

# A Universal Predictor-corrector Algorithm for Numerical Simulation of Generalized Fractional Differential Equations

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## Research Article

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# A universal predictor-corrector algorithm for numerical simulation of generalized fractional differential equations

Zaid Odibat\*

## Abstract

This study introduces some remarks on generalized fractional integral and differential operators, that generalize some familiar fractional integral and derivative operators, with respect to a given function. We briefly explain how to formulate representations of generalized fractional operators. Then, mainly, we propose a predictor-corrector algorithm for the numerical simulation of initial value problems involving generalized Caputo-type fractional derivatives with respect to another function. Numerical solutions of some generalized Caputo-type fractional derivative models have been introduced to demonstrate the applicability and efficiency of the presented algorithm. The proposed algorithm is expected to be widely used and utilized in the field of simulating fractional-order models.

**Keywords:** Fractional differential equation; generalized fractional derivative; Caputo derivative; predictor-corrector algorithm; numerical solution.

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## 1 Introduction

Fractional calculus theory has gained enormous attention, in recent years, and has shown remarkable development in modelling physical phenomena, with memory and non-locality characteristics, in many fields [1-11]. Fractional order derivatives of a given function include the entire history of the function; the next state of a fractional-order model depends not only on its current state but also on all of its historical states. Fractional integral and derivative operators have been treated as the preferred mathematical operators for characterization of memory in physical modeling issue [12-17]. In the literature, several fractional operators definitions have been proposed. A family of fractional derivatives, such as Riemann-Liouville, Hadamard, and Caputo derivatives are introduced using fractional integrals. The Riemann-Liouville fractional integral, which is one of the most studied definitions, of order  $\alpha > 0$  is defined by [1-6].

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad x > a. \quad (1)$$

Depending on the above fractional integral definition, the Riemann-Liouville and Caputo fractional derivatives of order  $\alpha > 0$  are defined by

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$${}^R D_{a+}^{\alpha} f(x) = D^m I_{a+}^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x (x-\xi)^{m-\alpha-1} f(\xi) d\xi, \quad x > a, \quad (2)$$

$${}^C D_{a+}^{\alpha} f(x) = I_{a+}^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-\xi)^{m-\alpha-1} f^{(m)}(\xi) d\xi, \quad x > a, \quad (3)$$

respectively, such that  $m-1 < \alpha \leq m$  and  $m \in \mathbb{N}$ . An alternative characterization of the Caputo differential operator can be introduced as a modification of Riemann-Liouville differential operator in the form

$${}^C D_{a+}^{\alpha} f(x) = {}^R D_{a+}^{\alpha} \left[ f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a+)}{k!} (x-a)^k \right], \quad x > a. \quad (4)$$

The Caputo differential operator is used extensively to model many physical phenomena in fractional differential calculus applications because it has many features that are similar to ordinary ones. The next type of fractional integral operators, which is related to our study, is the Hadamard fractional integral operator given by [3, 7]

$${}^H I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{\xi} \right)^{\alpha-1} f(\xi) \frac{d\xi}{\xi}, \quad \alpha > 0, \quad x > a. \quad (5)$$

The associated Hadamard fractional derivative operator of order  $\alpha > 0$  is given by

$${}^H D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \left( x \frac{d}{dx} \right)^m \int_a^x \left( \ln \frac{x}{\xi} \right)^{m-\alpha-1} f(\xi) \frac{d\xi}{\xi}, \quad \alpha > 0, \quad x > a, \quad (6)$$

where  $m = \lceil \alpha \rceil$ . Recently, Katugampola [18, 19] introduces a new fractional integral operator, that generalizes Riemann-Liouville integral operator, as follows

$$I_{a+}^{\alpha, \rho} f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \xi^{\rho-1} (x^{\rho} - \xi^{\rho})^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad x > a, \quad (7)$$

where  $\rho > 0$ . Consequently, for  $m-1 < \alpha \leq m$  where  $m \in \mathbb{N}$ , the Riemann-type and Caputo-type fractional derivative operators corresponding to Katugampola fractional integral operator, given in (7), are introduced in [18, 20] and [21] as,

$${}^R D_{a+}^{\alpha, \rho} f(x) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \left( x^{1-\rho} \frac{d}{dx} \right)^m \int_a^x \xi^{\rho-1} (x^{\rho} - \xi^{\rho})^{m-\alpha-1} f(\xi) d\xi, \quad x > a, \quad (8)$$

and

$${}^C D_{a+}^{\alpha, \rho} f(x) = \frac{\rho^{\alpha-m+1}}{\Gamma(m-\alpha)} \int_a^x s^{\rho-1} (x^{\rho} - s^{\rho})^{m-\alpha-1} \left( s^{1-\rho} \frac{d}{ds} \right)^m f(s) ds, \quad x > a, \quad (9)$$

respectively, where  $m = \lceil \alpha \rceil$  and  $\rho > 0$ . Furthermore, Osler [22] highlighted a useful extension of fractional integral and derivative operators by initiating fractional order integrals and derivatives of a given function  $f$  related to another function  $\sigma$  as

$$I_{a+}^{\alpha, \sigma(x)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \sigma'(\xi) (\sigma(x) - \sigma(\xi))^{\alpha-1} f(\xi) d\xi, \quad (10)$$

and

$$D_{a+}^{\alpha, \sigma(x)} f(x) = \frac{1}{\Gamma(m-\alpha)} \left( \frac{1}{\sigma'(x)} \frac{d}{dx} \right)^m \int_a^x \sigma'(\xi) (\sigma(x) - \sigma(\xi))^{m-\alpha-1} f(\xi) d\xi, \quad (11)$$

for  $a < x < b$  and  $m - 1 < \alpha \leq m$ , respectively. The concept of fractional integral  $I_{a+}^{\alpha, \sigma(x)} f(x)$ , for  $a < x < b$ , has been studied in [3], where  $\sigma$  is considered to be any monotone function on  $(a, b)$  that has a continuous derivative on  $(a, b)$ , and in [7], where  $\sigma$  is considered to be any increasing and positive monotone function on  $(a, b)$  that has a continuous derivative on  $(a, b)$ . Then the fractional derivative concept  $D_{a+}^{\alpha, \sigma(x)} f(x)$ , for  $a < x < b$ , has been introduced in [3, 7] under the additional condition that  $\sigma'(x) \neq 0$  on  $(a, b)$ . These conditions, as will be shown in the next section, can be adjusted in a more convenient manner. In [23], Almeida introduces a Caputo type fractional derivative with respect to the function  $\sigma(x)$  based on the fractional integral given in (10), which has characteristics similar to those of the Caputo derivative given in (3), as

$${}^C D_{a+}^{\alpha, \sigma(x)} f(x) = \frac{1}{\Gamma(m - \alpha)} \int_a^x \sigma'(\xi) (\sigma(x) - \sigma(\xi))^{m-\alpha-1} \left( \frac{1}{\sigma'(\xi)} \frac{d}{d\xi} \right)^m f(\xi) d\xi. \quad (12)$$

