The Case Against Cantor’s Diagonal Argument

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Research Article

Keywords: Cantor’s Diagonal Argument, fatal inconsistencies

Posted Date: July 10th, 2020

DOI: https://doi.org/10.21203/rs.3.rs-40722/v1

The Case Against Cantor's Diagonal Argument - v.3.4

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(Dated: July 8, 2020)

We present the case against Cantor's Diagonal Argument (CDA), exposing a number of fatal inconsistencies.

I. INTRODUCTION

It is part of the role of a scientist to guard against inadequate conceptions in their discipline. In the physical sciences, determining that certain physical evidence obtained from experiments is inconsistent with existing theory triggers the search for improved conceptions. In mathematics, the exposure of logical inconsistencies provides the incentive for improvement. We will not dwell on the mixture of human weaknesses and historical contingencies that can make this process of improvement somewhat inefficient.

In the following, we present a set of arguments exposing key flaws in the construction commonly known as Cantor's Diagonal Argument (CDA) found in his paper [2][5]. CDA is indeed beguiling, and its persistence for so long within an orthodox canon of pure mathematics is excusable to that degree.

In §II we provide a short outline of the main structure of CDA. Concerted critique of CDA then follows in §III. After that, in §IV a variant of CDA which we call the Functional CDA (FCDA) is discussed, which in some respects seems stronger than CDA. However, if CDA is false, so is FCDA.

No strong claim to the originality of our ideas and arguments is made. The content of this text was initially developed independently, but subsequent searches have revealed that a number of authors have detected the same flaws in CDA and have developed very similar arguments prior to this article. Since it will be easier to understand the similarities after reading this article, discussion of this literature is provided in Appendix D.

Throughout the paper we will adopt the following conventions regarding infinity. In analysis, the extended real numbers $\overline{\mathbb{R}}$ are formed as the set $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, where ∞ is a symbol, not a number. We adopt the interpretation of ∞ as meaning "beyond number". Since the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, with $\mathbb{N}^+ = \mathbb{N} \setminus 0$, are a subset of the reals, $\mathbb{N} \subset \mathbb{R}$, the extended naturals are similary defined by $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Thus, ∞ is a member of the extended naturals, but not a member of $\mathbb{N}$. The convention that $\forall r \in \mathbb{R}$, $\infty - r = \infty$ then implies $\forall n \in \mathbb{N}, \infty - n = \infty$. Hence we consider that $\forall n \in \mathbb{N}$, n is finite (as did Cantor [3]). From set theory we adopt the symbolic conventions that ω is the smallest infinite ordinal number greater than all $n \in \mathbb{N}$, and $\aleph_0$ the smallest infinite cardinal number. Given the set of extended natural numbers, taking $\omega \equiv \infty$ we define $\sup \mathbb{N} = \omega$, and $|\mathbb{N}| = |\omega| = \aleph_0$.

II. THE STRUCTURE OF CDA

We assume that the reader is familiar with CDA [2] (if not from the original German text then from a translation [5] or equivalents in recent texts, e.g. Theorem 2.11, in [12], or the version in [1]). A high level outline of the argument might note several stages:

1. Either the real numbers in an open interval (a, b) are countable or they are not.
2. Assume that they are countable.
3. Then they can be arranged in a table.
4. Then using digits on the diagonal of the table to create an "anti-diagonal" number provides a real number that cannot be in the original set.
5. Therefore there is at least one real number that has not been counted.
6. This contradicts the assumption that the real numbers in (a, b) are countable.

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In [2] Cantor specified an open interval (a, b), but one can equivalently consider the interval $I = [0, 1]$ or $[0, 1)$ without loss of generality: If $m = a + (b - a)/2$ it is simple to map (a, b) to $(a - m, m - a)$ centred on the origin which can be rescaled to $(-x, x)$ so that $|x| < 1$. Selecting the half interval $[0, x) \subset [0, 1]$ validates use of the closed interval. If $[0, 1]$ is (un)countable, then so is $[-1, 1]$, therefore $(-x, x)$ is, and hence any bounded open interval is too. Given this, the entries a_{ij} in the table FIG. 1 are typically taken to represent the digits of decimal fractions in $[0, 1]$ in some number base B. Cantor used an abstract code with $a_{ij} \in \{m, w\}$ which has an obvious mapping to binary $B_2 = \{0, 1\}$ but for the construction of CDA any finite base will do.

To allay concerns that some tacit questions about cardinality have been overlooked in forming the remapping above, we remark in addition that $\overline{\mathbb{R}}^+ = [0, \infty]$ is undeniably an infinite interval of the real number line, and if it is treated as a set then it must have infinite cardinality (even if its (un)countability has not yet been determined). It is therefore difficult to deny that $\overline{A} = [1, \infty]$ is also an infinite set with infinite cardinality. If that is admitted then the set $G = \{1/r : r \in \overline{A}\} = [0, 1] = I$ has the same cardinality as $\overline{A}$. If $\overline{A} \cup G = \overline{\mathbb{R}}^+$ then $|\overline{A}| + |G| = |\overline{\mathbb{R}}^+| = 2|I|$. By the conventions of analysis or those of the cardinal arithmetic associated with set theory $2.\infty = \infty$, and we can use $I = [0, 1]$ as an equivalent interval to study.

