beta}$ for $\beta < \alpha$. We define two functions ζ^L and ζ^U on I^* as follows:

$$\zeta^L(\beta) = \min\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\} \text{ and } \zeta^U(\beta) = \max\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\}.$$

It is clear to see that the functions ζ^L and ζ^U are continuous on I^* by the continuities regarding the canonical fuzzy intervals. We also see that $M_{\beta} = [\zeta^L(\beta), \zeta^U(\beta)]$. The continuities say that the family $\{M^{\bullet}_{\alpha} : \alpha \in I^* \text{ for } \alpha > 0\}$ will continuously shrink when α increases on I^*. For $\alpha \in I^*$ with $\alpha > 0$, it follows that

$$M^{\bullet}_{\alpha} = \bigcap_{k=1}^{\infty} M^{\bullet}_{\alpha_k}$$

for $0 < \alpha_k \uparrow \alpha$ with $\alpha_k \in I^*$ for all k. Using the proof of Theorem 3.6, we can similarly obtain the inclusion $(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}})_{\alpha} \subseteq M^{\bullet}_{\alpha}$. Therefore, we have $M^{\bullet}_{\alpha} = (\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}})_{\alpha}$ for $\alpha \in I^*$ with $\alpha > 0$. Moreover, for $\alpha \in I^*$ with $\alpha > 0$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_{\alpha} &= M^{\bullet}_{\alpha} = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_{\beta} = \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \zeta^L(\beta), \max_{\{\beta \in I^* : \beta \geq \alpha\}} \zeta^U(\beta)\right] \\ &= \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\}, \max_{\{\beta \in I^* : \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\}\right].\end{aligned}$$

For the 0-level set, we also have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_{0} &= \text{cl}\left(\bigcup_{\{\alpha \in I^{(\star DT)}_{\circledast} : \alpha > 0\}} \left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_{\alpha}\right) \text{ (referring to (3))} \\ &= \text{cl}\left(\bigcup_{\{\alpha \in I^* : \alpha > 0\}} M^{\bullet}_{\alpha}\right) \text{ (since } I^{(\star DT)}_{\circledast} = I^*) \\ &= \text{cl}\left(\bigcup_{\{\alpha \in I^* : \alpha > 0\}} \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \zeta^L(\beta), \max_{\{\beta \in I^* : \beta \geq \alpha\}} \zeta^U(\beta)\right]\right) \\ &= \left[\min_{\{\beta \in I^* : \beta \geq 0\}} \zeta^L(\beta), \max_{\{\beta \in I^* : \beta \geq 0\}} \zeta^U(\beta)\right] \\ &\quad \text{(using the nestedness and the continuities of functions } \zeta^L \text{ and } \zeta^U) \\ &= \left[\min_{\{\beta \in I^* : \beta \geq 0\}} \min\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\}, \max_{\{\beta \in I^* : \beta \geq 0\}} \min\left\{\tilde{\mathbf{a}}^L_{\beta} \bullet \tilde{\mathbf{b}}^L_{\beta}, \tilde{\mathbf{a}}^U_{\beta} \bullet \tilde{\mathbf{b}}^U_{\beta}\right\}\right].\end{aligned}$$

This completes the proof. ■

3.2.3 The Inner Product $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$

We shall study the inner product $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$ considering the family given in (19). According to the form of decomposition theorem, the membership function of inner product $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$ is defined by

$$\xi_{\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}}(z) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M^{\bullet}_{\alpha}}(z). \tag{29}$$

We need two useful lemmas.

Lemma 3.8. (Royden [17, p.161]) *Let X be a topological space, let K be a compact subset of X, and let f be a real-valued function defined on X. Suppose that f is upper semi-continuous. Then f assumes its maximum on a compact subset of X; that is, the supremum is attained in the following sense*

$$\sup_{x \in K} f(x) = \max_{x \in K} f(x).$$

Lemma 3.9. *Let $I = [0, \gamma]$ be a closed subinterval of $[0,1]$ for some $0 < \gamma \leq 1$. Suppose that the bounded real-valued functions $\zeta^L : I \to \mathbb{R}$ and $\zeta^U : I \to \mathbb{R}$ satisfying the following conditions:*

- *$\zeta^L(\alpha) \leq \zeta^U(\alpha)$ for each $\alpha \in I$;*
- *ζ^L is an increasing function and ζ^U is a decreasing function on I;*
- *ζ^L and ζ^U are left-continuous on $I \setminus \{0\} = (0, \gamma]$.*

Let $M_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$ for $\alpha \in I$. Then, for any fixed $x \in \mathbb{R}$, the following function

$$\zeta(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha}(x), & \text{if } \alpha \in I \text{ with } \alpha > 0 \end{cases}$$

is upper semi-continuous on I.

Theorem 3.10. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i = 1, \cdots, n$, and let*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

The family $\{M_\alpha : \alpha \in I^* \text{ for } \alpha > 0\}$ is taken by

$$M^{\bullet}_{\alpha} = \left[\min\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}, \max\left\{\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right\}\right].$$

Suppose that the supremum $\sup I^*$ is attained. Then $I^{(\dagger DT)}_{\circledast} = I^*$, and, for $\alpha \in I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M^{\bullet}_{\beta} \\ &= \left[\min_{\{\beta \in I^* : \beta \geq \alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta \in I^* : \beta \geq \alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right]. \end{aligned} \tag{30}$$

When $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ are taken to be nonnegative canonical fuzzy intervals for $i = 1, \cdots, n$, we have

$$\left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right].$$

Proof. The equality $I^{(\dagger DT)}_{\circledast} = I^*$ can be realized from Proposition 3.5. Next, we are going to show that

$$\left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M^{\bullet}_{\beta} \text{ for } \alpha \in I^* \text{ with } \alpha > 0. \tag{31}$$

Let

$$\zeta^L(\alpha) = \min_{\{\beta \in I^*:\beta\geq\alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\} \text{ and } \zeta^U(\alpha) = \max_{\{\beta \in I^*:\beta\geq\alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}.$$

Then $M^\bullet_\alpha = [\zeta^L(\alpha), \zeta^U(\alpha)]$. The continuities regarding the canonical fuzzy intervals show that the functions ζ^L and ζ^U are continuous on I^*. Using Lemma 3.9, given any fixed $x \in \mathbb{R}$, the following function

$$\zeta(\alpha) = \begin{cases} 0, & \text{if } \alpha = 0 \\ \alpha \cdot \chi_{M_\alpha}(x), & \text{if } \alpha \in I^* \text{ with } \alpha > 0 \end{cases}$$

is upper semi-continuous on I^*.

For $\alpha \in I^{(\dagger DT)}_{\circledast} = I^*$ with $\alpha > 0$, given any $\mathbf{z} \in (\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}})_\alpha$ with $\mathbf{z} \notin M^\bullet_\beta$ for all $\beta \in I^*$ with $\beta \geq \alpha$, we see that $\beta \cdot \chi_{M^\bullet_\beta}(\mathbf{z}) < \alpha$ for all $\beta \in I^*$. Since I^* is a compact set (a bounded and closed interval) and $\zeta(\beta) = \beta \cdot \chi_{M^\bullet_\beta}(\mathbf{z})$ is upper semi-continuous on I^* as described above, Lemma 3.8 says that the supremum of the function ζ is attained. Using (29), we have

$$\xi_{\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) = \sup_{\beta \in I^*} \zeta(\beta) = \sup_{\beta \in I^*} \beta \cdot \chi_{M^\bullet_\beta}(\mathbf{z}) = \max_{\beta \in I^*} \beta \cdot \chi_{M^\bullet_\beta}(\mathbf{z}) = \beta^* \cdot \chi_{M_{\beta^*}}(\mathbf{z}) < \alpha$$

for some $\beta^* \in I^*$, which shows that $\mathbf{z} \notin (\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}})_\alpha$. This contradiction says that there exists $\beta_0 \in I^*$ with $\beta_0 \geq \alpha$ satisfying $\mathbf{z} \in M^\bullet_{\beta_0}$. Therefore, we have the following inclusion

$$\left(\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}\right)_\alpha \subseteq \bigcup_{\{\beta \in I^*:\beta\geq\alpha\}} M^\bullet_\beta.$$

On the other hand, the following inclusion

$$\bigcup_{\{\beta \in I^*:\beta\geq\alpha\}} M^\bullet_\beta \subseteq \left\{\mathbf{z} \in \mathbb{R}^n : \sup_{\beta \in I^*} \beta \cdot \chi_{M^\bullet_\beta}(\mathbf{z}) \geq \alpha\right\} = \{\mathbf{z} \in \mathbb{R}^n : \xi_{\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}}(\mathbf{z}) \geq \alpha\} = (\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}})_\alpha$$

is obvious. This shows the equality (31). Using the continuities regarding the canonical fuzzy intervals, we can also obtain the equality (30).