On the other hand, the growing interest in applications involving fractional order differential models has motivated the improvement, development and extension of analytical and numerical methods for solving such models. The predictor-corrector (P-C) approach, presented in [24], is one of the most robust, stable and efficient methods that is widely used for the numerical solution of initial value problems (IVPs) with Caputo fractional derivatives. Some applications and simulations of this approach can be found in [25-31]. In [32], an optimized algorithm of the P-C approaches has been developed. The suggested algorithm, which implements the first-order Taylor series linearization of the nonlinear equation, provides approximate numerical solutions to Caputo-type IVPs with better accuracy than those obtained using the approach discussed in [24]. In [33], an adaptive algorithm of the P-C approaches to handle IVPs containing the generalized Caputo-type fractional derivative defined in (9) is introduced. The details and the mathematical derivation of the adaptive P-C algorithm is presented in [33].

In the present study, we provide precise conditions on the function  $\sigma$  to define fractional integrals with respect to the function  $\sigma$ , namely generalized fractional integrals. The conditions, that  $\sigma$  is considered to be any strictly increasing function that has a continuous derivative on  $(a, b)$ , are given in the sense that the generalized fractional integral, for  $a < x < b$ , is well defined. Accordingly, we introduce fractional derivative representations with respect to the function  $\sigma$ , namely generalized fractional derivatives, under the additional condition that  $\sigma'(x) \neq 0$  on  $(a, b)$ . We present the formulation of generalized fractional integral and derivative operators with respect to the function  $\sigma$  in a way that they can be considered as generalizations of many well known fractional operators. Then, basically, we suggest a universal P-C algorithm to simulate nonlinear IVPs containing generalized Caputo-type fractional derivatives with respect to another function numerically. Besides, some test problems are discussed to display the efficiency, performance, features and accuracy of the suggested algorithm. Throughout our work,  $\sigma$  will indicate a strictly increasing and differentiable function, where  $\frac{d}{dx} \sigma(x) = \delta(x) \neq 0$ , on  $(a, b)$ .

## 2 Generalized fractional operators

### 2.1 Generalized fractional integral operator

In this section we consider the generalization of fractional integral given in formula (10), where  $\sigma$  is assumed to be any strictly increasing function that has a continuous derivative on  $(a, b)$ , which will be called, in this work, the generalized fractional integral. The considered generalized fractional integral operator can be seen as a generalization of many well known definitions, such as Riemann-Liouville, Hadamard and Katugampola integral operators. We explain the mathematical derivation approach and then present some properties of the studied generalized integral operator concept. First, let's recall that the Riemann-Liouville integral operator, given in (1), can be seen as a direct extension of the Cauchy integral formula

$$\int_a^x d\xi_1 \int_a^{\xi_1} d\xi_2 \cdots \int_a^{\xi_{n-1}} f(\xi_n) d\xi_n = \frac{1}{(n-1)!} \int_a^x (x-\xi)^{n-1} f(\xi) d\xi, \quad (13)$$

where the integer  $n$  is replaced with any real  $\alpha > 0$ , using the property that  $\Gamma(n+1) = n!$ . Now, to derive the examined generalized fractional integral operator, consider for  $n \in \mathbb{N}$  and strictly increasing differentiable function  $\sigma(x)$  (such that  $\frac{d}{dx}\sigma(x) = \delta(x)$ ) on  $(a, b)$ , the  $n$ -fold integral formula

$$I_{a+}^{n, \sigma(x)} f(x) = \int_a^x \delta(\xi_1) d\xi_1 \int_a^{\xi_1} \delta(\xi_2) d\xi_2 \cdots \int_a^{\xi_{n-1}} \delta(\xi_n) f(\xi_n) d\xi_n. \quad (14)$$

In case of  $n = 2$ , exchanging the order of integration, using Dirichlet technique [5], we obtain

$$\int_a^x \delta(\xi_1) d\xi_1 \int_a^{\xi_1} \delta(\xi) f(\xi) d\xi = \int_a^x \delta(\xi) f(\xi) d\xi \int_{\xi}^x \delta(\xi_1) d\xi_1 = \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi)) f(\xi) d\xi. \quad (15)$$

Repeating the above iteration  $n - 1$  times, we get

$$I_{a+}^{n, \sigma(x)} f(x) = \frac{1}{(n-1)!} \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi))^{n-1} f(\xi) d\xi, \quad (16)$$

which suggests the following definition, when the integer  $n$  is replaced with any real  $\alpha > 0$ .

**Definition 1** *The generalized fractional integral operator of order  $\alpha$  with respect to the function  $\sigma(x)$ ,  $I_{a+}^{\alpha, \sigma(x)}$ , is defined (provided it exists) by*

$$I_{a+}^{\alpha, \sigma(x)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi))^{\alpha-1} f(\xi) d\xi, \quad a < x < b, \quad (17)$$

where  $\alpha > 0$ ,  $a \geq -\infty$ ,  $b \leq \infty$ , and the function  $\sigma(x)$  is assumed to ensure convergence of the integral.