III. CRITIQUE

It is interesting to pursue the alternative option in stage 1 of the argument outline in §II. (We have to presume that CDA does not hold at this stage.) We then assume that the real numbers in I are not countable (in the naïve pre-CDA sense). Since I is a set, by definition its elements must be unique within the set, so we can write $I = \{r_1, r_2, r_3, \ldots\}$, where no particular ordering of the elements is implied by the indexes of each r_i. Likewise, $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. We can imagine a process where we remove elements one at a time from I and append them to a new set, forming $\{r_1\} \mapsto 1$, $\{r_1, r_2\} \mapsto 2$, $\{r_1, r_2, r_3\} \mapsto 3$ and so on, in a manner that parallels the set theoretic construction of $\mathbb{N}$. If the real numbers in I were not countable we would expect to run out of elements of $\mathbb{N}$ before we had removed all the elements of I. But this implies an upper bound on $\mathbb{N}$ which by definition of the naturals (using the Axiom of Infinity) cannot occur.

Thus, in order for the reals in I to be uncountable (in the naïve sense), there would be no set of symbols (finite or infinite) that could be used to label all elements of I using a 1-to-1 map. For it must be the case that there is a 1-to-1 mapping between any infinite set of unique symbols and the natural numbers in $\mathbb{N}$. Those real numbers that cannot, thereby, be labelled uniquely (however one attempts to do it) in order that they may be distinguished from others effectively do not exist for mathematics. Therefore mathematics is restricted to using the subset of uniquely labelled members of the set of (supposedly uncountable) reals. The elementary mapping process above provides a labelling process that maps to $\mathbb{N}$. Therefore the real numbers used in mathematics are countable.

We will call the above the Choose-and-Count (C&C) argument. The fact that this line of argument exists and provides a conclusion that is at least as plausible as the "counter-intuitive" conclusion reached by CDA should immediately begin to give mathematicians pause for thought.

The C&C argument contains an idea which will be useful to bear in mind in what follows. We can imagine an abstract set of unique symbols used in a manner equivalent to a number base B_∞ which could constitute a positional number representation scheme with only one digit position. Such a scheme lacks a diagonal, so CDA cannot be applied to it. As above, the abstract base B_∞ is necessarily bijective with $\overline{\mathbb{N}}$, and the concept of "uncountability" is unnecessary. Call this the Abstract Base corollary.

Little needs to be said about stage 2 of the CDA outline. Stage 3 seems innocuous at first sight, but its use of abstract symbols in place of actual numbers becomes more misleading as the argument progresses. We note that, looking at actual numbers, one possibility is to order a subset Q of rational numbers, $Q \subset \mathbb{Q} \cap [0, 1]$, according to a scheme that mirrors the naturals. W.l.o.g., for expository convenience we will use the decimal base with digits in

$$
\begin{array}{l}
r_1 \mapsto a_{11} a_{12} a_{13} a_{14} \ldots \\
r_2 \mapsto a_{21} a_{22} a_{23} a_{24} \ldots \\
r_3 \mapsto a_{31} a_{32} a_{33} a_{34} \ldots \\
r_4 \mapsto a_{41} a_{42} a_{43} a_{44} \ldots \\
\quad \ldots
\end{array}
$$

FIG. 1. Assumed tabulation of the real numbers.

$B_{10} = \{0, 1, 2, \dots, 9\}$. Then Q is the subset of rationals whose decimal fraction representations have only zeros after a finite number of decimal places. See FIG. 2 and Appendix C.

$$
\begin{array}{c}
0 \mapsto 0000\dots = q_0 \mapsto 0.0000\dots \\
1 \mapsto 1000\dots = q_1 \mapsto 0.1000\dots \\
2 \mapsto 2000\dots = q_2 \mapsto 0.2000\dots \\
3 \mapsto 3000\dots = q_3 \mapsto 0.3000\dots \\
\dots \\
10 \mapsto 0100\dots = q_{10} \mapsto 0.0100\dots \\
11 \mapsto 1100\dots = q_{11} \mapsto 0.1100\dots \\
12 \mapsto 2100\dots = q_{12} \mapsto 0.2100\dots \\
13 \mapsto 3100\dots = q_{13} \mapsto 0.3100\dots \\
\dots
\end{array}
\tag{1}
$$

FIG. 2. An ordered tabulation of the real numbers $q_n \in Q \subset \mathbb{Q} \cap [0, 1]$.

The subset Q is conspicuously countable, and can be used to construct a rigorous challenge to the structure of CDA (see Appendix A). We can also ask, given this method by which to order these real numbers in Q, if there are any other real numbers in I that could not be placed in order using the same approach. It is hard to imagine that there could be, though this raises interesting issues concerning limits and number representation systems hinted at earlier (see Appendix C). If there are not in fact any such excluded numbers then the real numbers in I would be countable. (Proof of the bijective mapping between the limiting decimal fraction representations and $\mathbb{R}$ can be found in, e.g., [13].) One could then assert that $|B_{10}|^{|\mathbb{N}|} = 10^{\aleph_0} \equiv 10^{\infty} = \infty \equiv \aleph_0$.

In addition, this issue of table construction is independent of which base is chosen for number representations. According to this critique, irrespective of the choice of number base, the set of representations of real numbers in $[0, 1]$ cannot be expanded beyond countable infinity. This concurs with the idea that the cardinality of the continuum should not be altered by a different choice of number base.

The suspicion is, then, that the use of abstract notation for stage 3 of CDA is likely to mislead the unwary into believing that properties of positional number representation systems, such as the availability of a useful ordering and its countability, can be overlooked harmlessly.

Stage 4 of the CDA outline can be critiqued on several grounds. The first is that if we have all the numbers in $[0, 1]$ in our table *a priori*, then using the diagonal process to create a number that is not in the original set does precisely that: It creates a number that is not in $[0, 1]$. So why does the constructed anti-diagonal number seem to have a form that puts it within $[0, 1]$? Could it be that ignoring the zero ahead of the decimal point is a bit sneaky?

