

# Some asymptotic formulars for a Brownian motion whose dimension is variable with a regular variation from a parabolic domain to research the variation of biological species

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## Research

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# Some asymptotic formulars for a Brownian motion whose dimension is variable with a regular variation from a parabolic domain to research the variation of biological species

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## Abstract

Consider a Brownian motion with a regular variation starting at an interior point of a domain  $D$  in  $R^{d+1}$ ,  $d \geq 1$ , let  $\tau_D$  denote the first time the Brownian motion exits from  $D$ . Estimates with exact constants for the asymptotics of  $\log P(\tau_D > T)$  are given for  $T \rightarrow \infty$ , depending on the shape of the domain  $D$  and the order of the regular variation. Furthermore, the asymptotically equivalence are obtained. The problem is motivated by the early results of Lifshits and Shi, Li in the first exit time and Karamata in the regular variation. The methods of proof are based on their results and the calculus of variations.

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## 1. Introduction and main result

In the natural world, it is very important to maintain ecological stability. It is fatal to nature that the ecological chain is very likely to be destroyed when the ecology is affected by natural disasters or a series of external factors. It will lead to changes in the entire biological chain when the extinction or addition of a species in the ecological chain. In view of this phenomenon in the biological world, this paper studies the first exit time of Brownian motion. Each dimension of Brownian motion can be regarded as a change in the number of species, while the dimension of the Brownian motion can be seen as the species of the species in the ecology and it changes over time.

Throughout the paper, let  $\{B(t), t \geq 0\}$  be a standard  $d$ -dimensional Brownian motion and  $\{W(t), t \geq 0\}$  be a standard one-dimensional Brownian motion, such that  $B$  and  $W$  are independent, and  $\|x\| := (\sum_{i=1}^d x_i^2)^{1/2}$  be the Euclidean norm of  $x := (x_1, \dots, x_d) \in R^d$ .

In the last few years, in view of various motivations, two different types of asymptotic behaviours are studied. One type was usually called “the large deviation”, and the other was usually called “the small ball estimates”. The

“large deviation” has been studied by a large number of mathematicians and applied in mathematics. The “small ball estimates” receives also much research interest, yet relatively little is known. For the sake of clarity, let us recall some well-known results of small ball estimates. A general situation is:

$$\log P(\|W(t)\|_s < \varepsilon) \sim -\frac{\beta}{\varepsilon^\alpha}, \quad \varepsilon \rightarrow 0, \quad (1.1)$$

where  $\|\cdot\|_s$  is a semi-norm in the space of real of functions on  $[0, 1]$ , and  $\alpha > 0, \beta > 0$  are finite constants. Under some special norms, the well known results were obtained, such as  $L^2$ -norm, uniform sup-norm, Hölder-norm and so on. All the results related to small ball estimates can be seen in Kuelbs and Li [?] (1993), Novikov [?] (1979), Baldi and Roynette [?] (1992), Cameron and Martin [?] (1944), Chung [?] (1948).

In [?] (1999), applying a Gaussian correlation inequality into the small deviation, Li obtained a well-known asymptotic estimates for Brownian motion under weighted sup-norms as follows:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \log P \left( \sup_{0 < t < \infty} \frac{\|B(t)\|}{f(t)} < \varepsilon \right) = -\frac{j_{(d-2)/2}^2}{2} \int_0^\infty \frac{dt}{f^2(t)}, \quad (1.2)$$

where  $j_{(d-2)/2}$  is the smallest positive root of the Bessel function  $J_{(d-2)/2}$ .

Although these above probabilities are more general than (??),  $f(t)$  is still a nonrandom function. Li considered the first exit time problem and study the probability in [?] (2003) as follows:

$$P(f(\|B(s)\|) \leq W(s) + f(0) + 1, 0 \leq s \leq t), \quad (1.3)$$

where  $f(x)$  is a nondecreasing lower semicontinuous convex function on  $[0, \infty)$  with  $f(0)$  finite. The upper and lower estimates of log-probability in (??) were obtained by Gaussian technique firstly. However, in view of the generality of  $f(x)$ , the upper and lower estimates were not asymptotically equivalent.

Lifshits and Shi in [?] (2002) considered this problem and gave further restrictions and assumed  $f(t) = t^p$ , where  $p > 1$ . They obtained the lower and upper estimates of the following probability

$$P(\|B(s)\|^p \leq 1 + W(s), 0 \leq s \leq t), \quad (1.4)$$

and also provided the lower and upper estimates are asymptotically equivalent, thus improving Li’s estimates in this case.

It is the results of Lifshits, Shi and Li that motivate our study and we thank them for giving a copy of Li’s paper and useful discussions in their letters.

Returning to our first exit time problem in this paper, we develop the probabilities in (??) and (??), and consider more general case, namely,

$$P(\|B(s)\|^p \leq u_0 + h(s) + W(s), 0 \leq s \leq t), \quad (1.5)$$

where  $u_0 > 0$ ,  $h(x)$  is a continuous strictly positive function and also regularly varying at  $\infty$  of index  $r$ , namely,

$$\lim_{x \rightarrow \infty} h(\lambda x)/h(x) = \lambda^r, \quad \lambda > 0 \text{ and } 0 < r < p/2.$$

For example  $h(x) = x^{p/4}$  or  $(x \log(1+x))^{p/3}$  and so on.

The probability (??) is very useful. It is more general and complicated than (??). Studying it aims to understand the further relationship between the estimates of this probability and the regular variation  $h$ . Next, we give the main result of our paper as follows:

**Theorem 1.1.** *Let  $p > 1, 0 < r < p/2$ .*

1. *If  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = \infty$ , we have*

$$\begin{aligned} & \limsup_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} d(t)^{-\frac{p}{p+1}} \log P(\|B(s)\|^p \leq u_0 + h(s) + W(s), 0 \leq s \leq t) \\ & \leq -(p+1) \left( \frac{\pi c_1^p}{2^{p+3}(p-1)^{p-1}} \frac{\Gamma^2((p-1)/2)}{\Gamma^2(p/2)} \right)^{1/(p+1)}. \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} & \liminf_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} d(t)^{-\frac{2p}{p+1}} \log P(\|B(s)\|^p \leq u_0 + h(s) + W(s), 0 \leq s \leq t) \\ & \geq -(p+1) \left( \frac{c_2^{3p} \pi \Gamma^2((p-1)/2)}{8(p-1)^{p-1} \Gamma^2(p/2)} \right)^{\frac{1}{p+1}}. \end{aligned} \quad (1.7)$$

2. *If  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = 0$ , and*

$$(h^{1/p}(s))'' \leq 0, (h^{1/p}(s))' \geq 0 \text{ on } (0, \infty), \quad \sqrt{t}(h^{1/p}(t))' \rightarrow 0 \text{ as } t \rightarrow \infty,$$

we have

$$\limsup_{t \rightarrow \infty} t^{-1} h^{2/p}(t) d(t)^{-1} \log P(\|B(s)\|^p \leq u_0 + h(s) + W(s), 0 \leq s \leq t) \leq -\frac{c_1 p}{2(p-2r)} \quad (1.8)$$

and

$$\liminf_{t \rightarrow \infty} t^{-1} h^{2/p}(t) d^{-2}(t) \log P(\|W(t)\|^p \leq u_0 + h(s) + W(s), 0 \leq s \leq t) \geq -\frac{c_2^3 p}{2(p-2r)}. \quad (1.9)$$

where  $\Gamma(\cdot)$  denotes the usual gamma function, and  $c_1$  and  $c_2$  are strictly positive constants which are independent of  $p$  and  $t$ .