For the 0-level set, we have

$$\begin{aligned} \left(\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}\right)_0 &= \text{cl}\left(\bigcup_{\{\alpha \in I^{(\dagger DT)}_{\circledast}:\alpha>0\}} \left(\tilde{\mathbf{a}} \circledast^\dagger_{DT} \tilde{\mathbf{b}}\right)_\alpha\right) \text{ (referring to (3))} \\ &= \left[\min_{\{\beta \in I^*:\beta\geq 0\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta \in I^*:\beta\geq 0\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right] \\ &\quad \text{(using the nestedness, continuities and the equality (30)).} \end{aligned}$$

This completes the proof. ■

3.3 The Equivalences and Fuzziness

The equivalences among $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT} \tilde{\mathbf{b}}$ for $\circledast_{DT} \in \{\circledast^\diamond_{DT}, \circledast^\star_{DT}, \circledast^\dagger_{DT}\}$ will be presented below.

Theorem 3.11. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i = 1, \cdots, n$. Suppose that the different inner products $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}$ are obtained from Theorems 3.4 and 3.6, respectively. Assume that the supremum $\sup I^*$ is attained. Then*

$$I^{(EP)}_{\circledast} = I^{(\diamond DT)}_{\circledast} = I^* \text{ and } \tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast^\diamond_{DT} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left[\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}, \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}\right].$$

Theorem 3.12. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i = 1, \cdots, n$. Suppose that the different inner products $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$ are obtained from Theorems 3.7 and 3.10, respectively. Assume that the supremum $\sup I^*$ is attained. Then*

$$I^{(\star DT)}_{\circledast} = I^{(\dagger DT)}_{\circledast} = I^* \text{ and } \tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha &= \left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha \\ &= \left[\min_{\{\beta\in I^*:\beta\geq\alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}, \max_{\{\beta\in I^*:\beta\geq\alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}\right].\end{aligned}$$

Theorem 3.13. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be nonnegative canonical fuzzy intervals for $i = 1, \cdots, n$. Suppose that the different inner products $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$, $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$, $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$ are obtained from Theorems 3.4, 3.6, 3.7 and 3.10, respectively. Assume that the supremum $\sup I^*$ is attained. Then*

$$I^{(EP)}_{\circledast} = I^{(\diamond DT)}_{\circledast} = I^{(\star DT)}_{\circledast} = I^{(\dagger DT)}_{\circledast} = I^*$$

and

$$\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}} = \tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}.$$

Moreover, for $\alpha \in I^*$, we have

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left[\tilde{\mathbf{a}}^L_\alpha \bullet \tilde{\mathbf{b}}^L_\alpha, \tilde{\mathbf{a}}^U_\alpha \bullet \tilde{\mathbf{b}}^U_\alpha\right].$$

The equivalence between $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ cannot be guaranteed. However, based on the α-level sets, we can compare their fuzziness.

Definition 3.14. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy intervals with interval ranges $I_{\tilde{a}}$ and $I_{\tilde{b}}$, respectively. We say that $\tilde{a}$ is *fuzzier* than $\tilde{b}$ when $I_{\tilde{a}} = I_{\tilde{b}}$ and $\tilde{b}_\alpha \subseteq \tilde{a}_\alpha$ for all $\alpha \in I_{\tilde{a}}$ with $\alpha > 0$.

Theorem 3.15. *Let $\tilde{a}^{(i)}$ and $\tilde{b}^{(i)}$ be canonical fuzzy intervals for $i = 1, \cdots, n$. Suppose that $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ are obtained from Theorems 3.6 and 3.7, respectively. Assume that the supremum $\sup I^*$ is attained. Then $I^{(\diamond DT)}_{\circledast} = I^{(\star DT)}_{\circledast} = I^*$ and $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$.*

Proof. Given any $\alpha \in I^*$ with $\alpha > 0$, we see that

$$\min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y} \leq \min_{\{\beta\in I^*:\beta\geq\alpha\}} \min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\beta,\tilde{\mathbf{b}}_\beta)} \mathbf{x}\bullet\mathbf{y} \leq \min_{\{\beta\in I^*:\beta\geq\alpha\}} \min\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}$$

and

$$\max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y} \geq \max_{\{\beta\in I^*:\beta\geq\alpha\}} \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\beta,\tilde{\mathbf{b}}_\beta)} \mathbf{x}\bullet\mathbf{y} \geq \max_{\{\beta\in I^*:\beta\geq\alpha\}} \max\left\{\tilde{\mathbf{a}}^L_\beta \bullet \tilde{\mathbf{b}}^L_\beta, \tilde{\mathbf{a}}^U_\beta \bullet \tilde{\mathbf{b}}^U_\beta\right\}.$$

From Theorems 3.11 and 3.12, we obtain

$$\left(\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}\right)_\alpha \subseteq \left(\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}\right)_\alpha$$

for each $\alpha \in I^*$ with $\alpha > 0$. This shows that $\tilde{\mathbf{a}} \circledast^{\diamond}_{DT} \tilde{\mathbf{b}}$ is fuzzier than $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$, and the proof is complete. ■

4 The Second Type of Inner Product

The first type of inner product is directly based on the inner product of real vectors. Now, the second type of inner product will be based on the form of conventional inner product. First of all, we recall the addition and multiplication of fuzzy intervals.

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy intervals with membership functions $\xi_{\tilde{a}}$ and $\xi_{\tilde{b}}$, respectively. Given an aggregation function $\mathfrak{A} : [0,1] \times [0,1] \to [0,1]$, according to the extension principle, the membership functions of addition $\tilde{a} \oplus_{EP} \tilde{b}$ and multiplication $\tilde{a} \otimes_{EP} \tilde{b}$ are defined by

$$\xi_{\tilde{a}\oplus_{EP}\tilde{b}}(z) = \sup_{\{(x,y):z=x+y\}} \mathfrak{A}\left(\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\right)$$

and

$$\xi_{\tilde{a}\otimes_{EP}\tilde{b}}(z) = \sup_{\{(x,y):z=x\cdot y\}} \mathfrak{A}\left(\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\right)$$

for each $z \in \mathbb{R}$.

By referring to (17), (18) and (19), we can define the multiplication of $\tilde{a}$ and $\tilde{b}$ according to the form of decomposition theorem by considering three different families

$$\{M_\alpha^{\otimes} : \alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} \text{ with } \alpha > 0\}$$

as follows.

- In order to define the multiplication $\tilde{a} \otimes_{DT}^{\diamond} \tilde{b}$, we take

$$\begin{aligned} M_\alpha^{\otimes} &= \tilde{a}_\alpha \cdot \tilde{b}_\alpha = \left\{xy : x \in \tilde{a}_\alpha \text{ and } y \in \tilde{b}_\alpha\right\} \\ &= \left[\min\left\{\tilde{a}_\beta^L\tilde{b}_\beta^L, \tilde{a}_\beta^L\tilde{b}_\beta^U, \tilde{a}_\beta^U\tilde{b}_\beta^L, \tilde{a}_\beta^U\tilde{b}_\beta^U\right\}, \max\left\{\tilde{a}_\beta^L\tilde{b}_\beta^L, \tilde{a}_\beta^L\tilde{b}_\beta^U, \tilde{a}_\beta^U\tilde{b}_\beta^L, \tilde{a}_\beta^U\tilde{b}_\beta^U\right\}\right] \end{aligned} \tag{32}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$.