This line of counterargument carries over more forcefully to an aspect of CDA associated with modern set theory. If the table of reals in $[0, 1]$ is constructed in binary B_2, and the digit positions in each number can be labelled with members of $\mathbb{N}$ as an index set, then each binary number can be used to map the position of its ones to a subset of $\mathbb{N}$ (see [1]). In this case the table of all the real numbers in $[0, 1]$ maps to the powerset of the natural numbers $\mathcal{P}(\mathbb{N})$. Hence the set theory trope that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|} > \aleph_0$. Meanwhile, simple combinatorial arguments consistent with cardinal arithmetic can also be used to show instead that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| = |\mathbb{N}| = \aleph_0$ (see Appendix B). We argue that the inconsistency arises from CDA not having been sufficiently clear about what is being counted. CDA does not match numbers with numbers under a single number representation scheme. It matches digit position indexes of the number representation scheme against numbers represented in that scheme.

We can ask, when any finite positional number representation system with a base B_m where $m \geqslant 2$ and d digit positions must generate $s(d) = |B_m|^d$ unique representations, where $s(d) > d$ is always true, why it is that CDA prioritises the diagonal of a square and not that of a rectangle. If, instead, we construct the anti-diagonal number by counting down the table by d positions before each digit selection we find that any anti-diagonal number constructed (from the rectangle) is now within the set we have counted through, and therefore not outside the counted set. This Stretched Diagonal counterargument remains true as $d \to \infty$, irrespective of any ordering of number representations in the table.

In addition, if we allow the conceit that "completed" infinities exist, then the entire table for $s(\aleph_0) = |B_2|^{\aleph_0}$ exists. Then if we can have numbers with $\aleph_0$ symbol positions, what prevents us from having numbers with $2^{\aleph_0}$ symbol positions? Imagine treating the rectangular symbol table as a matrix and forming its transpose, then asking

if the rows of the transposed table are also mapped to number representations. If so, then we can form the table for $s(2^{\aleph_0}) = |B_2|^{2^{\aleph_0}}$, again form the transpose, and ask if the new rows are also number representations. There is no limit to how far this process can be iterated. The problem then arises that we are no longer sure what a number is, and we have scant grounds for claiming that there is something worth calling a limit within mathematics, without selecting ω by mere fiat.

Furthermore, the way in which the construction of the square anti-diagonal only counts digit positions and not the actual unique number representations in the table can be exposed with rigorous argument. See Appendix A. Therefore, not only is the choice of construction —the diagonal of a square —too arbitrary to make the idea that the anti-diagonal number is not a member of the original set a necessity, but combined with the failure of the same process to count unique representations of numbers in the table properly we are led to conclude that stage 5 of the CDA outline is an unjustified claim. The final claim in stage 6 cannot then be held true.

IV. FUNCTIONAL CDA

Cantor's example for a generalised CDA in the second half of [2] used functions where $x \mapsto f(x) \in \{0, 1\}$, $\forall x \in [0, 1]$. Cantor assumes the uncountability of the domain, but this might not be necessary for the argument. In [7], on pp.7-8, Kleene presents the functional version of CDA (FCDA), where the entries in the table consist of the unique bounded discrete functions whose domain and range are a countable set. This enumerability rules out unbounded values of the functions given a domain point x, so w.l.o.g. both domain and range can be assumed to be the natural numbers, e.g. $F = \{f_m : y = f_m(x); x, y \in \mathbb{N}^+\}$. The nature of the index m will be seen to be problematic here. The fact that x and y are in $\mathbb{N}^+$ means that this version of the CDA can be seen as one where each symbol position is in base $B_{\mathbb{N}}$. So according to set theory combinatorics the number of possible unique functions in the set should be $|B_{\mathbb{N}}|^{|\mathbb{N}|} = \aleph_0^{\aleph_0}$. At first sight FCDA seems like a much stronger argument: The initial set already seems to escape naïve notions of counting — unless some way to map the functions f_m to an ordered countable set is available.

We note that FCDA is still vulnerable to the Stretched Diagonal counterargument (despite the use of $B_{\mathbb{N}}$) insofar as whether the anti-diagonal function is found within the initial set or outside the subset accounted for by the square diagonal is dependent upon an arbitrary choice of pattern for diagonal construction. Yet this doesn't remove the obligation to give an account of the index m given the apparent size of the initial set F. Thus the problem with FCDA is not really about diagonal arguments, it is just one of the determination of the size of a very large set.

The C&C argument (and the Abstract Base corollary) can still be applied to F, again providing a strong argument that F is a countable set.

We can also consider the set $F = \{f_m : f_m \in \Lambda_{\mathbb{N}}\}$ as the set of bounded points of the multidimensional lattice $\Lambda_{\mathbb{N}}$. If it is possible to map each f_m to a unique point in the subset of the 2-D plane $I^2 = [0, 1] \times [0, 1]$ then by the arguments against CDA in §III F will be countable (from $|\mathbb{N}|^2 = \aleph_0.\aleph_0 = \aleph_0$).

One possible mapping is given by mapping each entry in the FCDA table according to:

$$y_{mb} = f_m(b) \mapsto \frac{1}{(b + if_m(b))^p} = \frac{1}{(z_{mb})^p}, \qquad \text{with } i = \sqrt{-1},\ b \in \mathbb{N}^+ \tag{2}$$

with $p \geqslant 2$ chosen large enough to ensure convergence when we form the series of these terms for each f_m.