It is easy to see the relationship among (??), (??) and Theorem ???. If  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = \infty$ , the Brownian motion  $W(s)$  is the dominant term

in (??), and the asymptotic behavior of  $\log P(\tau_D > t)$  is the same to the one in (??). Otherwise, the function  $h(s)$  is the dominant term in (??), and the asymptotic behavior is the same to the one in (??). Since there is not an exact constant in (??) when  $\int_0^\infty f^{-2}(s)ds = \infty$ , we do not obtain a precise estimate for  $1 < p \leq 2$ ,  $p/2 \leq r \leq 1$ . More work needs to be done in this direction, and it also seems a challenging problem.

The rest of the paper is organized as follows. In Section 2, we present several estimates of exit probabilities for moving boundaries. They are necessary for the proof of Theorem ???. In Section 3, combining the Lemmas in Section 2, the detailed proof of the upper estimates in Theorem ?? is given. Finally, In Section 4, we give the detailed proof of the lower estimates in Theorem ??.

## 2. Exit probability for moving boundary

To prove Theorem ??, in this section, we need the following results due to Shao and Wang [?] for the small ball probabilities in Gaussian field, Bingham for regular variation in [?] (1987), Novikov for the estimates of nonexit probability of a Wiener process to a moving boundary in [?] (1979), Berthet and Shi for the small ball estimates for d-dimensional Brownian motion under a weighted sup-norm in [?] (2000), Li [?] for the result for the first exit time of Brownian motion from a unbounded convex domain, Lifshits and Shi [?] for the first exit time of Brownian motion from a parabolic domain.

If the dimension is a constant  $d$ , we consider the probability

$$P(\sup_{0 \leq s \leq t} \|\tilde{B}(s)\| \leq x).$$

From Li [?], it is easy to see, for any  $x > 0$ ,

$$K^{-1} \exp \left\{ -\frac{j_v^2}{2x^2} \right\} \leq P \left( \sup_{0 \leq s \leq 1} |\tilde{B}(s)| \leq x \right) \leq K \exp \left\{ -\frac{j_v^2}{2x^2} \right\}$$

where  $K$  is a various positive constant, and  $j_v$  is the smallest positive zero of the Bessel function  $J_v$ ,  $v = (d-2)/2$ .

However, up to now, we don't know any result for the dimension  $d(t)$ . So, in this section, we give two lemmas for getting upper and lower bounds of the probability

$$P(\sup_{0 \leq s \leq t} \|B(s)\| \leq x).$$

**Lemma 2.1.** *Let  $d \geq 1$ ,  $\{Z(t); t \in R^d\}$  be a functional Levy Brownian fields of order  $\alpha$ ,  $0 < \alpha < 2$ , i.e.*

$$E(Z(s) - Z(t))^2 = \|s - t\|^{2\alpha},$$

for all  $s, t \geq 0$ . Then there exists  $0 < c_1 < \infty$  depending only on  $\alpha$  and  $d$  such that

$$P \left\{ \sup_{0 \leq t \leq 1} |Z(t)| \leq x \right\} \leq \exp \left( -\frac{c_1}{x^{d/\alpha}} \right), \quad (2.1)$$

for any  $0 < x < 1$ .

The argument of Lemma ?? was given in Shao and Wang [? ]. And they also gave a lower bound of small ball probability of Gaussian fields.

**Lemma 2.2.** *Let  $X = \{X(t); t \in [0, 1]^d\}$  be a Gaussian field with mean zero. Assume that there exists a non-decreasing function  $\sigma(x)$  on  $[0, 1]$  such that*

$$E|X(t) - X(s)|^2 \leq \sigma^2(\|t - s\|), \quad (2.2)$$

for every  $s, t \in [0, 1]^d$ .

*Suppose that  $\alpha > 0$ ,  $\sigma(x)/x^\alpha$  is non-decreasing on  $[0, 1]$  for some  $\alpha > 0$  and that*

$$\sigma(kh) \leq k^2 \sigma(h), \quad (2.3)$$

*for every  $0 < h \leq 1$  and integer  $k$  with  $0 \leq k \leq 1/h$ . Then there exists a positive constant  $c_2 = c_2(\alpha, d)$ , for any  $0 < x < 1$*

$$P \left\{ \sup_{0 \leq t \leq 1} |X(t)| \leq c_2 \sigma(x) \right\} \geq \exp \left( -\frac{c_2}{x^{d/\alpha}} \right). \quad (2.4)$$

In fact, Shao in [? ] obtained estimate of the probability  $P(\sup_{0 \leq s \leq t} \|B(s)\| \leq x)$  with exact constants. However, we don't care about the precise coefficients in this paper. So we just use the above Lemmas. By Lemma ?? and Lemma ??, we get the following Propositions.

**Proposition 2.1.**  *$\{B(s), 0 \leq s \leq t\}$  be a  $d(t)$ -dimensional Brownian motion,  $d(t) \geq 1$ . Then there exist constants  $0 < c_1 < c_2 < \infty$ , such that*

$$\exp \left( -\frac{c_2^3 t d^2(t)}{x^2} \right) \leq P \left( \sup_{0 \leq s \leq t} \|B(s)\| \leq x \right) \leq \exp \left( -\frac{c_1 t d(t)}{x^2} \right). \quad (2.5)$$

**Proof:** First, we use the independence between  $B_i$

$$\begin{aligned} P \left( \sup_{0 \leq s \leq t} \|B(s)\| \leq x \right) &\leq P \left( \sup_{0 \leq s \leq t} |B_i(s)| \leq x, i = 1, 2, \dots, d(t) \right) \\ &= P \left( \sup_{0 \leq s \leq 1} |B_1(s)| \leq \frac{x}{\sqrt{t}} \right)^{d(t)}. \end{aligned} \quad (2.6)$$

By Lemma ??, let  $d = 1, \alpha = 1/2$ , and  $Z(t)$  defined as  $B_1(t)$ , then (??) can be written as

$$P \left( \sup_{0 \leq s \leq 1} |B_1(s)| \leq \frac{x}{\sqrt{t}} \right) \leq \exp \left( -\frac{c_1 t}{x^2} \right). \quad (2.7)$$

