Supplementary Material for: Nanoscale Real-Time Detection of Quantum Vortices at Millikelvin Temperatures

A. Guthrie, S. Kafanov, M. T. Noble, Yu. A. Pashkin, G. R. Pickett,
V. Tsepelin, A. A. Dorofeev, V. A. Krupenin, and D. E. Presnov

I. DEVICE DESCRIPTION

The nano-electromechanical device system (NEMS) consists of a doubly-clamped aluminium-on-silicon nitride ($\mathrm{Al-on-Si_3N_4}$) composite nanobeam. The beam's dimensions are defined lithographically, with length $l = 70\,\mu\mathrm{m}$ and width $w = 200\,\mathrm{nm}$. The 100 nm thick $\mathrm{Si_3N_4}$ layer determines beam's mechanical properties, while Al layer allows to excite and measure beam motion magnetomotively. The combined thickness of the aluminium and silicon nitride layers is $t = 130\,\mathrm{nm}$, with a combined density of $3062\,\mathrm{kg\,m^{-3}}$. The vacuum frequency of the fundamental mode is determined experimentally to be $f_0 = 2.166\,\mathrm{MHz}$. The nanobeam is suspended roughly $d \sim 1\,\mu\mathrm{m}$ above the silicon substrate. The experiment is housed in a brass-experimental cell containing superfluid ^{4}He at a temperature of 10 mK, mounted to the mixing chamber of a cryogen-free dilution refrigerator.

II. MEASUREMENT SCHEME

The nanobeam response was probed using a magnetomotive detection scheme. Here, the Lorentz force driving the nanobeam is a result of an AC current passed through the nanobeam in a perpendicular magnetic field, which is supplied by a large external solenoid. The beam motion in the magnetic field produces a Faraday voltage that is detected by a drop in the transmitted signal. For characterisation of the nanobeam, a vector network analyser was used to both supply the AC current, and acquire the transmitted response measured as a function of frequency. The resulting Lorentzian resonance curve is fitted to obtain the nanobeam velocity, v, and force, F, using previously established methods [1].

To perform time dependent resonance tracking, two phase-sensitive lock-in measurement techniques were employed: single-frequency detection, and multi-frequency detection. Single-frequency detection was conducted using a signal generator to supply a fixed-frequency, constant AC signal to the nanobeam input, with the nanobeam output connected to a high-frequency (SR844) lock-in amplifier. With the driving frequency fixed on resonance, any change in the nanobeam resonance frequency is detected as a drop in the measured signal.

Simultaneous detection at multiple frequencies was performed using a multi-frequency lock-in amplifier (MLA) [2] in place of the signal generator and high-frequency lock-in. The MLA instrument operates by using a frequency comb composed from integer multiples n_i of a base tone f_b so that all measurement frequencies f_i satisfy $f_i = n_i f_b$. To be able to distinguish between tones the measurement time t_m must be larger then the inverse separation between frequencies $t_m > 1/f_b$. This constrains the time resolution and frequency spacing of the instrument, and faster measurements have the frequencies placed further apart. It is also pertinent to note that non-linearity of a resonator will cause mixing between the frequency tones although the use of low excitation drives avoids this problem [3].

For both resonance tracking techniques, an oscilloscope was used in conjunction with the lock-in demodulation in order to record vortex capture events. The lock-in demodulated signal at the beam's vortex free resonance was monitored by the oscilloscope, which would trigger the lock-in amplifier to record data when the signal strength fell sufficiently due to resonance frequency shift. For single frequency measurements fall and subsequent rise in the signal would then give the event lifetime. In multi-frequency measurements the recorded data was fitted with a Lorentzian peak to obtain the beam's resonate frequency as a function of time, and the lifetime was then found from this data.

Similarly, the tuning fork is measured with a vector network analyser, using an I-V converter [4] (transimpedance amplifier) to recover the signal which can then be used to find the fork velocity [5, 6]. The driving force on the fork can be found from the drive signal using well established techniques [5, 6].

III. FREQUENCY SHIFT DUE TO A TRAPPED VORTEX

A. Tension of the Beam in Vacuum

The beam's resonance frequencies can be modelled as the harmonics of a doubly clamped resonator [7]:

$$f_n = \frac{k_n^2}{\pi\sqrt{48}}\frac{w}{l^2}\sqrt{\frac{E}{\rho_{\mathrm{Al}}}}\sqrt{1+\gamma_n\left(\frac{l}{w}\right)^2\frac{T_0}{wtE}}, \qquad (1)$$

where η is the strain, w and l represent the width and length of the beam respectively. The coefficients k_n and γ_n have different values depending on the eigenmode of the resonance: $k_1 = 4.7300$, $\gamma_1 = 0.2949$, $k_2 = 7.8532$, $\gamma_2 = 0.1453$, and $k_{n\geqslant 3} = \pi(n+1/2)$, $\gamma_{n\geqslant 3} = 12(k_n - 2)/k_n^3$.

During fabrication, the $\mathrm{Si_3N_4}$ layer is pre-stressed to improve the mechanical properties of the beam. Using the measured value of $f_0 = 2.166\,\text{MHz}$, with Young's modulus $E = 70\,\text{GPa}$ we can estimate the intrinsic nanobeam tension using eq. (1) to be $T_0 = 5.6\,\mu\text{N}$.