Then for each f_m define $H : F \to \mathbb{C}$:

$$H(f_m) := \sum_{b \in \mathbb{N}^+} \frac{1}{(z_{mb})^p}. \tag{3}$$

Since for each m each sequence of the $(z_{mb})_{b\in\mathbb{N}^+}$ is unique by definition, with all z_{mb} bounded, and for each f_m the series in (3) is convergent, then $H(f_m)$ is unique for each index m.

Proof: For a given set of terms, a convergent sum is the same independent of the ordering of the terms. So $H(f_m)$ is unique iff the set $\zeta_m = \{z^p_{mb}\}$ is unique, and ζ_m is unique iff the set $\xi_m = \{z_{mb}\}$ is unique. Since the sequence $b = 1, 2, 3, \ldots$ is fixed, by (2) ξ_m is unique iff the sequence $(f_m(b))_{b\in\mathbb{N}^+}$ is unique. By assumption, $\forall m$, $(f_m(b))_{b\in\mathbb{N}^+}$ is unique, and the result follows. □

We can then form a mapping from:

$$H(f_m) = s_m + it_m \mapsto (\widetilde{s_m}, \widetilde{t_m}) \in \mathbb{R}^2, \tag{4}$$

where the terms in $(\widetilde{s_m}, \widetilde{t_m})$ have both been rescaled to lie in the interval $[-1, 1] \times [-1, 1]$. (Another rescaling would put all points inside I^2.)

Given that we already know that CDA is false, so $[-1, 1] \subset \mathbb{R}$ is countable, then since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph_0$ we have that F must be countable too. (Therefore we no longer need to worry about the status of the index m.)

For extra confirmation it would be good to show that the set of pairs $P = \{(\widetilde{s_m}, \widetilde{t_m})\}$ is totally disconnected, and can be suitably ordered. Ordering is taken care of by a small extension of the scheme given in §III and FIG. 2. We first prioritise positive over negative, then order those numbers s_m by the order of their mirror naturals. Then we order pairs $(\widetilde{s_m}, \widetilde{t_m})$ with s_m prioritised over t_m, with the subsets of t_m also ordered by their mirror naturals. So we have the four quadrants in order, $(+,+)$, $(+,-),(-,+),(-,-)$. Since we know that the members of each quadrant are countable, then from earlier arguments we have at most $4.\aleph_0 = \aleph_0$ for the cardinality of the whole region. Therefore the cardinality of P is at most $\aleph_0$.

To show the disconnectedness, we have to show that if any two functions f_k and f_j differ in only one position t by the smallest difference (given the assumption of the range and domain being mapped to $\mathbb{N}^+$) such that $|f_k(t) - f_j(t)| = 1$, then $|H(f_k) - H(f_j)| > 0$, for all pairs $k \neq j$. Consider the difference between the mapped terms,

$$\left| \frac{1}{(t + i f_k(t))^p} - \frac{1}{(t + i f_j(t))^p} \right| = \epsilon \tag{5}$$

By construction we have that $\forall t$, $1 \leqslant t < \infty$ and $\forall m$, $1 \leqslant f_m(t) < \infty$, hence $\forall (k, j)$, $k \neq j$, we have that the real number $\epsilon > 0$. Therefore P is a totally disconnected set.

V. CONCLUSION

We contend, on the basis of the evidence for inconsistencies and weaknesses presented, that CDA is false and so is FCDA. Moreover, there are stronger, simple arguments for adopting the view that all sets are countable: If sets by definition contain unique elements and a subset operator $A \subset B$ exists, then an enumeration can be constructed for all sets. It follows that two countably infinite sets cannot have cardinalities that differ, otherwise the "smaller" of the two sets would imply an upper bound on $\mathbb{N}$ that the "larger" did not observe. Therefore there is only one infinity, ∞.

The fact that CDA has existed as part of the canon of mathematics for so long is rather disconcerting given the evidence generated by detailed scrutiny. CDA may have a future role in mathematical pedagogy as a cautionary tale perhaps, but it should not inform the modern practice of mathematics.

Appendix A: CDA —A Broken Form

Applying the form of Cantor's Diagonal Argument (CDA) straightforwardly to $\mathbb{N}$ yields the contradictory claim that there are more natural numbers than natural numbers. The error of conflating counting the symbol positions of a positional q-ary number representation system with the numbers it represents is exposed.

In FIG. 1 we have the usual representation of the CDA table construct for the reals in $I = [0, 1]$ using some positional q-ary number representation system with base B_q.

From this, CDA is used to advocate that there are more reals than naturals.

In FIG. 3 we have a similar representation for the natural numbers in $\mathbb{N}$ left-padded with zeros. E.g. in base 10: $\ldots 0001 = 1; \ldots 0002 = 2$ and so on. The table representation need not contain these numbers in order (though ordering can be useful for exposition). They could be arbitrarily shuffled without loss of generality.

$$\begin{array}{r}
\ldots a_{14} a_{13} a_{12}\, a_{11} \rightarrow n_1 \\
\ldots a_{24} a_{23}\, a_{22}\, a_{21} \rightarrow n_2 \\
\ldots a_{34}\, a_{33}\, a_{32} a_{31} \rightarrow n_3 \\
\ldots\, a_{44}\, a_{43} a_{42} a_{41} \rightarrow n_4 \\
\ldots
\end{array}$$

FIG. 3.

We operate the form of the argument in CDA on the table in FIG. 3 in exactly the same way as the original argument purporting to apply to the reals, and obtain the claim that there are more naturals than there are naturals. A contradiction. The only difference between this application of CDA and a key subset of the original one is a prior assumption about what the symbol strings of the q-ary number representation system are supposed to represent.