Plugging (??) into (??), we have

$$P\left(\sup_{0 \leq s \leq t} \|B(s)\| \leq x\right) \leq \exp\left(-\frac{c_1 t d(t)}{x^2}\right). \quad (2.8)$$

Next, we consider the lower bound of the probability.

$$\begin{aligned} & P\left(\sup_{0 \leq s \leq t} \|B(s)\| \leq x\right) \\ & \geq P\left(\sup_{0 \leq s \leq t} |B_i(s)| \leq \frac{x}{\sqrt{d(t)}}, i = 1, 2, \dots, d(t)\right) \\ & = P\left(\sup_{0 \leq s \leq 1} |B_i(s)| \leq \frac{x}{\sqrt{t d(t)}}, i = 1, 2, \dots, d(t)\right). \end{aligned} \quad (2.9)$$

By Lemma ??, let  $\alpha = \frac{1}{2}$ ,  $d = 1$ ,  $\sigma(x) = \sqrt{x}$ , then the inequality (??) can be written as

$$P\left(\sup_{0 \leq s \leq 1} |B_i(s)| \leq \frac{x}{\sqrt{t d(t)}}\right) \geq \exp\left(-\frac{c_2^3 t d(t)}{x^2}\right). \quad (2.10)$$

Plugging (??) into (??)

$$P\left(\sup_{0 \leq s \leq t} \|B(s)\| \leq x\right) \geq \exp\left(-\frac{c_2^3 t d^2(t)}{x^2}\right). \quad (2.11)$$

Combining (??) and (??), we have the inequality (??).  $\square$

The inequality (??) is very important for the proof of the Theorem ??.

Similarly to Theorem 2.2 in Li [? ], using Proposition ?? we give a result for variable dimension  $d(t)$ .

**Proposition 2.2.** *Let  $g(t)$  be a continuous strictly positive function such that  $g''(t) \leq 0$  is continuous and  $g'(t) \geq 0$ . Then*

$$\begin{aligned} & P(\|B(s)\| \leq g(s), 0 \leq s \leq t) \\ & \geq \exp\left\{\frac{d(t)}{2} \frac{g(t)}{g(0)} - \frac{1}{2} \int_0^t (g'(s))^2 ds - c_2^3 d^2(t) \int_0^t \frac{1}{g^2(s) ds}\right\}, \end{aligned} \quad (2.12)$$

here  $c_2$  is a sufficiently large constant.

In particular, under the additional condition  $\sqrt{t}g'(t) \rightarrow 0$  as  $t \rightarrow \infty$ ,

$$\log P(\|B(s)\| \leq g(s), 0 \leq s \leq t) \geq -(1 + \delta) c_2^3 d^2(t) \int_0^t g^{-2}(s) ds. \quad (2.13)$$

for and  $\delta > 0$  and  $t$  large.

The proof of the Proposition ?? is based on proof Theorem 2.2 in [? ], we only replaced (2.18) in Li [? ] by the following inequality, using Brownian scaling property and (??), we have

$$\begin{aligned}
& P(\|\Xi(s)\| \leq g(s), 0 \leq s \leq t) \\
&= P\left(\left\|g(s) \int_0^t G(s)dB(s)\right\| \leq g(s), 0 \leq s \leq t\right) \\
&= P\left(\|B(s)\| \leq 1, 0 \leq s \leq \int_0^t \frac{ds}{g^2(s)}\right) \\
&= P\left(\sup_{0 \leq s \leq 1} |B(s)| \leq \left(\int_0^t \frac{ds}{g^2(s)}\right)^{-1/2}\right) \\
&\geq \exp\left\{-c_2^3 d^2(t) \int_0^t \frac{ds}{g^2(s)}\right\}. \tag{2.14}
\end{aligned}$$

The rest proof is similar to the proof of Theorem 2.2 in Li [? ].

**Lemma 2.3.** *A measurable function  $h > 0$  satisfying*

$$h(\lambda x)/h(x) \rightarrow \lambda^\rho \quad (x \rightarrow \infty) \quad \text{for any } \lambda > 0, \tag{2.15}$$

*is called regularly varying at  $\infty$  of index  $\rho$ . Furthermore,  $h$  also satisfies the following conditions:*

1. *if  $\rho > 0$ , the convergence in (??) is uniform in  $\lambda$  on each  $(0, b]$  ( $0 < b < \infty$ );*
2.  *$h(x) = x^\rho l(x)$ , where  $l(x)$  is a positive measurable function and said to be slowly varying, namely, satisfying*

$$l(\lambda x)/l(x) \rightarrow 1 \quad (x \rightarrow \infty) \quad \forall \lambda > 0; \tag{2.16}$$

$$\log l(x)/\log x \rightarrow 0 \quad \text{as } x \rightarrow \infty. \tag{2.17}$$

The study of functions  $h(x), l(x)$  and their properties such as (??-??), together with their wide-ranging applications, constitutes the theory of regular variation, instituted by Karamata in [? ] (1930) and subsequently developed by him and many others. All above results can also be easily found in [? ] (1987).

**Lemma 2.4.** *Let  $g(t)$  be an absolutely continuous function, with  $g(0) < 0$  and*

$$\int_0^t \dot{g}^2(s)ds < \infty \quad \text{for } t < \infty.$$

*For any  $p > 1$*

$$P(W(s) \geq g(s), 0 \leq s \leq t) \geq \left(\Phi_0\left(\frac{|g(0)|}{\sqrt{t}}\right)\right)^{p/(p-1)} \exp\left\{-\frac{p}{2} \int_0^t \dot{g}^2(s)ds\right\}.$$



Here

$$\Phi_0(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp\{-\frac{y^2}{2}\} dy = 2\Phi(x) - 1 \quad (x \geq 0),$$

where  $\dot{g}$  denotes the Radon-Nikodym derivative of  $g$  and  $\Phi(x)$  is the distribution function of a standard normal random variable.

The argument of Lemma (??) given by Novikov in [?] (1979) follows from well-known theorems concerning the equivalence of measures of Gaussian processes. Note that

$$\Phi_0(|g(0)|/\sqrt{t}) \sim |g(0)|(2/\pi t)^{1/2}, \quad t \rightarrow \infty. \quad (2.18)$$

Before giving the last Lemma, let us introduce some notation appeared in [?] (2002). Here and in the following,

$$B_{\theta,p} := \frac{1}{2} \inf_{f \in A_{\theta,p}^\uparrow} \int_0^1 \dot{g}^2(t) dt, \quad (2.19)$$

where  $A_{\theta,p}^\uparrow$  is the set of all nondecreasing functions in the set  $A_{\theta,p}$  defined by

$$A_{\theta,p} := \left\{ g : [0, 1] \rightarrow R_+, g(0) = 0, g \text{ absolutely continuous}, \int_\theta^1 g^{-\frac{2}{p}}(t) dt \leq 1 \right\}. \quad (2.20)$$

Combining the notation (??-??) and applying the classical Schilder large deviation theorem, Lifshits and Shi in [?] (2002) provided a useful upper estimate (see formula (2.15) in that paper).