- In order to define the multiplication $\tilde{a} \otimes_{DT}^{\star} \tilde{b}$, let M_β be a closed interval given by

$$M_\beta = \left[\min\left\{\tilde{a}_\beta^L\tilde{b}_\beta^L, \tilde{a}_\beta^U\tilde{b}_\beta^U\right\}, \max\left\{\tilde{a}_\beta^L\tilde{b}_\beta^L, \tilde{a}_\beta^U\tilde{b}_\beta^U\right\}\right].$$

  We take

$$M_\alpha^{\otimes} = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta \tag{33}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$.

- In order to define the multiplication $\tilde{a} \otimes_{DT}^{\dagger} \tilde{b}$, we take

$$M_\alpha^{\otimes} = \left[\min\left\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\right\}, \max\left\{\tilde{a}_\alpha^L\tilde{b}_\alpha^L, \tilde{a}_\alpha^U\tilde{b}_\alpha^U\right\}\right] \tag{34}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$.

Using the form of decomposition theorem, given $\otimes_{DT} \in \{\otimes_{DT}^{\diamond}, \otimes_{DT}^{\star}, \otimes_{DT}^{\dagger}\}$, the membership function of $\tilde{a} \otimes_{DT} \tilde{b}$ is defined by

$$\xi_{\tilde{a}\otimes_{DT}\tilde{b}}(z) = \sup_{\{\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^{\otimes}}(z), \tag{35}$$

where $M_\alpha^{\otimes}$ corresponds to the above three cases. We also have

$$\sup \mathcal{R}\left(\xi_{\tilde{a}\otimes_{DT}\tilde{b}}\right) = \sup_{z\in\mathbb{R}} \xi_{\tilde{a}\otimes_{DT}\tilde{b}}(z) = \sup_{z\in\mathbb{R}} \sup_{\{\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^{\otimes}}(z) = \sup\left(I_{\tilde{a}} \cap I_{\tilde{b}}\right).$$

We shall also define the addition of $\tilde{a}$ and $\tilde{b}$ based on the form of decomposition theorem by considering three different families

$$\{M_\alpha^\oplus : \alpha \in I_{\tilde{a}} \cap I_{\tilde{b}} \text{ with } \alpha > 0\}$$

as follows.

- In order to define the addition $\tilde{a} \oplus_{DT}^{\diamond} \tilde{b}$, we take

$$\begin{aligned} M_\alpha^\oplus &= \tilde{a}_\alpha + \tilde{b}_\alpha = \left\{x + y : x \in \tilde{a}_\alpha \text{ and } y \in \tilde{b}_\alpha\right\} \\ &= \left[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right] \end{aligned} \tag{36}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$.

- In order to define the addition $\tilde{a} \oplus_{DT}^{\star} \tilde{b}$, let M_β be a closed interval given by

$$M_\beta = \left[\min\left\{\tilde{a}_\beta^L + \tilde{b}_\beta^L, \tilde{a}_\beta^U + \tilde{b}_\beta^U\right\}, \max\left\{\tilde{a}_\beta^L + \tilde{b}_\beta^L, \tilde{a}_\beta^U + \tilde{b}_\beta^U\right\}\right] = \left[\tilde{a}_\beta^L + \tilde{b}_\beta^L, \tilde{a}_\beta^U + \tilde{b}_\beta^U\right].$$

  We take

$$M_\alpha^\oplus = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta \tag{37}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$. It is clear to see that

$$M_\alpha^\oplus = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} M_\beta = \bigcup_{\{\beta \in I^* : \beta \geq \alpha\}} \left[\tilde{a}_\beta^L + \tilde{b}_\beta^L, \tilde{a}_\beta^U + \tilde{b}_\beta^U\right] = \left[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right].$$

- In order to define the addition $\tilde{a} \oplus_{DT}^{\dagger} \tilde{b}$, we take

$$M_\alpha^\oplus = \left[\min\left\{\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right\}, \max\left\{\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right\}\right] = \left[\tilde{a}_\alpha^L + \tilde{b}_\alpha^L, \tilde{a}_\alpha^U + \tilde{b}_\alpha^U\right] \tag{38}$$

  for $\alpha \in I_{\tilde{a}} \cap I_{\tilde{b}}$ with $\alpha > 0$.

Then, we see that

$$\tilde{a} \oplus_{DT}^{\diamond} \tilde{b} = \tilde{a} \oplus_{DT}^{\star} \tilde{b} = \tilde{a} \oplus_{DT}^{\dagger} \tilde{b}.$$

In this case, we simply write $\tilde{a} \oplus_{DT} \tilde{b}$, and its membership function is defined by

$$\xi_{\tilde{a} \oplus_{DT} \tilde{b}}(z) = \sup_{\{\alpha \in_{\tilde{a}} \cap I_{\tilde{b}} : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^\oplus}(z), \tag{39}$$

We also have

$$\sup \mathcal{R}\left(\xi_{\tilde{a} \oplus_{DT} \tilde{b}}\right) = \sup_{z \in \mathbb{R}} \xi_{\tilde{a} \oplus_{DT} \tilde{b}}(z) = \sup_{z \in \mathbb{R}} \sup_{\{\alpha \in_{\tilde{a}} \cap I_{\tilde{b}} : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^\oplus}(z) = \sup\left(I_{\tilde{a}} \cap I_{\tilde{b}}\right).$$

Now, we are in a position to define the second type of inner product of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ as follows.

Definition 4.1. Let $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ be fuzzy intervals for $k = 1, \cdots, n$. The inner product between $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ is defined by

$$\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}} = \left(\tilde{a}^{(1)} \otimes_1 \tilde{b}^{(1)}\right) \oplus_1 \cdots \oplus_{n-1} \left(\tilde{a}^{(n)} \otimes_n \tilde{b}^{(n)}\right), \tag{40}$$

where the addition

$$\oplus_i \in \{\oplus_{EP}, \oplus_{DT}\} \text{ for } i = 1, \cdots, n-1. \tag{41}$$

and the multiplication

$$\otimes_j \in \{\otimes_{EP}, \otimes_{DT}^{\diamond}, \otimes_{DT}^{\star}, \otimes_{DT}^{\dagger}\} \text{ for } j = 1, \cdots, n. \tag{42}$$

The inner product $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$ depends on the choice of addition and multiplication according to (41) and (42), respectively. Therefore, it is completely different from the first type of inner product $\tilde{\mathbf{a}} \circledast \tilde{\mathbf{b}}$ for

$$\circledast \in \{\circledast_{EP}, \circledast_{DT}^{\diamond}, \circledast_{DT}^{\star}, \circledast_{DT}^{\dagger}\}.$$

We write $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_j \tilde{b}^{(j)}$ for $j = 1, \cdots, n$, which can also be regarded as the special case of first type of inner product $\tilde{c}^{(j)} = \tilde{a}^{(j)} \circledast \tilde{b}^{(j)}$ by considering $n = 1$. Let $I_{\tilde{c}^{(j)}}$ be the interval range of $\tilde{c}^{(j)}$. According to the results in Wu [15], we see that

$$I_{\tilde{c}^{(j)}} = I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}}. \tag{43}$$

Now, we have

$$\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}} = \tilde{c}^{(1)} \oplus_1 \cdots \oplus_{n-1} \tilde{c}^{(n)}. \tag{44}$$

Remark 4.2. According to the results in Wu [15], when each of the following supremum

$$\sup \left(I_{\tilde{c}^{(j)}} \cap I_{\tilde{c}^{(j+1)}} \right) \tag{45}$$

is attained for $j = 1, \cdots, n-1$, the additions $\oplus_{EP}$ and $\oplus_{DT}$ are equivalent. In this case, the inner product (44) can be simply written as

$$\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}} = \tilde{c}^{(1)} \oplus \cdots \oplus \tilde{c}^{(n)}.$$

The membership function $\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}$ is in a very general situation, since the addition $\oplus_i$ for $i = 1, \cdots, n-1$ and multiplication $\otimes_j$ for $j = 1, \cdots, n$ can be any operations in (41) and (42), respectively. However, we can use the Decomposition Theorem 2.1 to rewrite the membership function $\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}$ using its α-level sets.