**Remark 1** *The assumption that the function  $\sigma(x)$  must be strictly increasing on  $(a, b)$  is sufficient to ensure that  $(\sigma(x) - \sigma(\xi)) > 0$  when  $a < \xi < x$  and that  $\sigma(x)$  is injective, and so has an inverse on  $(a, b)$ .*

According to the concept of generalized fractional integral operator, introduced in Definition 1, we can simply draw the following integral operators as special cases

1. In the case of  $\sigma(x) = x$ , the generalized integral operator  $I_{a+}^{\alpha, x}$  reduces fully to the Riemann-Liouville integral operator  $I_{a+}^{\alpha}$ , given in (1).
2. In the case of  $\sigma(x) = \ln x$ , the generalized integral operator  $I_{a+}^{\alpha, \ln x}$ , where  $a \geq 0$ , reduces to the Hadamard integral operator  ${}^H I_{a+}^{\alpha}$ , given in (5).
3. In the case of  $\sigma(x) = x^{\rho}/\rho$ , the generalized integral operator  $I_{a+}^{\alpha, x^{\rho}/\rho}$ , where  $\rho > 0$  and  $a \geq 0$ , reduces to the Katugampola integral operator  $I_{a+}^{\alpha, \rho} f(x)$ , given in (7).
4. In the case of  $\sigma(x) = x^{\rho}$ , the generalized integral operator  $I_{a+}^{\alpha, x^{\rho}}$ , where  $\rho > 0$  and  $a \geq 0$ , reduces to the integral operator

$$I_{a+}^{\alpha, x^{\rho}} f(x) = \frac{\rho}{\Gamma(\alpha)} \int_a^x \xi^{\rho-1} (x^{\rho} - \xi^{\rho})^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad a < x < b. \quad (18)$$

5. In the case of  $\sigma(x) = \exp(\lambda x)$ , the generalized integral operator  $I_{a+}^{\alpha, \exp(\lambda x)}$ , where  $\lambda > 0$ , reduces to the integral operator

$$I_{a+}^{\alpha, \exp(\lambda x)} f(x) = \frac{\lambda}{\Gamma(\alpha)} \int_a^x \exp(\lambda \xi) (\exp(\lambda x) - \exp(\lambda \xi))^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad a < x < b. \quad (19)$$

**Remark 2** The Erdélyi-Kober fractional integral operator, which is defined as [34]

$$(I_{a+; \rho, \eta}^{\alpha} f)(x) = \frac{\rho x^{-\rho(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \xi^{\rho\eta+\rho-1} (x^{\rho} - \xi^{\rho})^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad 0 \leq a < x < b, \quad (20)$$

with  $\rho > 0$  and  $\eta \in \mathbb{R}$ , can be reformulated using the integral operator  $I_{a+}^{\alpha, x^{\rho}}$ , given (18), as

$$(I_{a+; \rho, \eta}^{\alpha} f)(x) = x^{-\rho(\alpha+\eta)} I_{a+}^{\alpha, x^{\rho}} x^{\rho\eta} f(x). \quad (21)$$

It is known that the Riemann-Liouville fractional integral operator given in (1) satisfies the semigroup property  $I_{a+}^{\alpha} I_{a+}^{\beta} = I_{a+}^{\alpha+\beta}$ , when  $\alpha, \beta > 0$ . Now, we give a similar semigroup property of the generalized fractional integral operator.

**Theorem 1** Let  $\alpha > 0$  and  $\beta > 0$ . Then, for  $a < x < b$ , [23]

$$I_{a+}^{\alpha, \sigma(x)} I_{a+}^{\beta, \sigma(x)} f(x) = I_{a+}^{\alpha+\beta, \sigma(x)} f(x). \quad (22)$$

Next, we investigate a useful relationship between the generalized fractional integral operator with respect to the function  $\sigma(x)$ ,  $I_{a+}^{\alpha, \sigma(x)}$ , and the Riemann-Liouville integral operator  $I_{a+}^{\alpha}$ . This relationship will be used to formulate classes of generalized fractional derivative operators.

**Theorem 2** Let  $\alpha > 0$ . Then, for  $a < x < b$ ,

$$I_{a+}^{\alpha, \sigma(x)} f(x) = [I_{\sigma(a)}^{\alpha} (f \circ \sigma^{-1})](\sigma(x)). \quad (23)$$

**Proof.** Implementing the change of variables  $u = \sigma(\xi)$  in the generalized integral formula of the function  $f$ , given in Definition 1, yields

$$I_{a+}^{\alpha, \sigma(x)} f(x) = \frac{1}{\Gamma(\alpha)} \int_{\sigma(a)}^{\sigma(x)} (\sigma(x) - u)^{\alpha-1} f(\sigma^{-1}(u)) du, \quad (24)$$

where  $\sigma^{-1}$  is the inverse function of  $\sigma$ . Using the fact that  $(I_b^{\alpha} g)(h(x)) = \frac{1}{\Gamma(\alpha)} \int_b^{h(x)} (h(x) - u)^{\alpha-1} g(u) du$ , formula (23) can be, thus, derived. This completes the proof.

## 2.2 Generalized fractional derivative operators

This section introduces, based on the representation of the generalized fractional integral operator presented in Definition 1, generalized forms of fractional derivative operators. These forms generalize the Riemann-Liouville, Caputo, Hadamard and Katugampola fractional derivative operators. Some of the generalized derivative operators will be derived based on the relation (23). If  $0 < \alpha \leq 1$ , we may introduce the generalized Riemann-type fractional derivative operator of order  $\alpha$  with respect to the function  $\sigma(x)$ ,  ${}^R D_{\sigma(a)}^{\alpha, \sigma(x)}$ , by replacing the operator  $I_{\sigma(a)}^{\alpha}$  in formula (23) with the composite operator  $D I_{\sigma(a)}^{1-\alpha}$ . One can easily recognize that, for  $0 < \alpha \leq 1$ ,

$$\begin{aligned}
({}^R D_{a+}^{\alpha, \sigma(x)} f)(x) &= [D I_{\sigma(a)}^{1-\alpha} (f \circ \sigma^{-1})](\sigma(x)), \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\delta(x)} \frac{d}{dx} \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi))^{-\alpha} f(\xi) d\xi, \quad a < x < b,
\end{aligned} \tag{25}$$

where  $\delta(x) \neq 0$  on  $(a, b)$ . In general, using the conceptual relationship  $({}^R D_{a+}^{\alpha, \sigma(x)} f)(x) = [D^m I_{\sigma(a)}^{m-\alpha} (f \circ \sigma^{-1})](\sigma(x))$ ,  ${}^R D_{a+}^{\alpha, \sigma(x)}$  is characterized as,

**Definition 2** *The generalized Riemann-type fractional derivative operator of order  $\alpha$  with respect to the function  $\sigma(x)$ ,  ${}^R D_{a+}^{\alpha, \sigma(x)}$ , is defined (provided it exists) by*

$$({}^R D_{a+}^{\alpha, \sigma(x)} f)(x) = \frac{1}{\Gamma(m-\alpha)} \left( \frac{1}{\delta(x)} \frac{d}{dx} \right)^m \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi))^{m-\alpha-1} f(\xi) d\xi, \quad a < x < b, \tag{26}$$

where  $a \geq -\infty$ ,  $b \leq \infty$ ,  $m-1 < \alpha \leq m$  and  $\delta(x) \neq 0$  on  $(a, b)$ .