**Lemma 2.5.** *Let  $p > 1$  and  $\theta > 0$ , we have*

$$\limsup_{\lambda \rightarrow \infty} \lambda^{-\frac{p}{p+1}} \log \mathbf{E} \left\{ \exp \left( -\lambda \int_\theta^1 M^{-\frac{2}{p}}(t) dt \right) \right\} \leq -(p+1) p^{-\frac{p}{p+1}} B_{\theta,p}^{\frac{1}{p+1}},$$

where  $M(t) := \sup_{0 \leq s \leq t} B(s)$ . In particular,  $B_{0,p} = \frac{\pi p^p \Gamma^2((p-1)/2)}{8(p-1)^{p-1} \Gamma^2(p/2)}$ .

Using the classical Schilder large deviation theorem in [?] (1998), the upper estimates of the log-probability in Lemma ?? was obtained by Lifshits and Shi in [?] (2002). By exploiting a theorem of Biane and Yor in [?] (1987) relating different additive functionals of Bessel processes, they obtained the identification of the limit of the upper bound.

Next, using Lemma (??)-(??) and Proposition (??)-(??), we provide the proofs of Theorem ?? in the following two sections.

### 3. Proof of upper bounds

Our upper bounds argument is a modification of the one appeared in [?] (2002). We need to pay special attention to the regular variation  $h(t)$ . In view of completeness, we give the detailed proof. For the sake of brevity, we write

$$\chi_{d,p,h}(t) = P\{\|B(t)\|^p \leq u_0 + h(s) + W(t), 0 \leq s \leq t\}.$$

For any  $\varepsilon \in (0, 1)$ , let  $t_0 = 0, t_i = (1 - \varepsilon)^{N-i}t, 1 \leq i \leq N$ . From Theorem 1.5.3 in [?] (1987), we know that any function  $h$  varying regularly with non-zero exponent is asymptotic to a monotone function. In particular  $l$  is slowly varying and  $r > 0$ ,  $x^r l(x)$  is asymptotic to a nondecreasing function. Without loss of generality, in this argument, we assume that  $h(t)$  is nondecreasing for  $t$  large. Thus, for  $t$  large, we have

$$\chi_{d,p,h}(t) \leq P \left\{ \sup_{t_{i-1} \leq s \leq t_i} \|B(s)\| \leq \left( u_0 + h(t_i) + \sup_{0 \leq s \leq t_i} W(s) \right)^{1/p}, \forall i \leq N \right\}. \quad (3.1)$$

Since  $B(t) \in R^d(t)$  has independent increments, we have

$$\begin{aligned} & P \left\{ \sup_{t_{i-1} \leq s \leq t_i} \|B(s)\| \leq a_i, 1 \leq i \leq N \right\} \\ &= P(\cap_{i=1}^{N-1} A_i, \|B(s)\| \leq a_N, s \in [t_{N-1}, t_N]) \\ &= E\{E(1_{\{\cap_{i=1}^{N-1} A_i\}} | B(t_{N-1}) = x) \\ &\quad \times P(\|B(s) - B(t_{N-1}) + x\| \leq a_N, s \in [t_{N-1}, t_N] | B(t_{N-1}) = x)\}, \end{aligned} \quad (3.2)$$

where

$$A_i = \{\|B(s)\| \leq a_i, t_{i-1} \leq s \leq t_i\}, \quad 1 \leq i \leq N.$$

By Anderson's inequality and the fact that  $W$  has stationary increments, we have for  $a_N \geq 0$ ,

$$\begin{aligned} & P(\|B(s) - B(t_{N-1}) + x\| \leq a_N, t_{N-1} \leq s \leq t_N) \\ &\leq P(\|B(s) - B(t_{N-1})\| \leq a_N, t_{N-1} \leq s \leq t_N) \\ &= P(\|B(s)\| \leq a_N, 0 \leq s \leq t_N - t_{N-1}). \end{aligned} \quad (3.3)$$

Plugging the inequality (??) into (??), by induction, we have

$$P \left\{ \sup_{t_{i-1} \leq s \leq t_i} \|B(s)\| \leq a_i, 1 \leq i \leq N \right\} \leq \prod_{i=1}^N P \left\{ \sup_{0 \leq s \leq t_i - t_{i-1}} \|B(s)\| \leq a_i \right\}. \quad (3.4)$$

By Proposition ??, we know that

$$P \left\{ \sup_{0 \leq s \leq t} \|B(s)\| \leq x \right\} \leq \exp \left( -\frac{c_1 d(t)}{x^2} t \right), \quad x \rightarrow 0^+, \quad (3.5)$$

Combining (??) and (??), we have

$$P \left\{ \sup_{t_{i-1} \leq s \leq t_i} \|B(s)\| \leq a_i, \forall i \leq N \right\} \leq \exp \left( -c_1 d(t) \sum_{i=1}^N \frac{t_i - t_{i-1}}{a_i^2} \right). \quad (3.6)$$

By conditioning on the Brownian motion  $W$ , and combining (??) and (??), we have

$$\begin{aligned} & \chi_{d,p,h}(t) \\ \leq & E \left\{ \exp \left( -c_1 d(t) \sum_{i=1}^N \frac{t_i - t_{i-1}}{[u_0 + h(t_i) + \sup_{0 \leq s \leq t_i} W(s)]^{2/p}} \right) \right\}. \end{aligned} \quad (3.7)$$

Note that  $M(t) := \sup_{0 \leq s \leq t} W(s)$ , where  $t \geq 0$ . Then, we obtain

$$\sum_{i=1}^N \frac{t_i - t_{i-1}}{[u_0 + h(t_i) + \sup_{0 \leq s \leq t_i} W(s)]^{2/p}} = \sum_{i=1}^N \frac{t_i - t_{i-1}}{[u_0 + h(t_i) + M(t_i)]^{2/p}}. \quad (3.8)$$

Combining the monotonicity of  $t \rightarrow M(t)$  and  $t \rightarrow h(t)$  for  $t$  large, we have

$$\begin{aligned} \sum_{i=1}^N \frac{t_i - t_{i-1}}{[u_0 + h(t_i) + M(t_i)]^{2/p}} & \geq \sum_{i=1}^{N-1} \frac{(1-\varepsilon)(t_{i+1} - t_i)}{[u_0 + h(t_i) + M(t_i)]^{2/p}} \\ & \geq (1-\varepsilon) \sum_{i=1}^{N-1} \int_{t_i}^{t_{i+1}} \frac{ds}{[u_0 + h(s) + M(s)]^{2/p}} \\ & = (1-\varepsilon) \int_{t_1}^t \frac{ds}{[u_0 + h(s) + M(s)]^{2/p}}. \end{aligned} \quad (3.9)$$