The interval range of $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$ is denoted by $I_{\circledcirc}$. Let $(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha$ be the α-level set of $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$. According to the Decomposition Theorem 2.1, the membership function is given by

$$\begin{aligned} \xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}(x) &= \sup_{\alpha \in \mathcal{R}(\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}})} \alpha \cdot \chi_{(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha}(x) = \max_{\alpha \in \mathcal{R}(\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}})} \alpha \cdot \chi_{(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha}(x) \\ &= \sup_{\alpha \in I_{\circledcirc}} \alpha \cdot \chi_{(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha}(x) = \max_{\alpha \in I_{\circledcirc}} \alpha \cdot \chi_{(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha}(x). \end{aligned}$$

The purpose is to obtain the α-level set $(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha$. We can see that

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}\right) = \sup_{x \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}(x) = \sup_{x \in \mathbb{R}} \sup_{\alpha \in I_{\circledcirc}} \alpha \cdot \chi_{(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha}(x) = \sup I_{\circledcirc} \equiv \alpha^\circ. \tag{46}$$

The definition of interval range says that

$$I_{\circledcirc} = \begin{cases} [0, \alpha^\circ], & \text{if the supremum } \alpha^\circ = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}\right) \text{ is attained} \\ [0, \alpha^\circ), & \text{if the supremum } \alpha^\circ = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}}\right) \text{ is not attained.} \end{cases} \tag{47}$$

By referring to (2), we have $(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha \neq \emptyset$ for $\alpha \in I_{\circledcirc}$, and $(\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}})_\alpha = \emptyset$ for $\alpha \notin I_{\circledcirc}$.

4.1 Using the Extension Principle

Now, we can take $\oplus_i = \oplus_{EP}$ in (41) for $i = 1, \cdots, n-1$ and $\otimes_j = \otimes_{EP}$ in (42) for $j = 1, \cdots, n$. In this case, the membership functions of $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_{EP} \tilde{b}^{(j)}$ are given by

$$\xi_{\tilde{c}^{(j)}}(z) = \sup_{\{(x,y): z = xy\}} \mathfrak{A}_j \left(\xi_{\tilde{a}^{(j)}}(x), \xi_{\tilde{b}^{(j)}}(y) \right) \tag{48}$$

for $j = 1, \cdots, n$, where each $\mathfrak{A}_j : [0,1] \times [0,1] \to [0,1]$ is an aggregation function defined on $[0,1]^2$ for $j = 1, \cdots, n$.

From (44), the membership function of $\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}$ is given by

$$\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}(z) = \sup_{\{(x_1,\cdots,x_n): z = x_1 + \cdots + x_n\}} \mathfrak{A}\left(\xi_{\tilde{c}^{(1)}}(x_1), \cdots, \xi_{\tilde{c}^{(n)}}(x_n)\right), \tag{49}$$

where $\mathfrak{A} : [0,1]^n \to [0,1]$ is an aggregation function defined on $[0,1]^n$. Next, we are going to study the α-level sets $(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}})_\alpha$.

Let $I_{\circledcirc}^{(EP)}$ be the interval range of $\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}$. From (46), we have

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}\right) = \sup_{x \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}(x) = \sup I_{\circledcirc}^{(EP)} \equiv \alpha^\circ. \tag{50}$$

Theorem 4.3. *Let $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ be fuzzy intervals for $k = 1, \cdots, n$. Suppose that the aggregation functions $\mathfrak{A}_j$ for $j = 1, \cdots, n$ and $\mathfrak{A}$ are taken by*

$$\mathfrak{A}_j(\alpha_1, \alpha_2) = \begin{cases} \min\{\alpha_1, \alpha_2\}, & \text{if } \alpha_1 \in \mathcal{R}(\xi_{\tilde{a}^{(j)}}) \text{ and } \alpha_2 \in \mathcal{R}(\xi_{\tilde{b}^{(j)}}) \\ \text{any expression}, & \text{otherwise}, \end{cases}$$

for $j = 1, \cdots, n$ and

$$\mathfrak{A}(\alpha_1, \cdots, \alpha_n) = \begin{cases} \min\{\alpha_1, \cdots, \alpha_n\}, & \text{if } \alpha_j \in \mathcal{R}(\xi_{\tilde{c}^{(j)}}) \text{ for } j = 1, \cdots, n \\ \text{any expression}, & \text{otherwise}, \end{cases}$$

Let

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

Assume that the supremum $\sup \mathcal{R}(\xi_{\tilde{a}^{(j)}})$, $\sup \mathcal{R}(\xi_{\tilde{b}^{(j)}})$ and $\sup \mathcal{R}(\xi_{\tilde{c}^{(j)}})$ are attained for $j = 1, \cdots, n$. Then, we have the following results.

(i) *We have*

$$\alpha^\circ = \sup I_{\circledcirc}^{(EP)} = \min\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\} = \sup I^*, \tag{51}$$

where α_j^* and β_j^* for $j = 1, \cdots, n$ are given in (4).

(ii) *The supremum $\sup I_{\circledcirc}^{(EP)}$ is attained if and only if the supremum $\sup I^*$ is attained.*

(iii) *We have $I_{\circledcirc}^{(EP)} = I^*$.*

Proof. To prove part (i), from (5) and (6), we immediately have

$$\min\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\} = \sup I^*.$$

Since the supremum $\sup \mathcal{R}(\xi_{\tilde{a}^{(j)}})$ and $\sup \mathcal{R}(\xi_{\tilde{b}^{(j)}})$ are attained, it follows that $\alpha_j^* \in \mathcal{R}(\xi_{\tilde{a}^{(j)}})$ and $\beta_j^* \in \mathcal{R}(\xi_{\tilde{b}^{(j)}})$ for $j = 1, \cdots, n$. From (48), we have

$$\gamma_j^* = \sup_{z \in \mathbb{R}} \xi_{\tilde{c}^{(j)}}(z) = \sup_{z \in \mathbb{R}} \sup_{\{(x,y): z = xy\}} \min\left\{\xi_{\tilde{a}^{(j)}}(x), \xi_{\tilde{b}^{(j)}}(y)\right\} \geq \min\left\{\alpha_j^*, \beta_j^*\right\}.$$

On the other hand, since

$$\min\left\{\xi_{\tilde{a}^{(j)}}(x), \xi_{\tilde{b}^{(j)}}(y)\right\} \leq \min\left\{\alpha_j^*, \beta_j^*\right\} \text{ for any } x, y \in \mathbb{R},$$

it follows that

$$\gamma_j^* = \sup_{z \in \mathbb{R}} \sup_{\{(x,y): z = xy\}} \min\left\{\xi_{\tilde{a}^{(j)}}(x), \xi_{\tilde{b}^{(j)}}(y)\right\} \leq \min\left\{\alpha_j^*, \beta_j^*\right\}.$$

Therefore, we obtain

$$\gamma_j^* = \min\left\{\alpha_j^*, \beta_j^*\right\}. \tag{52}$$

Let $\gamma_j^* = \sup \mathcal{R}(\xi_{\tilde{c}^{(j)}})$ for $j = 1, \cdots, n$. Since the supremum $\sup \mathcal{R}(\xi_{\tilde{c}^{(j)}})$ are attained for $j = 1, \cdots, n$, it follows that $\gamma_j \in \mathcal{R}(\xi_{\tilde{c}^{(j)}})$ for $j = 1, \cdots, n$. From (50) and (49), we have

$$\begin{aligned}\alpha^\circ &= \sup I_{\circledcirc}^{(EP)} = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}\right) = \sup_{z \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}(z) \\ &= \sup_{z \in \mathbb{R}} \sup_{\{(x_1, \cdots, x_n) : z = x_1 + \cdots + x_n\}} \min \left\{ \xi_{\tilde{c}^{(1)}}(x_1), \cdots, \xi_{\tilde{c}^{(n)}}(x_n) \right\} \\ &\geq \min \left\{ \gamma_1^*, \cdots, \gamma_n^* \right\} \\ &= \min \left\{ \alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^* \right\} \text{ (by (52)).}\end{aligned} \tag{53}$$

On the other hand, since

$$\min \left\{ \xi_{\tilde{c}^{(1)}}(x_1), \cdots, \xi_{\tilde{c}^{(n)}}(x_n) \right\} \leq \min \left\{ \gamma_1^*, \cdots, \gamma_n^* \right\} \text{ for any } x_1, \cdots, x_n \in \mathbb{R},$$

it follows that

$$\alpha^\circ = \sup_{z \in \mathbb{R}} \sup_{\{(x_1, \cdots, x_n) : z = x_1 + \cdots + x_n\}} \min \left\{ \xi_{\tilde{c}^{(1)}}(x_1), \cdots, \xi_{\tilde{c}^{(n)}}(x_n) \right\} \leq \min \left\{ \gamma_1^*, \cdots, \gamma_n^* \right\},$$

which proves part (i).