**Remark 3** *We can observe that in the case of  $\sigma(x) = x$  the generalized Riemann-type derivative operator  ${}^R D_{a+}^{\alpha, x}$  reduces to the Riemann-Liouville derivative operator given in (2), in the case of  $\sigma(x) = \ln x$  the generalized Riemann-type derivative operator  ${}^R D_{a+}^{\alpha, \ln x}$  reduces to the Hadamard derivative operator given in (6) and in the case of  $\sigma(x) = x^\rho/\rho$  the generalized Riemann-type derivative operator  ${}^R D_{a+}^{\alpha, x^\rho/\rho}$  reduces to the Katugampola derivative operator given in (8).*

Now, we turn our concern to introduce a Caputo-type of generalized fractional derivative operators. The generalized Caputo-type fractional derivative,  ${}^C D_{a+}^{\alpha, \sigma(x)}$ , can be obtained by replacing the operator  $I_{\sigma(a)}^\alpha$  in formula (23) with the composite operator  $I_{\sigma(a)}^{m-\alpha} D^m$ . So, using the conceptual relationship  $({}^C D_{a+}^{\alpha, \sigma(x)} f)(x) = [I_{\sigma(a)}^{m-\alpha} D^m (f \circ \sigma^{-1})](\sigma(x))$ ,  ${}^C D_{a+}^{\alpha, \sigma(x)}$  is characterized as,

**Definition 3** *The generalized Caputo-type fractional derivative operator of order  $\alpha$  with respect to the function  $\sigma(x)$ ,  ${}^C D_{a+}^{\alpha, \sigma(x)}$ , is defined (provided it exists) by*

$$({}^C D_{a+}^{\alpha, \sigma(x)} f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x \delta(\xi) (\sigma(x) - \sigma(\xi))^{m-\alpha-1} \left( \frac{1}{\delta(\xi)} \frac{d}{d\xi} \right)^m f(\xi) d\xi, \quad a < x < b, \tag{27}$$

where  $m-1 < \alpha \leq m$  and  $\delta(x) \neq 0$  on  $(a, b)$ .

**Remark 4** *In particular, if  $0 < \alpha < 1$  then, for  $a < x < b$ , the generalized Caputo-type fractional derivative reduces to*

$$({}^C D_{a+}^{\alpha, \sigma(x)} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\sigma(x) - \sigma(\xi))^{-\alpha} f'(\xi) d\xi. \tag{28}$$

In view of the following results, which have been proved in [23], we can observe that the properties of the generalized Caputo-type fractional derivative appear closer to those of ordinary derivatives than those of the generalized Riemann-type fractional derivative.

**Theorem 3** *Let  $m-1 < \alpha \leq m$  and  $f(x) \in C^m[a, b]$ . Then, for  $a < x < b$ ,*

$$({}^C D_{a+}^{\alpha, \sigma(x)} f)(x) = {}^R D_{a+}^{\alpha, \sigma(x)} \left\{ f(x) - \sum_{n=0}^{m-1} \frac{1}{n!} (\sigma(x) - \sigma(a))^n \left[ \left( \frac{1}{\delta(t)} \frac{d}{dt} \right)^n f(t) \right] \Big|_{t=a} \right\}, \tag{29}$$

and

$$I_{a+}^{\alpha, \sigma(x)} {}^C D_{a+}^{\alpha, \sigma(x)} f(t) = f(x) - \sum_{n=0}^{m-1} \frac{1}{n!} (\sigma(x) - \sigma(a))^n \left[ \left( \frac{1}{\delta(t)} \frac{d}{dt} \right)^n f(t) \right] \Big|_{t=a}. \tag{30}$$

### 3 The universal predictor-corrector algorithm

In this section, we develop an extended P-C algorithm, namely the universal P-C algorithm, to conveniently, effectively and accurately offer numerical solutions for IVPs involving generalized Caputo-type fractional derivatives. We will derive the main steps of the suggested algorithm to deal with the IVP

$$\begin{cases} {}^C D_{a+}^{\alpha, \sigma(x)} y(x) &= f(x, y(x)), & x \in (a, b], \\ y^{(k)}(a) &= y_0^k, & k = 0, 1, \dots, [\alpha], \end{cases} \quad (31)$$

where  ${}^C D_{a+}^{\alpha, \sigma(x)}$  is the generalized Caputo-type fractional derivative operator of order  $\alpha$  with respect to the function  $\sigma(x)$  given in Definition 3,  $a \geq 0$  and  $\delta(x) \neq 0$  on  $(a, b)$ . At first, for  $m-1 < \alpha \leq m$  and  $y \in C^m([a, b])$ , the IVP (31) is equivalent, referring to Theorem 3, to the integral equation

$$y(x) = v(x) + \frac{1}{\Gamma(\alpha)} \int_a^x \delta(s) (\sigma(x) - \sigma(s))^{\alpha-1} f(s, y(s)) ds, \quad (32)$$

where

$$v(x) = \sum_{n=0}^{m-1} \frac{1}{n!} (\sigma(x) - \sigma(a))^n \left[ \left( \frac{1}{\delta(t)} \frac{d}{dt} \right)^n f(t) \right] \Big|_{t=a}. \quad (33)$$

Let the functions  $f$  and  $\sigma$  be assumed so that a unique solution to IVP (31) exists in the interval  $[a, b]$ . To describe our algorithm, let us define a grid in the interval  $[a, b]$  with  $N+1$  non-equispaced nodes  $x_k$ ,  $k = 0, 1, \dots, N$ , given by

$$\begin{cases} x_0 &= a, \\ x_{k+1} &= \sigma^{-1}(\sigma(x_k) + h), & k = 0, 1, \dots, N-1, \end{cases} \quad (34)$$

where  $h = \frac{\sigma(b) - \sigma(a)}{N}$  and  $\sigma^{-1}$  is the inverse function of  $\sigma$ . Now, we are going to generate the approximation  $y_k \approx y(x_k)$  over the non-uniform grid  $\{x_k, k = 1, 2, \dots, N\}$  to simulate numerically the solution of the IVP (31). To do this, assuming that the numerical values of  $y(x)$  at  $x_0, x_1, \dots, x_k$  have been evaluated which are denoted as  $y_0, y_1, \dots, y_k$ , we need to calculate the approximation  $y_{k+1} \approx y(x_{k+1})$  by means of the integral

$$y(x_{k+1}) = v(x_{k+1}) + \frac{1}{\Gamma(\alpha)} \int_a^{x_{k+1}} \delta(s) (\sigma(x_{k+1}) - \sigma(s))^{\alpha-1} f(s, y(s)) ds. \quad (35)$$