Plugging (??) into (??), we have

$$\chi_{d,p,h}(t) \leq E \left\{ \exp \left( -(1-\varepsilon)c_1 d(t) \int_{t_1}^t \frac{ds}{[u_0 + h(s) + M(s)]^{2/p}} \right) \right\}. \quad (3.10)$$

Using the scaling property of  $W$  and variable substitution, we have

$$\begin{aligned} & \int_{t_1}^t \frac{ds}{[u_0 + h(s) + M(s)]^{2/p}} \\ & = t^{\frac{p-1}{p}} \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}}. \end{aligned} \quad (3.11)$$

Taking  $a = (1-\varepsilon)c_1 d(t) t^{\frac{p-1}{p}}$  and plugging (??) into (??), we have

$$\begin{aligned} & \chi_{d,p,h}(t) \\ \leq & E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \right\}. \end{aligned} \quad (3.12)$$

In the case of  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = \infty$ , we split the expectation in (??) into two parts

$$E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \right\} \quad (3.13)$$

$$\begin{aligned}
&\leq E \left\{ \exp \left( -a \int_{2(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \mathbf{1}_H \right\} \\
&\quad + E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^{2(1-\varepsilon)^{N-1}} \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \mathbf{1}_{H^c} \right\} \\
&= I + II.
\end{aligned}$$

where  $H = \{u_0 t^{-1/2} + t^{-1/2} h(t) \leq \varepsilon M(2(1-\varepsilon)^{N-1})\}$ . We have on  $H$ ,

$$\begin{aligned}
I &\leq E \left\{ \exp \left( -a \int_{2(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t) + M(s)]^{2/p}} \right) \mathbf{1}_H \right\} \\
&\leq E \left\{ \exp \left( -\frac{a}{(1+\varepsilon)^{2/p}} \int_{2(1-\varepsilon)^{N-1}}^1 \frac{ds}{M^{2/p}(s)} \right) \right\}
\end{aligned}$$

and on  $H^c$ ,

$$\begin{aligned}
II &\leq E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^{2(1-\varepsilon)^{N-1}} \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t) + M(s)]^{2/p}} \right) \mathbf{1}_{H^c} \right\} \\
&\leq \exp \left( -\frac{a(1-\varepsilon)^{N-1}}{(1+\varepsilon^{-1})^{2/p}} \frac{1}{[u_0 t^{-1/2} + t^{-1/2} h(t)]^{2/p}} \right). \tag{3.14}
\end{aligned}$$

For  $I$ , using Lemma ??, we can easily obtain

$$\begin{aligned}
&\inf_{\delta>0, \theta>0} \limsup_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} d(t)^{-\frac{p}{p+1}} \log \mathbf{E} \left\{ \exp \left( -(1-\delta) c_1 d(t) t^{\frac{p-1}{p}} 2 \int_{\theta}^1 M^{-\frac{2}{p}}(s) ds \right) \right\} \\
&\leq -(p+1)(c_1/p)^{-\frac{p}{p+1}} B_{0,p}^{\frac{1}{p+1}}, \tag{3.15}
\end{aligned}$$

where  $B_{0,p} = [\pi p^p \Gamma^2((p-1)/2)]/[8(p-1)^{p-1} \Gamma^2(p/2)]$ .

For  $II$ , using  $u_0 t^{-1/2} \leq \varepsilon t^{-1/2} h(t)$  for  $t$  large, we have

$$\frac{a(1-\varepsilon)^{N-1}}{(1+\varepsilon^{-1})^{2/p}} \frac{1}{[u_0 t^{-1/2} + t^{-1/2} h(t)]^{2/p}} \geq \frac{(1-\varepsilon)^N c_1 d(t)}{(1+\varepsilon^{-1})^{2/p} (1+\varepsilon)^{2/p}} t h^{-\frac{2}{p}}(t). \tag{3.16}$$

Plugging (??) into (??), we have

$$\limsup_{t \rightarrow \infty} t^{-1} d(t)^{-1} h^{2/p}(t) \log II \leq -\frac{c_1(1-\varepsilon)^{N+1}}{(1+\varepsilon^{-1})^{2/p} (1+\varepsilon)^{2/p}}. \tag{3.17}$$

In view of the fact  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = \infty$ , it is easy to see that

$$\lim_{t \rightarrow \infty} t^{(p-1)/(p+1)} d(t)^{p/(p+1)} / t h^{-2/p}(t) d(t) = 0.$$

Thus, comparing (??) with (??), it shows that (??) is the dominant term. Then we get the upper bound

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} d(t)^{-\frac{p}{p+1}} \log \chi_{d,p,h}(t) \\
& \leq \inf_{\delta > 0, \theta > 0} \limsup_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} \log \mathbf{E} \left\{ \exp \left( -(1-\delta) c_1 t^{\frac{p-1}{p}} \int_{\theta}^1 M^{-\frac{2}{p}}(s) ds \right) \right\} \\
& \leq -(p+1) \left( \frac{\pi c_1^p}{2^{p+3}(p-1)^{p-1}} \frac{\Gamma^2((p-1)/2)}{\Gamma^2(p/2)} \right)^{1/(p+1)}.
\end{aligned} \tag{3.18}$$

Then, we finish the proof of the upper estimates in (??).

Next, in the case of  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = 0$  and  $r < p/2$ , we still split the expectation in (??) into two parts,

$$\begin{aligned}
& E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \right\} \\
& = E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \mathbf{1}_F \right\} \\
& \quad + E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[u_0 t^{-1/2} + t^{-1/2} h(t \cdot s) + M(s)]^{2/p}} \right) \mathbf{1}_{F^c} \right\} \\
& = III + IV,
\end{aligned} \tag{3.19}$$

where  $F = \{u_0 t^{-1/2} + t^{-1/2} h(t) \leq \varepsilon^{-1} M(1)\}$ .

Next, dealing with *III* and *IV*, respectively. On the one hand, combining the monotonicity of  $M(t)$  and  $h(t)$  for  $t \rightarrow \infty$ ,

$$\begin{aligned}
III & \leq E \left\{ \exp \left( -a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{(1 + \varepsilon^{-1})^{2/p} M^{2/p}(1)} \right) \mathbf{1}_F \right\} \\
& \leq E \left\{ \exp \left( -\frac{a(1 - (1-\varepsilon)^{N-1})}{(1 + \varepsilon^{-1})^{2/p} M^{2/p}(1)} \right) \mathbf{1}_{\{M^{-1}(1) \leq \varepsilon^{-1} t^{1/2} h^{-1}(t)\}} \right\}.
\end{aligned} \tag{3.20}$$

It is convenient to recall the reflection principle of the Brownian motion

$$P(M^{-1}(1) \leq x) = P(M(1) \geq x^{-1}) = 2P(B(1) \geq x^{-1}). \tag{3.21}$$

Then, the derivative of (??) with respect to  $x$  is

$$P(M^{-1}(1) \in dx) = f_{M^{-1}(1)}(x) dx = \sqrt{\frac{2}{\pi}} x^{-2} \exp \left( -\frac{1}{2x^2} \right) dx, \quad x > 0. \tag{3.22}$$