To prove part (ii), the equalities (50) says that the supremun $\sup \mathcal{R}(\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}})$ is attained if and only if the supremum $\sup I_{\circledcirc}^{(EP)}$ is attained. Suppose that the supremum $\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}\right)$ is attained. From (53), there exists $z^* \in \mathbb{R}$ such that

$$\sup_{\{(x_1, \cdots, x_n) : z^* = x_1 + \cdots + x_n\}} \min \left\{ \xi_{\tilde{c}^{(1)}}(x_1), \cdots, \xi_{\tilde{c}^{(n)}}(x_n) \right\} = \alpha^\circ. \tag{54}$$

Since $\tilde{a}^{(j)}$ and $\tilde{b}^{(j)}$ are fuzzy intervals, it is well-known that the multiplication $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_{EP} \tilde{b}^{(j)}$ is also a fuzzy interval, i.e., the membership functions $\xi_{\tilde{c}^{(j)}}$ are upper semi-continuous for $j = 1, \cdots, n$. Therefore, the minimum function $\min \{\xi_{\tilde{c}^{(1)}}, \cdots, \xi_{\tilde{c}^{(n)}}\}$ is also upper semi-continuous. Since the set $\{(x_1, \cdots, x_n) : z^* = x_1 + \cdots + x_n\}$ is a compact set (a closed and bounded set), using Lemma 3.8, it follows that the supremum in (54) is attained. In other words, there exist $(x_1^*, \cdots, x_n^*)$ and $n_1 \in \{1, \cdots, n\}$ satisfying

$$\min \left\{ \xi_{\tilde{c}^{(1)}}(x_1^*), \cdots, \xi_{\tilde{c}^{(n)}}(x_n^*) \right\} = \alpha^\circ$$

and

$$\xi_{\tilde{c}^{(n_1)}}(x_{n_1}^*) = \alpha^\circ \leq \xi_{\tilde{c}^{(j)}}(x_i^*) \text{ for all } j = 1, \cdots, n. \tag{55}$$

Using (43), we see that

$$I^* = I_{\tilde{c}^{(1)}} \cap \cdots \cap I_{\tilde{c}^{(n)}}. \tag{56}$$

From (51) and (52), we also see that

$$\alpha^\circ = \min \left\{ \gamma_1^*, \cdots, \gamma_n^* \right\} = \sup I^*, \tag{57}$$

which also says that $\alpha^\circ = \gamma_{n_2}^*$ for some $n_2 \in \{1, \cdots, n\}$. Then, using (55), we have

$$\gamma_{n_2}^* = \alpha^\circ = \xi_{\tilde{c}^{(n_1)}}(x_{n_1}^*) \leq \xi_{\tilde{c}^{(n_2)}}(x_{n_2}^*) \leq \gamma_{n_2}^*,$$

which says that the supremum $\gamma_{n_2}^* = \sup \mathcal{R}(\xi_{\tilde{c}^{(n_2)}})$ is attained. Therefore, using (56) and (57), we obtain

$$I^* = I_{\tilde{c}^{(n_2)}} = [0, \alpha_{n_2}^*]$$

is a closed interval, which says that the supremum $\sup I^*$ is attained.

On the other hand, suppose that the supremum $\sup I^*$ is attained. Then, using (56) and (57) again, we have

$$I^* = [0, \gamma_{n_3}^*] = I_{\tilde{c}^{(n_3)}} \text{ for some } n_3 \in \{1, \cdots, n\}$$

and

$$\gamma^*_{n_3} = \min\left\{\gamma^*_1, \cdots, \gamma^*_n\right\} = \alpha^\circ,$$

which also says that the supremum $\gamma^*_{n_3} = \sup \mathcal{R}(\xi_{\tilde{c}^{(n_3)}})$ is attained, i.e., there exists $x^\circ_{n_3} \in \mathbb{R}$ satisfying

$$\xi_{\tilde{a}^{(n_3)}}(x^\circ_{n_3}) = \gamma^*_{n_3} = \alpha^\circ.$$

Since the supremum (54) is attained, there exists $(x^\circ_1, \cdots, x^\circ_n)$ such that its n_3-component is $x^\circ_{n_3}$ and

$$\alpha^\circ = \min\left\{\xi_{\tilde{c}^{(1)}}(x^\circ_1), \cdots, \xi_{\tilde{c}^{(n_3)}}(x^\circ_{n_3}) = \alpha^\circ, \cdots, \xi_{\tilde{c}^{(n)}}(x^\circ_n)\right\}.$$

In this case, we take $z^\circ \equiv x^\circ_1 + \cdots + x^\circ_n$, which says that the supremum $\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}}\right)$ is attained at z_0, which proves part (ii). Finally, part (iii) follows immediately from parts (j) and (ii). This completes the proof. ■

Now, we assume that $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ are taken to be canonical fuzzy intervals for $k = 1, \cdots, n$. We shall present the α-level sets $(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}})_\alpha$. Recall that the multiplication $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_i \tilde{b}^{(j)}$ in (40) for $j = 1, \cdots, n$ can be regarded as the special case of first type of inner product $\tilde{c}^{(j)} = \tilde{a}^{(j)} \circledast_{EP} \tilde{b}^{(j)}$ by considering $n = 1$. Suppose that the supremum $\sup(I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}})$ are attained for all $j = 1, \cdots, n$. Then, from Theorem 3.11, for any $\alpha \in I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}}$, we have

$$\begin{aligned}
\tilde{c}^{(j)}_\alpha = \left(\tilde{a}^{(j)} \otimes_{EP} \tilde{b}^{(j)}\right)_\alpha &= \left[\min_{(x,y) \in (\tilde{a}^{(j)}_\alpha, \tilde{b}^{(j)}_\alpha)} xy, \max_{(x,y) \in (\tilde{a}^{(j)}_\alpha, \tilde{b}^{(j)}_\alpha)} xy\right] \\
&= \left[\min\left\{\tilde{a}^L_{j\alpha}\tilde{b}^L_{j\alpha}, \tilde{a}^L_{j\alpha}\tilde{b}^U_{j\alpha}, \tilde{a}^U_{j\alpha}\tilde{b}^L_{j\alpha}, \tilde{a}^U_{j\alpha}\tilde{b}^U_{j\alpha}\right\}, \max\left\{\tilde{a}^L_{j\alpha}\tilde{b}^L_{j\alpha}, \tilde{a}^L_{j\alpha}\tilde{b}^U_{j\alpha}, \tilde{a}^U_{j\alpha}\tilde{b}^L_{j\alpha}, \tilde{a}^U_{j\alpha}\tilde{b}^U_{j\alpha}\right\}\right] \\
&\equiv \left[\tilde{c}^L_{j\alpha}, \tilde{c}^U_{j\alpha}\right].
\end{aligned} \tag{58}$$

Suppose that the supremum in (45) are attained for $j = 1, \cdots, n-1$. Then, Remark 4.2 says that the operations $\oplus_{EP}$ and $\oplus_{DT}$ are equivalent. The equality (56) also says that the the supremum $\sup I^*$ is attained. Therefore, Theorem 4.3 says that $I^{(EP)}_{\circledcirc} = I^*$. Moreover, according to (44) and the results in Wu [15], for any $\alpha \in I^{(EP)}_{\circledcirc} = I^*$, we have

$$\begin{aligned}
\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha &= \tilde{c}^{(1)}_\alpha + \cdots + \tilde{c}^{(n)}_\alpha = \left[\tilde{c}^L_{1\alpha} + \cdots + \tilde{c}^L_{n\alpha}, \tilde{c}^U_{1\alpha} + \cdots + \tilde{c}^U_{n\alpha}\right] \\
&\equiv \left[\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)^L_\alpha, \left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)^U_\alpha\right],
\end{aligned} \tag{59}$$

and

$$\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)^L_\alpha = \sum_{j=1}^n \tilde{c}^L_{j\alpha} \text{ and } \left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)^U_\alpha = \sum_{j=1}^n \tilde{c}^U_{j\alpha}. \tag{60}$$

Therefore, we can calculate the α-level sets $(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}})_\alpha$ according to the above formulas.