Therefore we contend that CDA is an erroneous form of argument, and its claimed result in respect of the reals is unfounded. We provide a more rigorous argument below.

Proof: Form the bijective map from naturals $\mathbb{N}$ to the set of finite symbol strings $S = \{s_i : s_i = (a_{ij})_{j\in\mathbb{N}},\ i \in \mathbb{N}\}$, formed using each $a_{ij} \in \{\text{`0'}, \text{`1'}, \ldots \text{`9'}\}$.

$$f : \mathbb{N} \to S$$

Without loss of generality, we study the $s_i \in S$ placed in a table such that $f^{-1}(s_i) = i$.

$$\begin{aligned} \ldots a_{03}a_{02}a_{01}\,a_{00} &= s_0 \\ \ldots a_{13}a_{12}\,a_{11}\,a_{10} &= s_1 \\ \ldots a_{23}\,a_{22}\,a_{21}a_{20} &= s_2 \\ \ldots\, a_{33}\,a_{32}a_{31}a_{30} &= s_3 \\ \ldots& \end{aligned} \tag{A1}$$

We then select all the digit symbols along the diagonal to form $d = (a_{ii})_{i\in\mathbb{N}}$.

$$\begin{aligned} \ldots 000\,0 &= s_0 \\ \ldots 00\,0\,1 &= s_1 \\ \ldots 0\,0\,02 &= s_2 \\ \ldots\,0\,003 &= s_3 \\ \ldots& \end{aligned} \tag{A2}$$

We notice that $d = s_0$, the symbol string composed entirely of zeros, with $f^{-1}(s_0) = 0$.

We can form the Cantorian replacement symbol string b by substituting for every symbol '0' in d a symbol from $B = \{\text{`1'}, \ldots, \text{`9'}\}$. So that $b = (b_j)_{j\in\mathbb{N}}$ and w.l.o.g. we set $b_j = \text{`1'}$, $\forall j \in \mathbb{N}$.

We also consider the symbol strings p_k whose initial $k + 1$ symbols are '1' with the remainder being '0',

$$p_k = f\left(\sum_{j=0}^{k} 10^j\right) \tag{A3}$$

as well as the symbol strings, $d_k = f(10^k), \forall k \in \mathbb{N}$.

We note that for every row i in the table, $f^{-1}(s_i) < f^{-1}(p_i)$, and that s_i contains more '0' symbols than p_i does. Moreover, $f^{-1}(s_k) < f^{-1}(p_i)$, $k \leqslant i$, $\forall i \in \mathbb{N}$.

The gap between any i and $f^{-1}(d_i)$ is given by the function,

$$g(i) = f^{-1}(d_i) - f^{-1}(s_i) = 10^i - i\ . \tag{A4}$$

which provides an indirect measure of the number of '0' symbols between the '1' in d_i and the leftmost symbol from B in s_i.

We see also that,

$$\left(f^{-1}(p_i) - i\right) > g(i), \forall i \in \mathbb{N}\ . \tag{A5}$$

Since $f^{-1}(p_i) \leqslant f^{-1}(b)$ for all i, by implication $b \notin S$, since it differs from every s_i. Yet $f^{-1}(b) \in \mathbb{N}$ by construction. So there is a contradiction.

We draw the conclusion that CDA does not count the naturals properly.

Given, the mapping $F : [0, 1] \to T$ from the real decimal fractions in $I = [0, 1]$ to the set of symbol strings of countably infinite length T, it is clear that the elements of S map bijectively to the countable subset of T representing the terminating decimal fractions. If CDA does not count correctly on S then it also does not count correctly on that subset of T.

Appendix B: $\mathcal{P}(\mathbb{N})$ and Countability

A simple application of combinatorics with set theory suggests that the real numbers are countable.

It appears to be possible to challenge the claim that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| \tag{B1}$$

using combinatorics.

The number of elements k in a powerset of any finite set A is given by:

$$k = |\mathcal{P}(A)| = \sum_{n=0}^{|A|} \binom{|A|}{n} = 2^{|A|} \tag{B2}$$

where $\binom{a}{b}$ is the binomial coefficient.

We observe that,

$$\log(n!) = \sum_{j=1}^{n} \log(j) \tag{B3}$$

Then,

$$\log \binom{n}{p} = \log \left(\frac{n!}{p!(n-p)!} \right) = \sum_{j=(n-p+1)}^{n} \log(j) - \sum_{r=1}^{p} \log(r) \tag{B4}$$

So that for all positive integers $p < n$,

$$\lim_{n\to\infty} \sum_{j=(n-p+1)}^{n} \log(j) - \sum_{r=1}^{p} \log(r) = \lim_{n\to\infty} \log(n) = \infty \equiv \aleph_0 \tag{B5}$$

If we let $|A| \to |\omega| = \aleph_0$ in (B2) then,

$$k \to \sum_{n=1}^{\aleph_0} \aleph_0 \to \aleph_0 \tag{B6}$$

since $n.\aleph_0 = \aleph_0$ for any $n \in \mathbb{N}$, and $\aleph_0 \oplus \aleph_0 = \aleph_0$ by the rules of cardinal arithmetic.