Combining (??) and (??), we have

$$III \leq \int_0^{\varepsilon^{-1} t^{1/2} h^{-1}(t)} \exp \left( -\frac{a(1 - (1-\varepsilon)^{N-1}) x^{2/p}}{(1 + \varepsilon^{-1})^{2/p}} \right) f_{M^{-1}(1)}(x) dx. \tag{3.23}$$

Note that, for  $t \rightarrow \infty$ ,

$$\begin{aligned}
& \int_0^{\varepsilon^{-1}t^{1/2}h^{-1}(t)} x^{-2} \exp\left(-\frac{a(1-(1-\varepsilon)^{N-1})x^{2/p}}{(1+\varepsilon^{-1})^{2/p}} - \frac{1}{2x^2}\right) dx \quad (3.24) \\
& \leq \int_0^{\varepsilon^{-1}t^{1/2}h^{-1}(t)} x^{-2} \\
& \quad \times \exp\left(-\frac{(1-\varepsilon)^2(1-(1-\varepsilon)^{N-1})j_{(d-2)/2}^2 x^{2/p}}{2(1+\varepsilon^{-1})^{2/p}} - \frac{1}{2x^2}\right) dx \\
& \sim \frac{t^{1/2}}{\varepsilon h(t)} \exp\left(-\frac{(1-\varepsilon)^2(1-(1-\varepsilon)^{N-1})j_{(d-2)/2}^2}{2(1+\varepsilon)^{2/p}} \frac{t^{\frac{1}{p}}}{h^{\frac{2}{p}}(t)} - \frac{\varepsilon^2 h^2(t)}{2t}\right).
\end{aligned}$$

Plugging (??) into (??) yields, for any  $\delta > 0$ ,

$$\begin{aligned}
III & \leq (1+\delta) \sqrt{\frac{2}{\pi}} \frac{t^{1/2}}{\varepsilon h(t)} \quad (3.25) \\
& \quad \times \exp\left(-\frac{(1-\varepsilon)^2(1-(1-\varepsilon)^{N-1})j_{(d-2)/2}^2}{2(1+\varepsilon)^{2/p}} \frac{t^{\frac{1}{p}}}{h^{2/p}(t)} - \frac{\varepsilon^2 h^2(t)}{2t}\right).
\end{aligned}$$

On the other hand, using the monotonicity of  $M(t)$  and  $h(t)$  for  $t \rightarrow \infty$  again, and  $(1+\varepsilon)u_0 t^{-1/2} < \varepsilon t^{-1/2} h(t)$  for  $t \rightarrow \infty$ , we have

$$\begin{aligned}
& IV \\
& \leq E \left\{ -\exp\left(\int_{(1-\varepsilon)^{N-1}}^1 \frac{adt}{[(1+\varepsilon)u_0 t^{-\frac{1}{2}} + t^{-\frac{1}{2}}(h(t \cdot t) + \varepsilon h(t))]^{\frac{2}{p}}}\right) \mathbf{1}_{F^c} \right\} \\
& \leq \exp\left(-a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[t^{-1/2}(h(t \cdot s) + 2\varepsilon h(t))]^{2/p}}\right) \\
& \leq \exp\left(-a \int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[(1+\delta)t^{-1/2}h(t)(s^r + 2\varepsilon)]^{2/p}}\right), \quad (3.26)
\end{aligned}$$

where we apply the condition 1 of Lemma (??) into the last inequality in (??). If  $\lim_{t \rightarrow \infty} h(t \cdot s)/h(t) = s^r$ , and this convergence is uniform in  $s \in [(1-\varepsilon)^{N-1}, 1]$ , then, for any  $\delta > 0$  and  $t$  large, we have

$$(1-\delta)h(t)s^r \leq h(t \cdot s) \leq (1+\delta)h(t)s^r. \quad (3.27)$$

Next, on the one hand, using

$$s^r + 2\varepsilon \leq (s + 2\varepsilon/r)^r \quad \text{if } s \in [0, 1 - 2\varepsilon/r] \text{ and } 0 < r \leq 1,$$

we have

$$\int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[(1+\delta)t^{-1/2}h(t)(s^r + 2\varepsilon)]^{2/p}} \quad (3.28)$$

$$\begin{aligned}
&\geq \int_{(1-\varepsilon)^{N-1}}^{1-2\varepsilon/r} \frac{ds}{[(1+\delta)t^{-1/2}h(t)(s+2\varepsilon/r)^r]^{2/p}} \\
&= \int_{(1-\varepsilon)^{N-1}+2\varepsilon/r}^1 \frac{ds}{[(1+\delta)t^{-1/2}h(t)s^r]^{2/p}} \\
&= [1 - ((1-\varepsilon)^{N-1} + 2\varepsilon/r)^{1-2r/p}] \frac{t^{1/p}h^{-2/p}(t)p}{(1+\delta)^{2/p}(p-2r)}.
\end{aligned}$$

Plugging (??) into (??), we have

$$\begin{aligned}
&\inf_{\varepsilon>0, \delta>0} \limsup_{t \rightarrow \infty} t^{-1}h^{2/p}(t)d^{-1}(t) \log IV \\
&\leq \inf_{\varepsilon>0, \delta>0} \left( -(1-\varepsilon)^2 [1 - ((1-\varepsilon)^{N-1} + 2\varepsilon/r)^{1-2r/p}] \frac{c_1 p}{(1+\delta)^{2/p}(p-2r)} \right) \\
&= -\frac{c_1 p}{(p-2r)}.
\end{aligned}$$

On the other hand, using variable substitution  $y^r = s^r + 2\varepsilon$  if  $p > 2$  and  $1 < r < p/2$ , we have

$$\begin{aligned}
&\int_{(1-\varepsilon)^{N-1}}^1 \frac{ds}{[(1+\delta)t^{-1/2}h(t)(s^r+2\varepsilon)]^{2/p}} \tag{3.29} \\
&= \frac{t^{\frac{1}{p}}h^{-\frac{2}{p}}(t)}{(1+\delta)^{\frac{2}{p}}} \int_{((1-\varepsilon)^{r(N-1)}+2\varepsilon)^{1/r}}^{(1+2\varepsilon)^{1/r}} \frac{1}{y^{2r/p}} \frac{y^{r-1}}{(y^r-2\varepsilon)^{1-\frac{1}{r}}} dy \\
&\geq \frac{t^{\frac{1}{p}}h^{-\frac{2}{p}}(t)}{(1+\delta)^{\frac{2}{p}}} \int_{((1-\varepsilon)^{r(N-1)}+2\varepsilon)^{1/r}}^1 \frac{1}{y^{2r/p}} dy \\
&= [1 - ((1-\varepsilon)^{r(N-1)} + 2\varepsilon)^{(p-2r)/rp}] \frac{t^{1/p}h^{-2/p}(t)}{(1+\delta)^{2/p}(1-2r/p)}.
\end{aligned}$$

Similarly, plugging (??) into (??), we also have

$$\inf_{\varepsilon>0, \delta>0} \limsup_{t \rightarrow \infty} t^{-1}h^{2/p}d^{-1}(t)(t) \log IV \leq -\frac{c_1 p}{(p-2r)}. \tag{3.30}$$

Note that  $\lim_{t \rightarrow \infty} th^{-2/p}(t)/t^{-1}h^2(t) = 0$  from the fact  $\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = 0$ . Comparing (??) with (??), it implies that (??) is the dominant term. Then, we obtain the upper bound

$$\limsup_{t \rightarrow \infty} t^{-1}h^{2/p}(t)d(t)^{-1} \log \chi_{d,p,h}(t) \leq -\frac{c_1 p}{2(p-2r)}. \tag{3.31}$$

Combining (??) and (??), we finish the whole proof of the upper estimates in Theorem ??.