Performing the substitution

$$u = \sigma(s), \quad (36)$$

yields

$$y(x_{k+1}) = v(x_{k+1}) + \frac{1}{\Gamma(\alpha)} \int_{\sigma(a)}^{\sigma(x_{k+1})} (\sigma(x_{k+1}) - u)^{\alpha-1} f(\sigma^{-1}(u), y(\sigma^{-1}(u))) du, \quad (37)$$

or equivalently

$$y(x_{k+1}) = v(x_{k+1}) + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^k \int_{\sigma(x_n)}^{\sigma(x_{n+1})} (\sigma(x_{k+1}) - u)^{\alpha-1} f(\sigma^{-1}(u), y(\sigma^{-1}(u))) du, \quad (38)$$



Next, by applying the trapezoidal rule with respect to the weight function  $(\sigma(x_{k+1}) - \cdot)^{\alpha-1}$  to the integrals given on the right-hand side of Eq. (38) (i.e. replacing the function  $f(\sigma^{-1}(u), y(\sigma^{-1}(u)))$  by its piecewise linear interpolating polynomial with nodes chosen at the  $\sigma(x_n)$ ,  $n = 0, 1, \dots, k+1$ ), we get

$$\int_{\sigma(x_n)}^{\sigma(x_{n+1})} (\sigma(x_{k+1}) - u)^{\alpha-1} f(\sigma^{-1}(u), y(\sigma^{-1}(u))) du \approx \frac{h^\alpha}{\alpha(\alpha+1)} \left\{ \left( (k-n)^{\alpha+1} - (k-n-\alpha)(k-n+1)^\alpha \right) f(x_n, y(x_n)) \right. \\ \left. + \left( (k-n+1)^{\alpha+1} - (k-n+\alpha+1)(k-n)^\alpha \right) f(x_{n+1}, y(x_{n+1})) \right\}. \quad (39)$$

Hence, substituting the approximations given in Eq. (39) in to Eq. (38) and collecting like terms, we obtain the corrector formula for  $y(x_{k+1})$ ,  $k = 0, 1, \dots, N-1$ , as

$$y(x_{k+1}) \approx v(x_{k+1}) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{n=0}^k a_{n,k+1} f(x_n, y(x_n)) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(x_{k+1}, y(x_{k+1})), \quad (40)$$

where

$$a_{n,k+1} = \begin{cases} k^{\alpha+1} - (k-\alpha)(k+1)^\alpha & \text{if } j = 0, \\ (k-n+2)^{\alpha+1} + (k-n)^{\alpha+1} - 2(k-n+1)^{\alpha+1} & \text{if } 1 \leq n < k. \end{cases} \quad (41)$$

The principal step of our algorithm is to exchange the quantity  $y(x_{k+1})$  appeared on the right hand side of Eq. (40) with the predictor value  $y^P(x_{k+1})$ . The predictor value  $y^P(x_{k+1})$  can be gained by implementing the Adams-Bashforth method to the integral Eq. (37). In this case, if we replace the function  $f(\sigma^{-1}(u), y(\sigma^{-1}(u)))$  by the quantity  $f(x_n, y(x_n))$  at each integral in Eq. (38), we obtain

$$y^P(x_{k+1}) \approx v(x_{k+1}) + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^k \int_{\sigma(x_n)}^{\sigma(x_{n+1})} (\sigma(x_{k+1}) - u)^{\alpha-1} f(x_n, y(x_n)) du. \quad (42)$$

That is

$$y^P(x_{k+1}) \approx v(x_{k+1}) + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{n=0}^k [(k+1-n)^\alpha - (k-n)^\alpha] f(x_n, y(x_n)). \quad (43)$$

Consequently, the universal P-C algorithm, for providing the numerical approximation  $y_{k+1} \approx y(x_{k+1})$ , is finally outlined by the rule

$$y_{k+1} = v(x_{k+1}) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{n=0}^k a_{n,k+1} f(x_n, y_n) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(x_{k+1}, y_{k+1}^p), \quad (44)$$

where  $y_n \approx y(x_n)$ ,  $n = 0, 1, \dots, k$ , and the predicted value  $y_{k+1}^p \approx y^P(x_{k+1})$  can be evaluated as expressed in Eq. (43) with the weights  $a_{n,k+1}$  being determined according to Eq. (41).

**Remark 5** One can simply observe that in the case of  $\sigma(x) = x$ , our P-C algorithm reduces to the P-C approach presented in [24] and in the case of  $\sigma(x) = x^\rho/\rho$ , where  $\rho > 0$ , our P-C algorithm reduces to the adaptive P-C algorithm presented in [33].

**Remark 6** Firstly, it is clear, from the P-C rule given in Eq. (44), that the approximation  $y_{k+1}$  is based on the complete data values  $(y_0, y_1, \dots, y_k)$ . This means that the generalized Caputo-type fractional derivative, introduced in Definition 3, holds memory effects. Secondly, in view of the results presented in

[35], we can notice that the universal P-C algorithm features are similar to those of the classic Adams-Bashforth-Moulton method and the P-C algorithm of [24]. Hence, our P-C algorithm behaves satisfactorily according to its numerical stability.

## 4 Applications

This section deals with generalized Caputo-type fractional differential equations that can be viewed as a fractional extension of some known equations. Further, it investigates the efficiency and performance of the universal P-C algorithm, introduced in the previous section, using some cases of the function  $\sigma(x)$ . Numerical solutions to the considered problems are provided by implementing the proposed P-C algorithm. The numerical simulation results are performed using the symbolic software Mathematica.