In slow motion, we can construct the limit for k iteratively, since,

$$k_0 = \lim_{n\to\aleph_0} \binom{n}{0} = 1, \tag{B7}$$

$$k_1 = k_0 + \lim_{n\to\aleph_0} \binom{n}{1} = \aleph_0, \tag{B8}$$

and for all $m > 1$,

$$k_m = k_{m-1} \oplus \lim_{n \to \aleph_0} \binom{n}{m} = \aleph_0. \tag{B9}$$

Hence,

$$k = \lim_{m \to \aleph_0} k_m = \aleph_0. \tag{B10}$$

Hence,

$$|\mathcal{P}(\mathbb{N})| = \aleph_0 = |\mathbb{N}|. \tag{B11}$$

In textbook set theory, we have

$$|\mathbb{R}| = |\{0,1\}^{|\mathbb{N}|}| = |\{0,1\}^{\omega}| = |2^{\omega}| = 2^{|\omega|} = 2^{\aleph_0} = |\mathcal{P}(\mathbb{N})|. \tag{B12}$$

But from the combinatoric argument earlier,

$$|\mathcal{P}(\mathbb{N})| = \aleph_0, \tag{B13}$$

hence we have that $|\mathbb{R}| = \aleph_0$.

For consistency, it must be the case that, $2^{\aleph_0} = \aleph_0$. Given the symbolic equality $\aleph_0 = \infty$, we see the equivalence with conventional analysis that $2^{\infty} = \infty$.

Meanwhile, $|\mathbb{R}| = \aleph_0$ implies that $\mathbb{R}$ is countably infinite.

Appendix C: Mirror Naturals

When using a positional number representation scheme in a finite base B_y, it follows as a corollary of the C&C argument that for every real $r \in I$ there exists a unique natural $n \in \mathbb{N}$ generated by reflecting r in the decimal point.

Given the bijective mirror mapping $M_1 : I \to \mathbb{N}$ we can form the mapping, $M_2 : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ using,

$$M_2(n_1, n_2) \mapsto (n_1 + M_1^{-1}(n_2)) = r \in \mathbb{R}^+ \tag{C1}$$

According to modern set theory, $\mathbb{N} \times \mathbb{N} \equiv \mathbb{N}$ (by the same argument that makes the rationals countable).

Likewise, the extension $M_3 : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$ is readily formed by,

$$M_3(z_1, n_2) = \begin{cases} M_2(z_1, n_2) : z_1 \geqslant 0 \\ -M_2(|z_1|, n_2) : z_1 < 0 \end{cases} \tag{C2}$$

which maps to all of $\mathbb{R}$.

Whether one agrees that the above mapping to $\mathbb{R}$ is bijective might yet hinge upon a nuance of the convergence of series in the limit. (See also the discussion in Appendix E). For example, the B_{10} decimal representation of a number $r \in I$ is a shorthand for the terms of its series construction. For a terminating decimal, this is:

$$r = 0.a_1 a_2 a_3 \ldots a_n = \sum_{k=1}^{n} a_k 10^{-k} = s_n \tag{C3}$$

In the case of a non-terminating decimal $r \in I$, the series of partial sums s_n approximating r must converge according to $\lim_{n\to\infty} s_n = r$. It follows from the construction of the series and the Limit of Terms test for convergence that,

$$\lim_{k\to\infty} a_k 10^{-k} = 0 \tag{C4}$$

It also follows from our definition of the mirror mapping M_1 that any decimal place of r that exists implies a corresponding digit of n, and vice versa. The nuance of the argument is whether one can interpret (C4) to imply the existence of a final decimal place with a zero in it.

But consider to the contrary the view that (C4) does not imply the existence of such a digit. Then either there is a final digit before it that exists and is significant and can be mapped under M_1, or the notion of limit ceases to mean limit.

If one is happy with the interpretation, then this limiting zero is discounted under the mapping M_1 and a unique mirror natural corresponds to the remaining significant figures of r under the mapping.

It is worth remarking that this dilemma over the status of the limit only arises for a number representation scheme with a finite base B_y. If we invoke the Abstract Base corollary from §III and use a symbol set B_∞ in a representation scheme that effectively only has one digit position, then all symbols have the same status in the scheme.

Appendix D: Literature Review

The authors of the preprint [8] have helpfully created a collection of all proofs for uncountability found in the literature, with a useful list of references. If we use the modern distinction between finite ($n \in \mathbb{N}$), countably infinite ($\aleph_0$), and uncountable ($2^{\aleph_0} > \aleph_0$), it is useful to ask whether these proofs (excepting CDA) change their conclusions if the real numbers are assumed to have only countably infinite cardinality from the outset. (Extra remarks about the modern terminology are given in Appendix E.) We provide supplementary critiques of these claimed proofs in [6].

Meanwhile, a small number of contemporary authors have published documents online pre-dating our article that contain cogent critiques of Cantor's arguments for the existence higher infinities. Among these, [10] contains useful counterarguments against several of Cantor's theorems, one of which is very similar in form to our approach in Appendix A.

A longer text that predates our article by almost a decade, [11], presents a lengthy mixture of logic and mathematics perhaps best regarded as a flawed magnum opus containing much of value. It astutely identifies inconsistencies in the proofs of both CDA and the Powerset Theorem, and identifies avenues of counterargument that are conceptually similar to the Stretched Diagonal argument and the use of the sum of binomial coefficients in Appendix B respectively. Beyond the conclusion that the reals are a countably infinite set the paper then continues on to assess whether Gödel's First Incompleteness Theorem is also false. However, in our view, some errors in a few of the proofs, and a nagging ambiguity of notation that sometimes conflates $\mathbb{N}$ with $\overline{\mathbb{N}}$ in a few contexts, mean that some of the conclusions are not robustly acquired (even if one agrees with them). We hope the author will correct these issues in a future version.