#### 4. Proof of lower bounds

Our lower bounds argument is based on the calculus of variations and Li's profile function method from [?] (2003). We also need to pay special attention to the regular variation  $h(t)$ . Let  $g \in \Lambda^2$ , where  $\Lambda^2$  is the Hilbert space of all absolutely continuous functions  $x$  such that  $x(0) = 0$ ,  $\int_0^1 \dot{x}^2(t)dt < \infty$ .

In the case

$$\lim_{t \rightarrow \infty} t^{p/(p+1)}/h(t) = \infty, \quad (4.1)$$

we have

$$\begin{aligned} & \chi_{d,p,h}(T) \\ & \geq P\{\|B(s)\|^p \leq t^{p/(p+1)}g(s/t) + h(s) < u_0 + h(s) + W(s), 0 \leq s \leq t\} \\ & \geq P\{\|B(s)\| \leq t^{1/(p+1)}g^{1/p}(s/t), 0 \leq s \leq t\} \\ & \quad \times P\{t^{p/(p+1)}g(s/t) < u_0 + W(s), 0 \leq s \leq t\}. \end{aligned} \quad (4.2)$$

In the second inequality of (??), we can remove the regular variation  $h(t)$  in the first probability without loss of the correct asymptotic rate, since  $t^{p/(p+1)}g(s/t)$  is the dominant term from the fact (??).

Next, dealing with the above two probabilities in (??), respectively. Using (??) in Proposition ??, we can easily obtain

$$\lim_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} \log P\{\|B(t)\| \leq t^{\frac{1}{p+1}}g^{\frac{1}{p}}(s/t), 0 \leq s \leq t\} = -(1+\delta)c_2^3 d^2(t) \int_0^1 g^{-\frac{2}{p}}(t)dt. \quad (4.3)$$

Let  $q > 1$ , by taking  $p = q$  in Lemma ??, using (??), we have

$$\liminf_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} \log P\{t^{p/(p+1)}g(s/t) < u_0 + W(s), 0 \leq s \leq t\} \geq -\frac{q}{2} \int_0^1 \dot{g}^2(t)dt. \quad (4.4)$$

Taking  $c^* = 2c_2^3 d^2(t)$  and combining (??) and (??), letting  $q \rightarrow 1, \delta \rightarrow 0$  such that maximizing the following term on the right hand side, we have

$$\liminf_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} \log \chi_{d,p,h}(t) \geq -\frac{1}{2} \inf_{\{g: g \in \Lambda^2\}} \left( \int_0^1 \frac{c^*}{g^{2/p}(s)} + \dot{g}^2(s)ds \right). \quad (4.5)$$

The right-hand side of (??) can be evaluated by classical methods of the calculus of variations. Without loss of generality we may assume  $g(s) \geq 0$ . A necessary condition for  $g(s)$  is such that for all  $y \in \Lambda^2$

$$\int_0^1 \left( -\frac{2}{p} \frac{c^* y(s)}{g^{2/p+1}(s)} + 2\dot{g}(s)\dot{y}(s) \right) ds = 0 \quad (4.6)$$

((??) is the derivative of the functional  $\int_0^1 c^*/g^{2/p}(s) + \dot{g}^2(s)ds$  at  $g$  applied to  $y$ , i.e.,

$$\left( \frac{\partial}{\partial \varepsilon} \int_0^1 \frac{c^*}{(g + \varepsilon y)^{2/p}} + (\dot{g} + \varepsilon \dot{y})^2 ds \right)_{\varepsilon=0}$$



(for the details see Diedonné [?] ] (1960)). Partial integration of the first term yields

$$\int_0^1 \left( \int_t^1 -\frac{2}{p} \frac{c^*}{g^{2/p+1}(s)} ds \dot{g}(t) + 2\dot{g}(t)\dot{g}(t) \right) dt = 0,$$

therefore

$$\int_t^1 \frac{2}{p} \frac{c^*}{g^{2/p+1}(s)} ds = 2\dot{g}(t), \quad (4.7)$$

which shows that  $\dot{g}$  has a continuous derivative and also  $\dot{g}(1) = 0, \dot{g}(t) > 0$ . Multiplying (??) with  $\dot{g}(t)$  and integrating, partial integration of the right-hand side term yields

$$\int_0^1 \frac{c^*}{g^{2/p}(t)} dt = p \int_0^1 \dot{g}^2(t) dt. \quad (4.8)$$

Differentiating (??), multiplying with  $\dot{g}(t)$  and integrating again yields

$$\frac{c^*}{g^{2/p}(t)} - \dot{g}^2(t) = \frac{c^*}{g^{2/p}(1)} - \dot{g}^2(1) = \frac{c^*}{g^{2/p}(1)}. \quad (4.9)$$

Separation of variables and integration yields

$$t = \int_0^{g(t)} \frac{du}{\sqrt{c^*} \sqrt{u^{-2/p} - g^{-2/p}(1)}}, \quad (4.10)$$

so that

$$1 = \int_0^{g(1)} \frac{du}{\sqrt{c^*} \sqrt{u^{-2/p} - g^{-2/p}(1)}} = \frac{g^{1+1/p}(1)}{\sqrt{c^*}} \frac{p}{2} \frac{\Gamma((p+1)/2)\Gamma(1/2)}{\Gamma(p/2+1)}. \quad (4.11)$$

Now, on the one hand, integrating both sides in (??),

$$\int_0^1 \frac{c^*}{g^{2/p}(t)} dt - \int_0^1 \dot{g}^2(t) dt = \frac{c^*}{g^{2/p}(1)}. \quad (4.12)$$

On the other side, using (??) and (??)