**Example 1.** Our first problem assumes the IVP

$${}^C D_{0+}^{\alpha, \sigma(x)} y(x) = 2y(x) - y^2(x) + 1, \quad y(0) = A, \quad x > 0, \quad (45)$$

where  ${}^C D_{0+}^{\alpha, \sigma(x)}$  is the generalized Caputo-type fractional derivative operator, introduced in Definition 3,  $A \in \mathbb{R}$ ,  $0 < \alpha \leq 1$  and  $\sigma(x) = x^\rho$  such that  $\rho > 0$ . To simulate numerically the IVP (45) over the interval  $[a, b] = [0, T]$ , on the basis of our P-C algorithm, we have  $x_0 = 0$ ,  $x_{k+1} = (x_k^\rho + h)^{1/\rho}$  ( $k = 0, 1, \dots, N-1$ ) and  $h = \frac{T^\rho}{N}$ , for some  $N \in \mathbb{N}$ . Hence, the approximation  $y_{k+1} \approx y(x_{k+1})$  can be described by the formula

$$y_{k+1} = A + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{n=0}^k a_{n, k+1} (2y_n - y_n^2 + 1) + \frac{h^\alpha}{\Gamma(\alpha + 2)} (2y_{k+1}^P - (y_{k+1}^P)^2 + 1), \quad (46)$$

with the weights  $a_{n, k+1}$  being determined according to Eq. (41), and

$$y_{k+1}^P = A + \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{n=0}^k [(k+1-n)^\alpha - (k-n)^\alpha] (2y_n - y_n^2 + 1). \quad (47)$$

Here, our P-C algorithm aims at searching numerical solutions to the IVP (45) at the nodes  $\{(kh)^{1/\rho}, k = 1, 2, \dots, N\}$ . Tables 1 and 2 show approximate solutions to the IVP (45), when  $A = 0$ , using our universal P-C algorithm for different values of  $\alpha$ ,  $\rho$  and  $N$  at  $t = 0.5$  and  $t = 2$ , respectively. In case of  $\alpha = 1$  and  $\rho = 1$ , the exact values of the solution to the IVP (45), when  $A = 0$ , at  $t = 0.5$  and  $t = 2$  are  $y(0.5) = 0.75601439$  and  $y(2) = 2.35777165$ , respectively. In this case, from the numerical values displayed in Tables 1 and 2, we can easily see that the numerical solutions provided using our algorithm are definitely highly compatible with the exact solution. For other cases, from the convergence of numerical values presented in Tables 1 and 2, we can realize the numerical stability characteristic of the proposed P-C algorithm. Certainly, the accuracy of the approximate solutions presented using our P-C algorithm can be enhanced when  $N$  becomes large.

In order to describe the dynamic behavior of the model given in Eq. (45) according to the parameters  $\alpha$  and  $\rho$  versus the variable  $x$ , we display in Fig. 1 numerical solutions of the IVP (45) with  $T = 2$  and  $N = 1000$  computed using our P-C algorithm, when  $A = 0$ , for some specific values of  $\alpha$  and  $\rho$ .

**Example 2.** Our second problem considers the IVP

$${}^C D_{0+}^{\alpha, \sigma(x)} y(x) + \beta y(x) + \gamma y^3(x) = 0, \quad y(0) = 1/2, \quad y'(0) = 0, \quad x > 0, \quad (48)$$

where  ${}^C D_{0+}^{\alpha, \sigma(x)}$  is the generalized Caputo-type fractional derivative operator, introduced in Definition 3,  $\beta, \gamma \in \mathbb{R}$ ,  $1 < \alpha \leq 2$  and  $\sigma(x) = \exp(\lambda x)$  such that  $\lambda > 0$ . To simulate numerically the IVP (48) over the interval  $[a, b] = [0, T]$ , according to our P-C algorithm, we have  $x_0 = 0$ ,  $x_{k+1} = \frac{1}{\lambda} \ln(\exp(\lambda x_k) + h)$

( $k = 0, 1, \dots, N - 1$ ) and  $h = \frac{\exp(\lambda T) - 1}{N}$ , for some  $N \in \mathbb{N}$ . Hence, the approximation  $y_{k+1} \approx y(x_{k+1})$  can be described by the formula

$$y_{k+1} = \frac{1}{2} - \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^k a_{j,k+1} (\beta y_j + \gamma y_j^3) - \frac{h^\alpha}{\Gamma(\alpha + 2)} (\beta y_{k+1}^P + \gamma (y_{k+1}^P)^3), \quad (49)$$

with the weights  $a_{n,k+1}$  being determined according to Eq. (41), and

$$y_{k+1}^P = \frac{1}{2} - \frac{h^\alpha}{\Gamma(\alpha + 1)} \sum_{j=0}^k b_{j,k+1} (\beta y_j + \gamma y_j^3). \quad (50)$$

Now, our P-C algorithm aims at searching numerical solutions to the IVP (48) at the nodes  $\{\frac{1}{\lambda} \ln(kh + 1), k = 1, 2, \dots, N\}$ . Tables 3 and 4 show approximate solutions to the IVP (48), when  $\beta = 2$  and  $\gamma = 1$ , using our universal P-C algorithm for different values of  $\alpha$ ,  $\lambda$  and  $N$  at  $t = 0.5$  and  $t = 2$ , respectively. As concluded in the previous example, from the convergence of numerical values presented in Tables 3 and 4, we can observe that our P-C algorithm behaves acceptably with respect to its numerical stability.

The dynamic behavior of the model given in Eq. (48) according to the parameters  $\alpha$  and  $\lambda$  versus the variable  $x$  is described in Fig. 2. This figure displays numerical solutions of the IVP (48) with  $T = 2.5$  and  $N = 1000$  computed using our P-C algorithm, when  $\beta = 2$  and  $\gamma = 1$ , for some specific values of  $\alpha$  and  $\lambda$ . It is obvious, from Fig. 2, that the obtained solutions when  $\alpha < 2$  oscillate with decreasing amplitude to zero and the behavior of the model given in Eq. (48) is very similar to the behavior of the damped harmonic oscillator.

## 5 Concluding remarks

This paper introduces a survey of formations of generalized fractional integrals and derivatives with respect to another function. A novel P-C algorithm to produce numerical solutions to IVPs with generalized Caputo-type fractional differential equations is developed. The proposed algorithm is universal in the sense that it works with any possible choice of the function  $\sigma(x)$  under the prescribed conditions. The obtained attractive numerical results recommend that the proposed P-C algorithm works successfully in handling IVPs with generalized Caputo-type fractional derivatives conveniently, directly and accurately. Because we have many possible options for the function  $\sigma(x)$  within the proposed algorithm, we hope to draw the attention of researchers to consider and employ the proposed P-C algorithm in providing approximate solutions to many nonlinear models including generalized fractional derivatives.

## 6 Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this article.

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Table 1. Numerical solutions to the IVP (45), when  $A = 0$ , at  $t = 0.5$ .