4.2 Using the Form of Decomposition Theorem

Now, we can take $\oplus_i = \oplus_{DT}$ in (41) for $i = 1, \cdots, n-1$ and $\otimes_j = \otimes^\diamond \in \{\otimes^\diamond_{DT}, \otimes^\star_{DT}, \otimes^\dagger_{DT}\}$ in (42) for $j = 1, \cdots, n$. In this case, by referring to (35), the membership functions of $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_{DT} \tilde{b}^{(j)}$ are given by

$$\xi_{\tilde{c}^{(j)}}(z) = \sup_{\{\alpha \in I_{\tilde{c}^{(j)}} : \alpha > 0\}} \alpha \cdot \chi_{M^{\otimes_j}_\alpha}(z), \tag{61}$$

for $j = 1, \cdots, n$, where $I_{\tilde{c}^{(j)}}$ is given in (43), and $M^{\otimes_j}_\alpha$ are given in (32), (33) or (34) regarding $\tilde{a}^{(j)}$ and $\tilde{b}^{(j)}$ for $j = 1, \cdots, n$.

Recall that

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}} = I_{\tilde{c}^{(1)}} \cap \cdots \cap I_{\tilde{c}^{(n)}}.$$

By referring to (39), the membership function of inner product (44) is given by

$$\xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}(z) = \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^{\oplus}}(z), \tag{62}$$

where $M_\alpha^{\oplus}$ are given in (36), (37) or (38) by replacing $\tilde{a}_\alpha^L + \tilde{b}_\alpha^L$ and $\tilde{a}_\alpha^U + \tilde{b}_\alpha^U$ as

$$\sum_{j=1}^{n} \tilde{a}_{j\alpha}^L + \sum_{j=1}^{n} \tilde{b}_{j\alpha}^L \text{ and } \sum_{j=1}^{n} \tilde{a}_{j\alpha}^U + \sum_{j=1}^{n} \tilde{b}_{j\alpha}^U,$$

respectively, since we assume to take all the same addition $\oplus_i = \oplus_{DT}$ for $i = 1, \cdots, n-1$.

Let $I_{\circledcirc}^{(DT)}$ be the interval range of $\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}$. From (46), we have

$$\sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}\right) = \sup_{x \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}(x) = \sup I_{\circledcirc}^{(DT)} \equiv \alpha^\circ. \tag{63}$$

Next, we shall also present the α-level sets $(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}})_\alpha$.

Theorem 4.4. *Let $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ be fuzzy intervals for $k = 1, \cdots, n$, and let*

$$I^* = I_{\tilde{a}^{(1)}} \cap \cdots \cap I_{\tilde{a}^{(n)}} \cap I_{\tilde{b}^{(1)}} \cap \cdots \cap I_{\tilde{b}^{(n)}}.$$

The following statements hold true.

(i) *We have*

$$\alpha^\circ = \sup I_{\circledcirc}^{(DT)} = \min\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\} = \sup I^*,$$

where α_j^* and β_j^* for $j = 1, \cdots, n$ are given in (4).

(ii) *Suppose that the supremum $\sup I^*$ is attained. Then, the supremum $\sup I_{\circledcirc}^{(DT)}$ is attained and $I_{\circledcirc}^{(DT)} = I^*$.*

Proof. To prove part (i), from (5) and (6), we immediately have

$$\min\{\alpha_1^*, \cdots, \alpha_n^*, \beta_1^*, \cdots, \beta_n^*\} = \sup I^*. \tag{64}$$

From (62) and (63), we also have

$$\alpha^\circ = \sup I_{\circledcirc}^{(DT)} = \sup \mathcal{R}\left(\xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}\right) = \sup_{z \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}(z) = \sup_{z \in \mathbb{R}} \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^\bullet}(z) = \sup I^*, \tag{65}$$

which proves part (i).

To prove part (ii), since the supremum $\sup I^*$ is attained, from (64), we have $I^* = [0, \alpha^\circ]$. From (65), we also have

$$\alpha^\circ = \sup_{z \in \mathbb{R}} \sup_{\{\alpha \in I^* : \alpha > 0\}} \alpha \cdot \chi_{M_\alpha^\bullet}(z),$$

which says that we can take $z \in M_{\alpha^\circ}^\bullet \subset \mathbb{R}$. Therefore, the supremum of the range $\mathcal{R}(\xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}})$ is attained with

$$\alpha^\circ = \sup_{z \in \mathbb{R}} \xi_{\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}}(z),$$

which also says that the supremum $\sup I_{\circledcirc}^{(DT)}$ is attained. From (47) and part (i), it follows that $I_{\circledcirc}^{(DT)} = I^*$. This completes the proof. ■

Now, we assume that $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ are taken to be canonical fuzzy intervals for $k = 1, \cdots, n$. We shall present the α-level sets $(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}})_\alpha$. Recall that the multiplication $\tilde{c}^{(j)} = \tilde{a}^{(j)} \otimes_i \tilde{b}^{(j)}$ in (40) for $j = 1, \cdots, n$ can be regarded as the special case of first type of inner product $\tilde{c}^{(j)} = \tilde{a}^{(j)} \circledast_{DT} \tilde{b}^{(j)}$ by considering $n = 1$. Suppose that the supremum $\sup(I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}})$ are attained for all $j = 1, \cdots, n$. Then, we can obtain two different kinds of α-level sets $\tilde{c}_\alpha^{(j)}$ from Theorems 3.11 and 3.12.

- From Theorem 3.11, for any $\alpha \in I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}}$, we have

$$\begin{aligned}\tilde{c}_\alpha^{(j)} &= \left(\tilde{a}^{(j)} \otimes_{DT}^{\diamond} \tilde{b}^{(j)}\right)_\alpha = \left[\min_{(x,y)\in(\tilde{a}_\alpha^{(j)}, \tilde{b}_\alpha^{(j)})} xy, \max_{(x,y)\in(\tilde{a}_\alpha^{(j)}, \tilde{b}_\alpha^{(j)})} xy\right] \\ &= \left[\min\left\{\tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U\right\}, \max\left\{\tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U\right\}\right] \\ &\equiv \left[\tilde{c}_{j\alpha}^L, \tilde{c}_{j\alpha}^U\right].\end{aligned} \tag{66}$$

- From Theorem 3.12, for any $\alpha \in I_{\tilde{a}^{(j)}} \cap I_{\tilde{b}^{(j)}}$, we also have

$$\tilde{c}_\alpha^{(j)} = \left(\tilde{a}^{(j)} \otimes_{DT}^{\star} \tilde{b}^{(j)}\right)_\alpha = \left(\tilde{a}^{(j)} \otimes_{DT}^{\dagger} \tilde{b}^{(j)}\right)_\alpha \tag{67}$$

$$\begin{aligned}&= \left[\min_{\{\beta\in I^*:\beta\geq\alpha\}} \min\left\{\tilde{a}_{j\beta}^L \tilde{b}_{j\beta}^L, \tilde{a}_{j\beta}^U \tilde{b}_{j\beta}^U\right\}, \max_{\{\beta\in I^*:\beta\geq\alpha\}} \max\left\{\tilde{a}_{j\beta}^L \tilde{b}_{j\beta}^L, \tilde{a}_{j\beta}^U \tilde{b}_{j\beta}^U\right\}\right] \quad (68)\\ &= \left[\min\left\{\min_{\{\beta\in I^*:\beta\geq\alpha\}} \tilde{a}_{j\beta}^L \tilde{b}_{j\beta}^L, \min_{\{\beta\in I^*:\beta\geq\alpha\}} \tilde{a}_{j\beta}^U \tilde{b}_{j\beta}^U\right\},\right. \\ &\qquad \left.\max\left\{\max_{\{\beta\in I^*:\beta\geq\alpha\}} \tilde{a}_{j\beta}^L \tilde{b}_{j\beta}^L, \max_{\{\beta\in I^*:\beta\geq\alpha\}} \tilde{a}_{j\beta}^U \tilde{b}_{j\beta}^U\right\}\right] \\ &\equiv \left[\tilde{c}_{j\alpha}^L, \tilde{c}_{j\alpha}^U\right].\end{aligned}$$