In addition, [4] presents a carefully constructed train of arguments. The expository use of slightly unconventional terminology comparing "writable" (for finite) and "unwritable" (for infinite) early on in respect of number representation schemes can be forgiven as cosmetic, as it does not undermine essential properties or insights. In §§1-8 the paper describes a process for constructing counterexamples to CDA that exploit specific orderings of the real numbers in the table to create anti-diagonals that contradict CDA even when infinite limits are taken. In §9 and onward the process is generalised and metamathematical model theory is invoked to argue against forms of the Continuum Hypothesis. Many of the arguments in [4] represent more formal statements of arguments and conclusions we have presented in the body of our article. For instance, in [4] the relationship between the Abstract Base corollary and positional number representation schemes is treated in detail throughout.

We hope our article complements the work in these other papers, and provides additional grounds for further progress.

Appendix E: Terminology and Infinity

Using language, the choice of terminology affects the inferences that can be made. It should be noted that the modern distinctions between finite ($n \in \mathbb{N}$), countably infinite ($\aleph_0$), and uncountable (of order $2^{\aleph_0} > \aleph_0$) that arose out of the works of Cantor and his contemporaries have consequences which must be handled with some care. For

example, if $\forall n \in \mathbb{N}$, n is finite, and countable infinity is (conventionally) defined by $\forall n \in \mathbb{N}$, $\omega - n = \omega$, then very strictly speaking, since $\mathbb{N}$ always contains only finite numbers, $\mathbb{N}$ cannot ever be an infinite set. Yet providing a cardinality for $\mathbb{N}$ requires setting $\sup \mathbb{N} = \omega = \sup \overline{\mathbb{N}}$, and then letting the cardinality operator behave according to $|\mathbb{N}| = |\overline{\mathbb{N}}| = |\omega| = \aleph_0$. Continuing in this precise vein, a set S is only countably infinite if it can be bijectively mapped to $\overline{\mathbb{N}}$. We are then confronted by the idea that sets that merely map bijectively to $\mathbb{N}$ are merely countable, and not countably infinite. To pass from the finite to the infinite one must imagine a "transfinite" journey unto ω.

The debate in mathematics over whether the limit operation $\lim_{n\to\infty} f(n)$ meant that the limit was definitely acquired as n progresses towards infinity rather than at $n = \infty$ preceded Cantor. The limits of convergent approximations of irrationals by continued fractions were one example discussed. Several of Cantor's proofs prior to CDA are more concerned with asserting that, when considering unending sequences of nested intervals of $\mathbb{R}$, given as $A_0 \supset A_1 \supset A_2 \ldots$, one can claim,

$$S = \bigcap_{n\in\mathbb{N}} A_n \qquad \neq \qquad S^* = \bigcap_{n=0}^{\omega} A_n \tag{E1}$$

Then arguing that only the statement on the right side of (E1) provides a unique limit point, $S^* = \{p\}$. This approach seems strong, but actually it only trades off uncertainty as to whether the statement on the left side ever converges to a unique limit, in favour of a certainty that convergence definitely occurs somewhere within the inherent vagueness of ω (veiling a contradiction, as exhibited at the end of Appendix C).

However, observe that adopting the claim in favour of the right side of (E1) does not automatically imply that the cardinality of $\mathbb{R}$ must be greater than $\aleph_0$. So the contemporary mathematical use of the word uncountable might not entirely align in meaning with its usage prior to 1880, and similarly with the term "transfinite".

We also remark that if one agrees that CDA and the claim that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$ are both untenable, forcing acceptance that $\mathbb{R}$ is countably infinite, while still insisting that the contents of $\mathbb{N}$ are only ever finite, then possible resolutions of the paradox of exact-yet-infinite limits involve shifts of conceptual boundaries, and suggestions might include:

- Accepting that $\sup \mathbb{N} = \infty = \sup \overline{\mathbb{N}}$ implies that infinity can remain a "potential infinity" never actually used and that this is sufficient to guarantee convergence to a single point for the statement on the left side of (E1). That is, asserting that the totality of $\mathbb{N}$ is used provides sufficient guarantee. This approach is compatible with the idea of mirror naturals (as in Appendix C). The mirror naturals for irrationals in $I = [0, 1]$ are then just very large natural numbers (possibly a subset of prime numbers) much closer to ∞ than to one. But notice how this approach weakens the ability to show (assuming it was necessary to do so) that the set $H(F)$ from §IV is disconnected.

- A similar approach (that would need to be more formally delineated) invoking the existence of "transfinite" numbers, τ, included in an extension to $\mathbb{N}$, that are bigger than any finite number while being strictly less than ∞. Limits that exist are then acquired by sequences iterating into the transfinite zone. This approach creates the reciprocal concept, from $1/\tau$, of the infinitesimals, and one enters the realm of "non-standard" analysis. While not as simple to implement as the approach above, non-standard analysis is known to simplify some aspects of using calculus and it avoids having to make strong claims about the properties of irrationals. If one agrees with the conclusions of this paper, a sketch of the minimal necessary embeddings might be:

$$\begin{aligned} {}^* &: \mathbb{N} \to {}^*\mathbb{N} \\ \mathbb{Z}^* &= \pm\mathbb{N}^* \\ {}^* &: \mathbb{Q} \to {}^*\mathbb{Q} \\ {}^*\mathbb{Q} &\cup \{0\} \simeq \mathbb{R} \end{aligned} \tag{E2}$$

  With no need to extend $\mathbb{R}$. Hopefully this would provide maximal compatibility with notation used for limits in analysis in the past, even if a change to the underlying semantics has occurred. A difficulty with this approach is that $\mathbb{N}$ appears to have a hard upper bound within $\mathbb{N}^*$ and questions can be raised about precisely where that is (though see below for a related option).