$N$	$\alpha = 1, \rho = 1$	$\alpha = 0.95, \rho = 0.75$	$\alpha = 0.95, \rho = 1.2$	$\alpha = 0.9, \rho = 0.75$	$\alpha = 0.9, \rho = 1.2$
10	0.75523284	1.01733981	0.69730404	1.09508694	0.76853201
20	0.75581919	1.01843651	0.69798213	1.09651085	0.76951431
40	0.75596563	1.01872276	0.69816124	1.09689801	0.76978433
80	0.75600221	1.01879757	0.69820845	1.09700316	0.76985833
160	0.75601135	1.01881714	0.69822088	1.09703170	0.76987857
320	0.75601363	1.01882225	0.69822415	1.09703944	0.76988410
640	0.75601420	1.01882359	0.69822500	1.09704153	0.76988561
1280	0.75601435	1.01882394	0.69822523	1.09704210	0.76988602
2560	0.75601438	1.01882403	0.69822529	1.09704225	0.76988613

Table 2. Numerical solutions to the IVP (45), when  $A = 0$ , at  $t = 2$ .

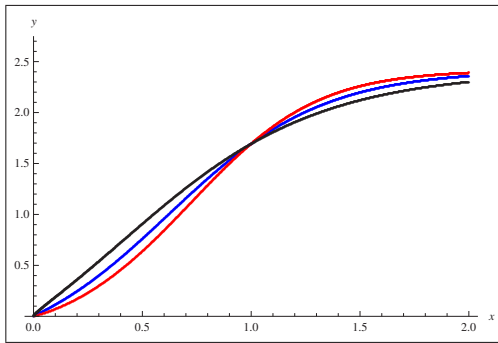
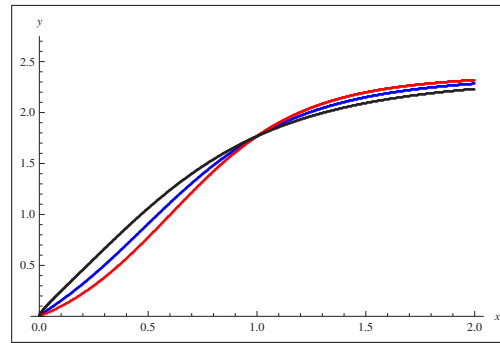
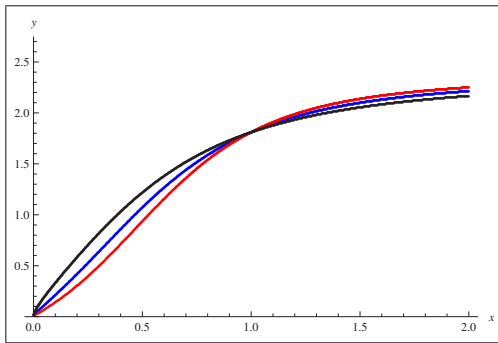
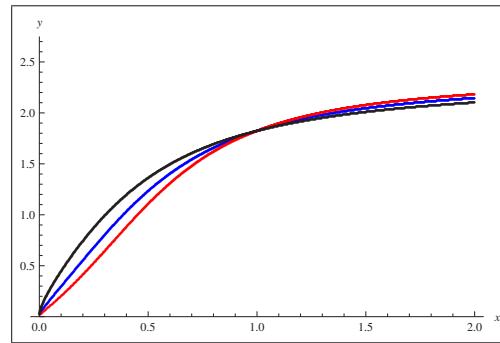
$N$	$\alpha = 1, \rho = 1$	$\alpha = 0.95, \rho = 0.75$	$\alpha = 0.95, \rho = 1.2$	$\alpha = 0.9, \rho = 0.75$	$\alpha = 0.9, \rho = 1.2$
10	2.34549527	2.23643698	2.34048739	2.20387374	2.30441202
20	2.35530727	2.24463114	2.35122618	2.21233698	2.31593360
40	2.35721255	2.24647572	2.35307547	2.21428310	2.31797138
80	2.35763805	2.24692294	2.35348404	2.21477003	2.31843796
160	2.35773897	2.24703533	2.35358296	2.21489673	2.31855511
320	2.35776357	2.24706407	2.35360784	2.21493033	2.31858567
640	2.35776964	2.24707149	2.35361422	2.21493932	2.31859379
1280	2.35777115	2.24707341	2.35361586	2.21494173	2.31859596
2560	2.35777153	2.24707390	2.35361629	2.21494238	2.31859655

Table 3. Numerical solutions to the IVP (48), when  $\beta = 2$  and  $\gamma = 1$ , at  $t = 0.5$ .

$N$	$\alpha = 2, \lambda = 1$	$\alpha = 1.9, \lambda = 0.75$	$\alpha = 1.9, \lambda = 0.5$	$\alpha = 1.8, \lambda = 0.75$	$\alpha = 1.8, \lambda = 0.5$
10	0.28453233	0.37047360	0.44513328	0.35020543	0.43284561
20	0.28434407	0.37037456	0.44510308	0.35007096	0.43279921
40	0.28429848	0.37034943	0.44509521	0.35003568	0.43278650
80	0.28428728	0.37034298	0.44509315	0.35002627	0.43278299
160	0.28428450	0.37034132	0.44509261	0.35002374	0.43278202
320	0.28428381	0.37034089	0.44509246	0.35002304	0.43278174
640	0.28428364	0.37034077	0.44509243	0.35002286	0.43278167
1280	0.28428360	0.37034075	0.44509242	0.35002280	0.43278165
2560	0.28428359	0.37034074	0.44509241	0.35002279	0.43278164

Table 4. Numerical solutions to the IVP (48), when  $\beta = 2$  and  $\gamma = 1$ , at  $t = 2$ .

$N$	$\alpha = 2, \lambda = 1$	$\alpha = 1.9, \lambda = 0.75$	$\alpha = 1.9, \lambda = 0.5$	$\alpha = 1.8, \lambda = 0.75$	$\alpha = 1.8, \lambda = 0.5$
10	-0.48639354	0.20438941	-0.36801975	0.14602780	-0.31639411
20	-0.54001454	0.16273870	-0.36434938	0.10892030	-0.31194764
40	-0.50634958	0.15488230	-0.36346293	0.10121662	-0.31081956
80	-0.50100509	0.15318882	-0.36323467	0.09936997	-0.31051265
160	-0.50009719	0.15278468	-0.36317420	0.09888518	-0.31042653
320	-0.49991416	0.15268158	-0.36315798	0.09875097	-0.31040203
640	-0.49987307	0.15265427	-0.36315361	0.09871278	-0.31039502
1280	-0.49986332	0.15264691	-0.36315242	0.09870178	-0.31039300
2560	-0.49986095	0.15264491	-0.36315210	0.09869859	-0.31039242

(a)  $\alpha = 1$ (b)  $\alpha = 0.9$ (c)  $\alpha = 0.8$ (d)  $\alpha = 0.7$ Figure 1: Plots of numerical solutions to the IVP (45), when  $T = 2$  and  $N = 1000$ : (Black)  $\rho = 0.8$ ; (Blue)  $\rho = 1$ ; (Red)  $\rho = 1.2$ .