Suppose that the supremum $\sup I^*$ is attained. Theorem 4.4 says that $I_{\circledcirc}^{(DT)} = I^*$. Moreover, according to (44) and the results in Wu [15], for any $\alpha \in I_{\circledcirc}^{(DT)} = I^*$, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha &= \tilde{c}_\alpha^{(1)} + \cdots + \tilde{c}_\alpha^{(n)} = \left[\tilde{c}_{1\alpha}^L + \cdots + \tilde{c}_{n\alpha}^L, \tilde{c}_{1\alpha}^U + \cdots + \tilde{c}_{n\alpha}^U\right] \\ &\equiv \left[\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^L, \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^U\right],\end{aligned} \tag{69}$$

where the α-level sets $\tilde{c}_\alpha^{(j)}$ can be taken from (66) or (68) for $j = 1, \cdots, n$, and

$$\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^L = \sum_{j=1}^n \tilde{c}_{j\alpha}^L \text{ and } \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^U = \sum_{j=1}^n \tilde{c}_{j\alpha}^U. \tag{70}$$

Therefore, we can calculate two kinds of α-level sets $(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}})_\alpha$ using (66) or (68) for $j = 1, \cdots, n$.

4.3 Comparison of Fuzziness

By referring to Definition 3.14, we are gong to compare the fuzziness between the first type of inner product $\tilde{\mathbf{a}} \circledast \tilde{\mathbf{b}}$ for $\circledast \in \{\circledast_{EP}, \circledast_{DT}^{\diamond}, \circledast_{DT}^{\star}, \circledast_{DT}^{\dagger}\}$ and the second type of inner product $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$.

Theorem 4.5. *Let $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ be canonical fuzzy intervals for $k = 1, \cdots, n$ such that the supremum $\sup I^*$ is attained. Suppose that the first type of inner products $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ are obtained from Theorems 3.4 and 3.6, respectively. Then, we have the following results.*

(i) *Assume that the second type of inner product is taken by*

$$\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}} = \left(\tilde{a}^{(1)} \otimes_{EP} \tilde{b}^{(1)}\right) \oplus_{EP} \cdots \oplus_{EP} \left(\tilde{a}^{(n)} \otimes_{EP} \tilde{b}^{(n)}\right) \tag{71}$$

Then, the first type of inner products $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ *and* $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ *are fuzzier than the second type of inner product* $\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}$ *in the sense of*

$$\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha \subseteq \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha$$

for $\alpha \in I^* = I_{\circledcirc}^{(EP)} = I_{\circledast}^{(EP)} = I_{\circledast}^{(\diamond DT)}$.

(ii) *Assume that the second type of inner product is taken by*

$$\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}} = \left(\tilde{a}^{(1)} \otimes_{DT}^{\diamond} \tilde{b}^{(1)}\right) \oplus_{DT} \cdots \oplus_{DT} \left(\tilde{a}^{(n)} \otimes_{DT}^{\diamond} \tilde{b}^{(n)}\right). \tag{72}$$

Then, the first type of inner products $\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}$ *and* $\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}$ *are fuzzier than the second type of inner product* $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$ *in the sense of*

$$\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha \subseteq \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha$$

for $\alpha \in I^* = I_{\circledcirc}^{(DT)} = I_{\circledast}^{(EP)} = I_{\circledast}^{(\diamond DT)}$.

Proof. It is clear to see that

$$\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n) = \sum_{j=1}^n \left(\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} x_j y_j \right) \tag{73}$$

and

$$\max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n) = \sum_{j=1}^n \left(\max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} x_j y_j \right), \tag{74}$$

since their objective functions are separable. Now, we have

$$\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} x_j y_j \leq \min \left\{ \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U \right\}$$

and

$$\max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} x_j y_j \geq \max \left\{ \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U \right\}$$

which imply, by referring (73) and (74),

$$\min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n) \leq \sum_{j=1}^n \left(\min \left\{ \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U \right\} \right) \tag{75}$$

and

$$\max_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} (x_1 y_1 + \cdots + x_n y_n) \geq \sum_{j=1}^n \left(\max \left\{ \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^L \tilde{b}_{j\alpha}^U, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^L, \tilde{a}_{j\alpha}^U \tilde{b}_{j\alpha}^U \right\} \right) \tag{76}$$

To prove part (i), we obtain

$$\begin{aligned} &\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^L = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha^L = \min_{(\mathbf{x},\mathbf{y}) \in (\tilde{\mathbf{a}}_\alpha, \tilde{\mathbf{b}}_\alpha)} \mathbf{x} \bullet \mathbf{y} \text{ (using Theorem 3.11)} \\ &\quad \leq \left(\tilde{a}^{(1)} \otimes_{EP} \tilde{b}^{(1)}\right)_\alpha^L + \cdots + \left(\tilde{a}^{(n)} \otimes_{EP} \tilde{b}^{(n)}\right)_\alpha^L = \left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha^L \text{ (using (58), (60), (71) and (75)).} \end{aligned}$$

Using (76), we can similarly obtain

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^U = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha^U = \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}$$
$$\geq \left(\tilde{a}^{(1)} \otimes_{EP} \tilde{b}^{(1)}\right)_\alpha^U + \cdots + \left(\tilde{a}^{(n)} \otimes_{EP} \tilde{b}^{(n)}\right)_\alpha^U = \left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha^U.$$

For $\alpha \in I^*$, it follows that

$$\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left[\left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha^L, \left(\tilde{\mathbf{a}} \circledcirc_{EP} \tilde{\mathbf{b}}\right)_\alpha^U\right] \text{ (using (59))}$$
$$\subseteq \left[\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^L, \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^U\right] = \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha.$$

To prove part (ii), we obtain

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^L = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha^L = \min_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y} \text{ (using Theorem 3.11)}$$
$$\leq \left(\tilde{a}^{(1)} \otimes_{DT}^{\diamond} \tilde{b}^{(1)}\right)_\alpha^L + \cdots + \left(\tilde{a}^{(n)} \otimes_{DT}^{\diamond} \tilde{b}^{(n)}\right)_\alpha^L = \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^L \text{ (using (66), (70), (72) and (75))}$$

Using (76), we can similarly obtain

$$\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^U = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha^U = \max_{(\mathbf{x},\mathbf{y})\in(\tilde{\mathbf{a}}_\alpha,\tilde{\mathbf{b}}_\alpha)} \mathbf{x}\bullet\mathbf{y}$$
$$\geq \left(\tilde{a}^{(1)} \otimes_{DT}^{\diamond} \tilde{b}^{(1)}\right)_\alpha^U + \cdots + \left(\tilde{a}^{(n)} \otimes_{DT}^{\diamond} \tilde{b}^{(n)}\right)_\alpha^U = \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^U.$$

For $\alpha \in I^*$, it follows that

$$\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha = \left[\left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^L, \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha^U\right] \text{ (using (69))}$$
$$\subseteq \left[\left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^L, \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha^U\right] = \left(\tilde{\mathbf{a}} \circledast_{EP} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\diamond} \tilde{\mathbf{b}}\right)_\alpha.$$