$$\begin{aligned} \int_0^1 \frac{c^*}{g^{2/p}(t)} dt &= \int_0^{g(1)} \frac{\sqrt{c^*}}{u^{2/p} \sqrt{u^{-2/p} - g^{-2/p}(1)}} du \\ &= \frac{\sqrt{c^*} p g^{1-1/p}(1)}{2} \frac{\Gamma((p-1)/2)\Gamma(1/2)}{\Gamma(p/2)} \\ &= p \left( \frac{(c^*)^p \pi \Gamma^2((p-1)/2)}{2^2 (p-1)^{p-1} \Gamma^2(p/2)} \right)^{1/(p+1)}. \end{aligned} \quad (4.13)$$

Note that from (??),

$$\frac{c^*}{g^{2/p}(1)} = (p-1) \left( \frac{(c^*)^p \pi \Gamma^2((p-1)/2)}{2^2 (p-1)^{p-1} \Gamma^2(p/2)} \right)^{1/(p+1)}. \quad (4.14)$$

Using (??)-(??), or combining (??) and (??), it is easy to obtain

$$\liminf_{t \rightarrow \infty} t^{-\frac{p-1}{p+1}} d(t)^{-\frac{2p}{p+1}} \log \chi_{d,p,h}(t) \geq -(p+1) \left( \frac{c_2^{3p} \pi \Gamma^2((p-1)/2)}{8(p-1)^{p-1} \Gamma^2(p/2)} \right)^{\frac{1}{p+1}}. \quad (4.15)$$

Then, we finish the proof of the lower estimates in (??).

Next, in the case

$$\lim_{t \rightarrow \infty} t^{p/(p+1)} / h(t) = 0, \quad r < p/2, \quad (4.16)$$

it is similar to the proof above, the key step is to find an appropriate profile function. From (??), we assume that there exists  $m$  such that  $p/(p+1) \leq r < m < p/2$ . We have for  $g \in \Lambda^2$

$$\begin{aligned} & \chi_{d,p,h}(t) \\ & \geq P\{\|B(s)\|^p \leq t^{1-m/p} g(s/t) + h(s) < u_0 + h(s) + W(s), 0 \leq s \leq t\} \\ & \geq P\{\|B(s)\|^p \leq h(s), 0 \leq s \leq t\} \\ & \quad \times P\{t^{1/2-m/p} g(s) < u_0 T^{-1/2} + W(s), 0 \leq s \leq 1\}. \end{aligned} \quad (4.17)$$

For the second inequality in (??), we can remove  $t^{1-m/p} g(s/t)$  in the first probability without loss of the correct asymptotic rate, since  $h(s)$  is the dominant term in this probability from the fact (??). By scaling, we get the second probability.

Next, dealing with the two probabilities above in (??), respectively. Using Proposition ??, we have

$$\log P(\|B(t)\|^p \leq h(s), 0 \leq s \leq t) \geq -(1+\delta) c_2^3 d^2(t) \int_0^T h^{-\frac{2}{p}}(s) ds \quad (4.18)$$

for  $\delta > 0$  and  $t$  sufficiently large. Using the condition (??) and (??) in Lemma (??), and differentiating the numerator and denominator in the following formula respectively, we have

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{\int_0^t h^{-2/p}(s) ds}{(1-2r/p)^{-1} t h^{-2/p}(t)} \\ & = \lim_{t \rightarrow \infty} \frac{t^{-2r/p} l^{-2/p}(t)}{t^{-2r/p} l^{-2/p}(t) - 2p^{-1}(1-2r/p)^{-1} t^{1-2r/p} l^{-1-2/p}(t) l'(t)} \\ & = \frac{1}{1 - 2p^{-1}(1-2r/p)^{-1} \lim_{t \rightarrow \infty} (\log l(t))' / (\log t)'} \\ & = 1, \end{aligned} \quad (4.19)$$

where we use the fact

$$\lim_{t \rightarrow \infty} (\log l(t))' / (\log t)' = \lim_{t \rightarrow \infty} t l'(t) / l(t) = 0. \quad (4.20)$$

Since it is easy to verify for the simplest non-trivial examples  $l(t) = \log t$  or  $\log \log t$  and complicated examples

$$l(t) = \exp\{(\log t)/\log \log t\} \text{ or } \exp\{(\log t)^{1/3} \cos((\log t)^{1/3})\}$$

appeared in Bingham [?] (1987), we omit the detailed proof of (??).

By the equation (??), for  $t$  large, we have, for  $\delta > 0$ ,

$$\int_0^t h^{-\frac{2}{p}}(s) ds \leq (1 + \delta) \frac{th^{-2/p}(t)}{1 - 2r/p}. \quad (4.21)$$

Combining (??) and (??), we have

$$\begin{aligned} & \liminf_{t \rightarrow \infty} t^{-1} h^{2/p}(t) d^{-2}(s) \log P\{\|B(s)\|^p \leq h(s), 0 \leq s \leq t\} \quad (4.22) \\ & \geq \sup_{\delta > 0} -(1 + \delta)^2 \frac{c_2^3}{(1 - 2r/p)} \\ & = -\frac{pc_2^3}{2(p - 2r)}. \end{aligned}$$

Let  $q > 1$ , by taking  $p = q$  in Lemma ??, using (??), we have

$$\liminf_{t \rightarrow \infty} t^{-(1-2m/p)} \log P\{t^{1/2-m/p} g(s) < u_0 t^{-1/2} + W(s), 0 \leq s \leq 1\} \geq -\frac{q}{2} \int_0^1 \dot{g}^2(s) ds. \quad (4.23)$$

Thus, comparing (??) with (??), in view of  $\lim_{t \rightarrow \infty} t^m/h(t) = \infty$  from the fact  $r < m$ , it shows that (??) is the dominant term. Thus, we have

$$\liminf_{t \rightarrow \infty} t^{-1} h^{2/p}(t) d^{-2}(s) \log \chi_{d,p,h}(t) \geq -\frac{pc_2^3}{2(p - 2r)}. \quad (4.24)$$

Combining (??) and (??), we complete the whole proof of lower estimates in Theorem ??.

## 5. Conclusion

This paper proves that the estimation of the upper and lower bounds during the first exit time of the Brownian motion is meaningful for the species ecological chain problem in the biological world.

## 6. Ethics approval and consent to participate

Not applicable

## 7. Consent for publication

Not applicable

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## 9. Availability of data and material

This article is intended to prove the theoretical study of species in the ecosystem, without data.

## 10. Competing interests

The authors declare that they have no competing interests.

## 11. Authors' contributions

The first author(Xiaoming Li) conceived the innovation of this paper and provided relevant lemmas to prove the theory. The second author(Wenbin Che) and the third author(Jingjun Zhang) gave detailed proof of the main conclusion of the paper.

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## Appendix

### A.1. Explanation of mathematical symbols and formulas

Abbreviations	Full name
$B^{(*)}$	Brownian motion
$P^{(*)}$	Probability function
$\  * \ $	Euclidean norm
$E^*$	Expectation
$* \lim$	limit
$\liminf$	limit inferior
$\limsup$	limit superior
$* \max$	maximum
$* \min$	minimum
$* \inf$	infimum
$* \sup$	supremum

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