For comparison, consider an approach which concertedly avoids explicitly defining any relative subsets of $\mathbb{N}$ as finite or transfinite, but which tacitly acknowledges that transfinites (indefinitely large but less than infinity) are only invoked whenever a limit of a sequence is sought. Instead of (E2), we could write $n \to \infty$, $1/n \to 0$, and use Dedekind

cuts to define $\mathbb{R}$ [9]. Then $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ retain their classical type definitions. When $\mathbb{R}$ has countably infinite cardinality (to use set theoretic terminology), instead of (E1) we take care to write:

$$S_m = \bigcap_{n=0}^{m} A_n \qquad \neq \qquad \lim_{m\to\infty} S_m = S^* = \bigcap_{n=0}^{\infty} A_n \tag{E3}$$

That is, the status of a specified value for m on the left side of (E3) must never be conflated with a limit process. In this approach, the left side of (E1) is not considered to be a valid statement because there is neither a specified upper bound on n nor the invocation of a limit process. For consistency, this approach may require that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are treated only as type descriptors and not as set extensions.

Meanwhile, the paradox of exact-yet-seemingly-infinite limits found in Appendix C perhaps has a resolution through acceptance of the idea that the mirror natural number corresponding to an irrational $r \in I$ represented in some finite base B_y is (silently) a transfinite natural, hidden within a limit process; technically inaccessible, and therefore unable to support further strong claims about its properties.

Providing thorough assessment of the relative merits or drawbacks of such approaches is, however, beyond the scope of this paper.

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[1] George S. Boolos and Richard C. Jeffrey. *Computability and Logic.* Cambridge University Press, 3rd edition, 1989.
[2] Georg Cantor. Ueber eine elementare Frage der Mannigfaltigkeitslehre. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1:75–78, 1891.
[3] Georg Cantor. *Contributions to the Founding of the Theory of Transfinite Numbers.* Dover, 1915. English Translation and Introduction: Philip E. B. Jourdain.
[4] E. Coiras. Counterexamples to Cantorian Set Theory. 2014. `https://arxiv.org/abs/1404.6447`.
[5] Google Translate™, DeepL™, and Peter P. Jones. A Translation of G. Cantor's "Ueber eine elementare Frage der Mannigfaltigkeitslehre". August 2019. `https://www.researchgate.net/publication/335364685_A_Translation_of_G_Cantor's_Ueber_eine_elementare_Frage_der_Mannigfaltigkeitslehre`.
[6] Peter P. Jones. Supplementary Critiques of Claimed Proofs for Uncountability, 2020. `https://www.researchgate.net/publication/341568678_Supplementary_Critiques_of_Claimed_Proofs_for_Uncountability`.
[7] S. C. Kleene. *Introduction to Metamathematics.* Ishi Press International, 2009. 2009 reprint of the original 1952 edition.
[8] Christina Knapp and Cesar E. Silva. The Uncountability of the Unit Interval. 2014. `https://arxiv.org/abs/1209.5119`.
[9] E. Landau. *Foundations of Analysis.* Chelsea Publishing, 1966. English Translation: F. Steinhardt.
[10] W. Mueckenheim. On Cantor's Important Proofs. 2006. `https://arxiv.org/abs/0306200`.
[11] J. A. Perez. Addressing Mathematical Inconsistency: Cantor and Gödel Refuted. 2010. `https://arxiv.org/abs/1002.4433`.
[12] Walter Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 3rd edition, 1976.
[13] V. A. Zorich. *Mathematical Analysis I.* Springer, 2004. English Translation: Roger Cooke.

Figures

$$
\begin{aligned}
r_1 &\mapsto \boxed{a_{11}}\, a_{12} a_{13} a_{14} \ldots \\
r_2 &\mapsto a_{21} \boxed{a_{22}}\, a_{23} a_{24} \ldots \\
r_3 &\mapsto a_{31} a_{32} \boxed{a_{33}}\, a_{34} \ldots \\
r_4 &\mapsto a_{41} a_{42} a_{43} \boxed{a_{44}} \ldots \\
&\ldots
\end{aligned}
$$

Figure 1

Assumed tabulation of the real numbers.

$$
\begin{aligned}
0 &\mapsto 0000\ldots = q_0 \mapsto 0.0000\ldots \\
1 &\mapsto 1000\ldots = q_1 \mapsto 0.1000\ldots \\
2 &\mapsto 2000\ldots = q_2 \mapsto 0.2000\ldots \\
3 &\mapsto 3000\ldots = q_3 \mapsto 0.3000\ldots \\
&\ldots \\
10 &\mapsto 0100\ldots = q_{10} \mapsto 0.0100\ldots \\
11 &\mapsto 1100\ldots = q_{11} \mapsto 0.1100\ldots \\
12 &\mapsto 2100\ldots = q_{12} \mapsto 0.2100\ldots \\
13 &\mapsto 3100\ldots = q_{13} \mapsto 0.3100\ldots \\
&\ldots
\end{aligned}
$$

Figure 2

An ordered tabulation of the real numbers qn □ Q □ Q □ [0, 1].

$$\ldots a_{14}a_{13}a_{12}\boxed{a_{11}} \to n_1$$

$$\ldots a_{24}a_{23}\boxed{a_{22}}a_{21} \to n_2$$

$$\ldots a_{34}\boxed{a_{33}}a_{32}a_{31} \to n_3$$

$$\ldots \boxed{a_{44}}a_{43}a_{42}a_{41} \to n_4$$

$$\ldots$$

Figure 3

A similar representation for the natural numbers in N left-padded with zeros.