B. Hydrodynamic Shift of the Beam Frequency in Liquid ^{4}He

In liquid ^{4}He, the nanobeam fundamental frequency will be shifted due to displacement of fluid by the beam. At 10 mK the normal-component is negligible, and we can ignore effects due to hydrodynamic clamping. The hydrodynamic displacement can be modelled as an increase in the effective mass of the beam, which thus shifts the resonance frequency from the vacuum state [1]:

$$\left(\frac{f_0}{f_H}\right)^2 = 1 + \beta\frac{\rho_H}{\rho_b} \qquad (2)$$

where ρ_H is the density of helium and β geometric constant. The resonance frequency for our beam in liquid helium at 10 mK is shifted by 50 kHz from vacuum to $f_H = 2.116\,\text{MHz}$. The geometric constant can therefore be calculated as $\beta = 0.46$.

C. Acoustic damping

Acoustic damping is a frequency dependent damping source for oscillators active at all temperatures. However the magnetomotive damping of the beam at 5 T is an order of magnitude higher then for acoustic damping [8] so we can neglect its affects here.

D. Effects of a Trapped Vortex on the Beam

In order to minimise its energy the trapped vortex will align its core along the nanobeam. The presence of a trapped vortex along the length of the nanobeam will give rise to two forces, both of which act to increase the frequency of the nanobeam. The relative magnitude of these shifts will be estimated here.

1. *Interactions between the nanobeam, vortex and surface*

The interaction between the vortex and substrate can be calculated by using the method of an image vortex, *i.e.* one should remove the surface from consideration and assume that the vortex interacts with a parallel image-vortex which is located a distance $2d$ from it.

The interaction force per unit length between two vortexes is given by:

$$\boldsymbol{f} = \boldsymbol{j} \times \boldsymbol{\kappa},$$

where $|\boldsymbol{\kappa}| = h/m_{^4\text{He}} = 9.92 \times 10^{-8}\,\text{m}^2\,\text{s}^{-1}$ is the circulation quanta in ^{4}He; $|\boldsymbol{j}| = \rho_{^4\text{He}} v_s$ is the flow density created by image-vortex on the place of the beam; the linear velocity of the superfluid on the distance r from the vortex core is given by $v_s = \kappa/(2\pi r)$.

The final expression for the repulsive force between the vortex trapped on the beam and silicon surface is given by

$$|\boldsymbol{F}| = \frac{1}{4\pi}\frac{l}{d}\kappa^2\rho_{^4\text{He}}.$$

Substituting our experimental parameters, the force per unit of length and the total attractive force are

$$|\boldsymbol{f}| = 115\,\text{nN}\,\text{m}^{-1}; \qquad |\boldsymbol{F}| = 8.04\,\text{pN}.$$

Under the action of this force, the beam will sag. Associating an origin with one of the clamped ends of the nanobeam, one can describe such sagging by the function:

$$z(x) = \frac{1}{2}\frac{|\boldsymbol{f}|}{E}\left(\frac{x}{t}\right)^2\frac{(l-x)^2}{tw},$$

The maximum displacement at the centre of the beam is:

$$z_{\max}\left(\frac{l}{2}\right) = \frac{1}{32}\frac{|\boldsymbol{f}|}{E}\frac{l^4}{wt^3}$$

in our case the maximum sagging will be $z_{\max} = 0.734\,\text{pm}$.

The maximum tension will be at $x = 0$ and $x = l$:

$$T_{\max} = \frac{1}{2}|\boldsymbol{f}|\frac{l^2}{t}$$

in our case $T_{\max} = 6.51\,\text{nN}$. The total tension acting on the nanobeam is now given by $T_{\text{tot}} = T_0 + T_{\max}$. Using T_{tot} as the value of the tension, the expected frequency due to a trapped vortex is found by substituting the result of eq. (1) into eq. (2) as

$$f_v = 2.117\,\text{MHz}, \qquad (3)$$

and corresponds to a frequency shift of $f_v - f_H = 1\,\text{kHz}$ which is comparable to what was observed.

2. *Magnus Force on the Beam*

The value of the Magnus force per unit length in a superfluid liquid is given by:

$$\boldsymbol{r} = -\rho_{^4\text{He}}\,\boldsymbol{\kappa} \times \boldsymbol{v}_b,$$

where v_b is the beam velocity. Assuming that $v_b \sim 10^{-2}\,\text{m}\,\text{s}^{-1}$ one can get:

$$|\boldsymbol{r}| = 124.09\,\text{nN}\,\text{m}^{-1},$$

this value is similar to the repulsive force from the surface. However, the sign of the Magnus force depends on the direction of motion. Thus, the half period of the beam oscillations, the Magnus force will be summed with the repulsive force from the surface, while over another half of the period Magnus force will be subtracted from the repulsive force. This leads to a significant diminishing of the observed frequency shift from the Magnus force, observable as a small, velocity dependent contribution to the frequency shift. At a nanobeam velocity of $v_b \sim 10^{-2}\,\mathrm{m\,s^{-1}}$, the corresponding frequency shift due to the Magnus force was $\sim 10\,\mathrm{Hz}$, much less than the contribution from the substrate interaction.

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