This completes the proof. ■

Theorem 4.6. *Let $\tilde{a}^{(k)}$ and $\tilde{b}^{(k)}$ be canonical fuzzy intervals for $k = 1, \cdots, n$ such that the supremum $\sup I^*$ is attained. Suppose that the first type of inner products $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$ are obtained from Theorems 3.7 and 3.10, respectively. We also assume that the second type of inner product is taken by*

$$\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}} = \left(\tilde{a}^{(1)} \otimes_{DT}^{\star} \tilde{b}^{(1)}\right) \oplus_{DT} \cdots \oplus_{DT} \left(\tilde{a}^{(n)} \otimes_{DT}^{\star} \tilde{b}^{(n)}\right) \tag{77}$$

or

$$\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}} = \left(\tilde{a}^{(1)} \otimes_{DT}^{\dagger} \tilde{b}^{(1)}\right) \oplus_{DT} \cdots \oplus_{DT} \left(\tilde{a}^{(n)} \otimes_{DT}^{\dagger} \tilde{b}^{(n)}\right). \tag{78}$$

Then the second type of inner product $\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}$ is fuzzier than the first type of inner products $\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}$ in the sense of

$$\left(\tilde{\mathbf{a}} \circledast_{DT}^{\star} \tilde{\mathbf{b}}\right)_\alpha = \left(\tilde{\mathbf{a}} \circledast_{DT}^{\dagger} \tilde{\mathbf{b}}\right)_\alpha \subseteq \left(\tilde{\mathbf{a}} \circledcirc_{DT} \tilde{\mathbf{b}}\right)_\alpha$$

for $\alpha \in I^* = I_{\circledcirc}^{(DT)} = I_{\circledast}^{(\star DT)} = I_{\circledast}^{(\dagger DT)}$.

Proof. Now, we have

$$\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta}+\cdots+\tilde{a}^L_{n\beta}\tilde{b}^L_{n\beta}\geq\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\min\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}$$

and

$$\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}+\cdots+\tilde{a}^U_{n\beta}\tilde{b}^U_{n\beta}\geq\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\min\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}$$

which imply

$$\begin{aligned}\min\left\{\tilde{\mathbf{a}}^L_\alpha\bullet\tilde{\mathbf{b}}^L_\alpha,\tilde{\mathbf{a}}^U_\alpha\bullet\tilde{\mathbf{b}}^U_\alpha\right\}&=\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta}+\cdots+\tilde{a}^L_{n\beta}\tilde{b}^L_{n\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}+\cdots+\tilde{a}^U_{n\beta}\tilde{b}^U_{n\beta}\right\}\\&\geq\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\min\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}.\end{aligned}\tag{79}$$

We can similarly obtain

$$\max\left\{\tilde{\mathbf{a}}^L_\alpha\bullet\tilde{\mathbf{b}}^L_\alpha,\tilde{\mathbf{a}}^U_\alpha\bullet\tilde{\mathbf{b}}^U_\alpha\right\}\leq\max\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\max\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}.\tag{80}$$

Therefore, we have

$$\begin{aligned}\left(\tilde{\mathbf{a}}\circledast^\star_{DT}\tilde{\mathbf{b}}\right)^L_\alpha&=\left(\tilde{\mathbf{a}}\circledast^\dagger_{DT}\tilde{\mathbf{b}}\right)^L_\alpha=\min_{\{\beta\in I^*:\beta\geq\alpha\}}\min\left\{\tilde{\mathbf{a}}^L_\alpha\bullet\tilde{\mathbf{b}}^L_\alpha,\tilde{\mathbf{a}}^U_\alpha\bullet\tilde{\mathbf{b}}^U_\alpha\right\}\text{ (using Theorem 3.12)}\\&\geq\min_{\{\beta\in I^*:\beta\geq\alpha\}}\left[\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\min\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}\right]\text{ (using (79))}\\&\geq\min_{\{\beta\in I^*:\beta\geq\alpha\}}\min\left\{\tilde{a}^L_{1\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{1\beta}\right\}+\cdots+\min_{\{\beta\in I^*:\beta\geq\alpha\}}\min\left\{\tilde{a}^L_{n\beta}\tilde{b}^L_{1\beta},\tilde{a}^U_{1\beta}\tilde{b}^U_{n\beta}\right\}\\&=\left(\tilde{a}^{(1)}\otimes^\star_{DT}\tilde{b}^{(1)}\right)^L_\alpha+\cdots+\left(\tilde{a}^{(n)}\otimes^\star_{DT}\tilde{b}^{(n)}\right)^L_\alpha\text{ (using (67) and (68))}\\&=\left(\tilde{a}^{(1)}\otimes^\dagger_{DT}\tilde{b}^{(1)}\right)^L_\alpha+\cdots+\left(\tilde{a}^{(n)}\otimes^\dagger_{DT}\tilde{b}^{(n)}\right)^L_\alpha=\left(\tilde{\mathbf{a}}\circledcirc_{DT}\tilde{\mathbf{b}}\right)^L_\alpha\text{ (using (70), (77) and (78))}\end{aligned}$$

Using (80), we can similarly obtain

$$\begin{aligned}\left(\tilde{\mathbf{a}}\circledast^\star_{DT}\tilde{\mathbf{b}}\right)^U_\alpha=\left(\tilde{\mathbf{a}}\circledast^\dagger_{DT}\tilde{\mathbf{b}}\right)^U_\alpha&=\max_{\{\beta\in I^*:\beta\geq\alpha\}}\max\left\{\tilde{\mathbf{a}}^L_\alpha\bullet\tilde{\mathbf{b}}^L_\alpha,\tilde{\mathbf{a}}^U_\alpha\bullet\tilde{\mathbf{b}}^U_\alpha\right\}\\&\leq\left(\tilde{a}^{(1)}\otimes^\star_{DT}\tilde{b}^{(1)}\right)^U_\alpha+\cdots+\left(\tilde{a}^{(n)}\otimes^\star_{DT}\tilde{b}^{(n)}\right)^U_\alpha\\&=\left(\tilde{a}^{(1)}\otimes^\dagger_{DT}\tilde{b}^{(1)}\right)^U_\alpha+\cdots+\left(\tilde{a}^{(n)}\otimes^\dagger_{DT}\tilde{b}^{(n)}\right)^U_\alpha=\left(\tilde{\mathbf{a}}\circledcirc_{DT}\tilde{\mathbf{b}}\right)^U_\alpha.\end{aligned}$$

For $\alpha\in I^*$, it follows that

$$\begin{aligned}\left(\tilde{\mathbf{a}}\circledast^\dagger_{DT}\tilde{\mathbf{b}}\right)_\alpha=\left(\tilde{\mathbf{a}}\circledast^\star_{DT}\tilde{\mathbf{b}}\right)_\alpha&=\left[\left(\tilde{\mathbf{a}}\circledast^\star_{DT}\tilde{\mathbf{b}}\right)^L_\alpha,\left(\tilde{\mathbf{a}}\circledast^\star_{DT}\tilde{\mathbf{b}}\right)^U_\alpha\right]\\&\subseteq\left[\left(\tilde{\mathbf{a}}\circledcirc_{DT}\tilde{\mathbf{b}}\right)^L_\alpha,\left(\tilde{\mathbf{a}}\circledcirc_{DT}\tilde{\mathbf{b}}\right)^U_\alpha\right]=\left(\tilde{\mathbf{a}}\circledcirc_{DT}\tilde{\mathbf{b}}\right)_\alpha.\end{aligned}$$

This completes the proof. ■

Remark 4.7. We have the following observations.

- Theorem 4.5 says that when the second type inner product $\tilde{\mathbf{a}}\circledcirc\tilde{\mathbf{b}}$ is taken by (71) or (72), we prefer to take this second type of inner products rather than the first type of inner product $\tilde{\mathbf{a}}\circledast_{EP}\tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}}\circledast^\diamond_{DT}\tilde{\mathbf{b}}$ because of the issue of fuzziness.

- Theorem 4.6 says that when the second type inner product $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$ is taken by (77) or (78), we prefer to take the first type of inner products $\tilde{\mathbf{a}} \circledast^{\star}_{DT} \tilde{\mathbf{b}}$ and $\tilde{\mathbf{a}} \circledast^{\dagger}_{DT} \tilde{\mathbf{b}}$ rather than the second type of inner product $\tilde{\mathbf{a}} \circledcirc \tilde{\mathbf{b}}$ because of the issue of fuzziness.

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