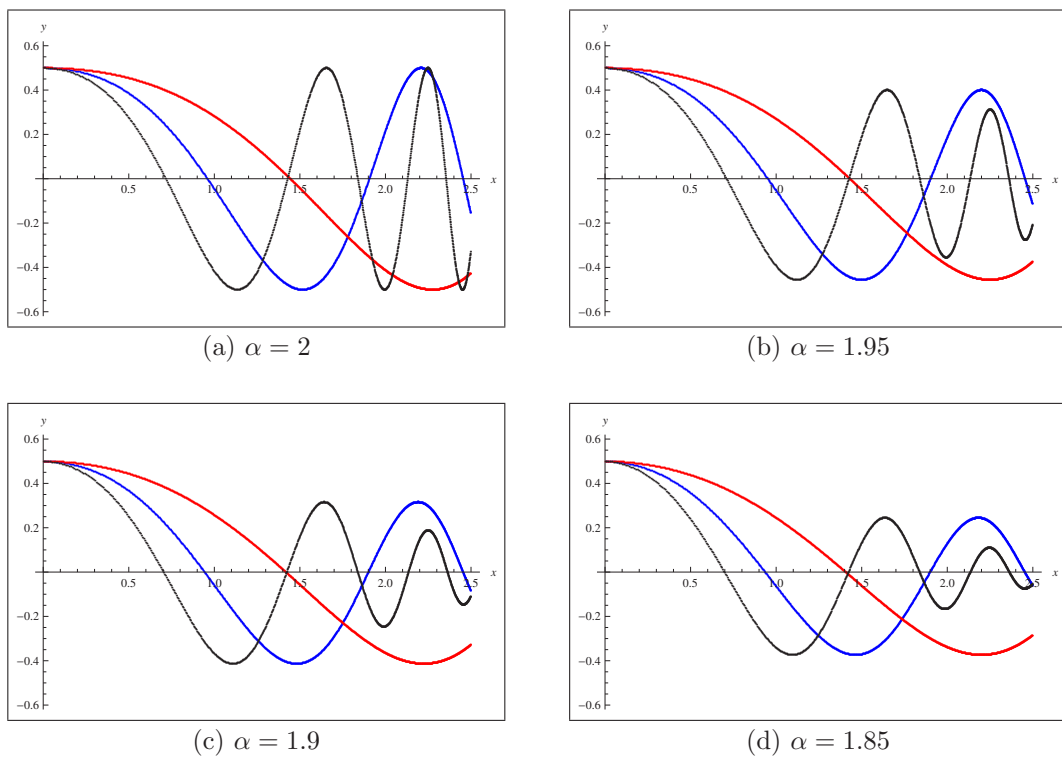
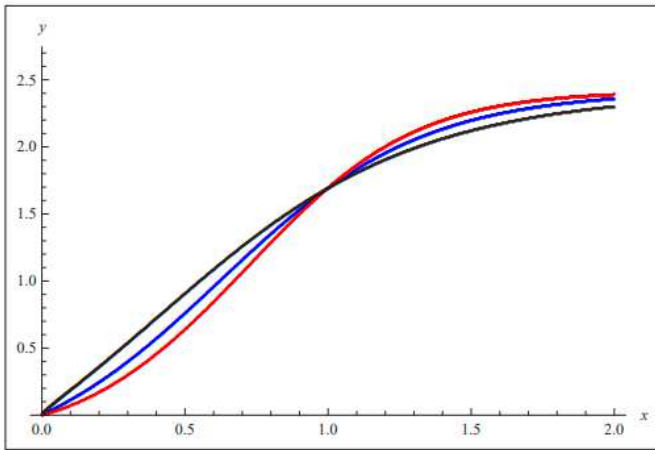


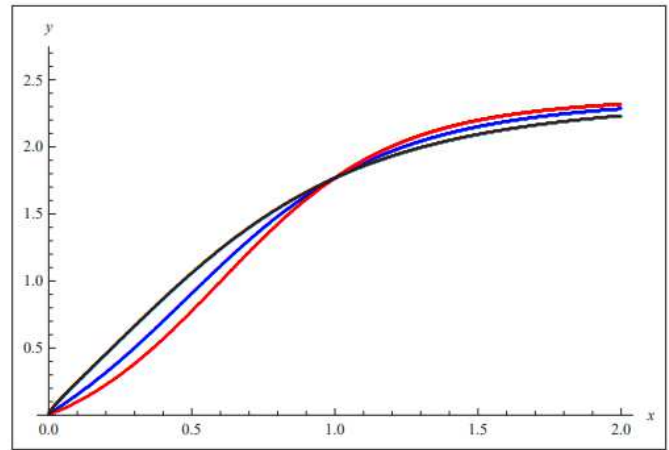
Figure 2: Plots of numerical solutions to the IVP (48), when  $T = 2.5$  and  $N = 1000$ : (*Black*)  $\lambda = 1$ ; (*Blue*)  $\lambda = 0.75$ ; (*Red*)  $\lambda = 0.5$ .



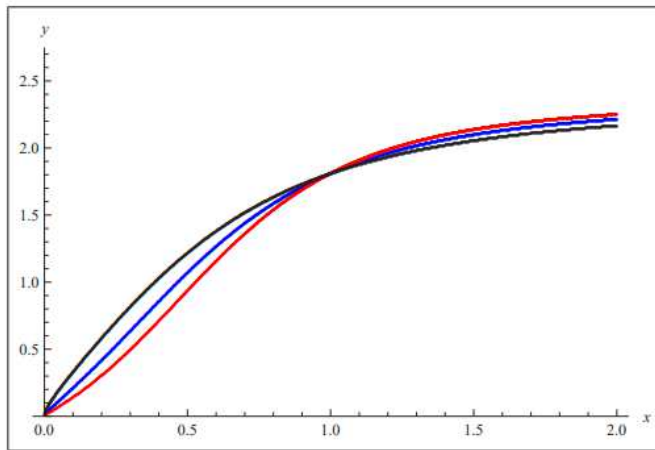
# Figures



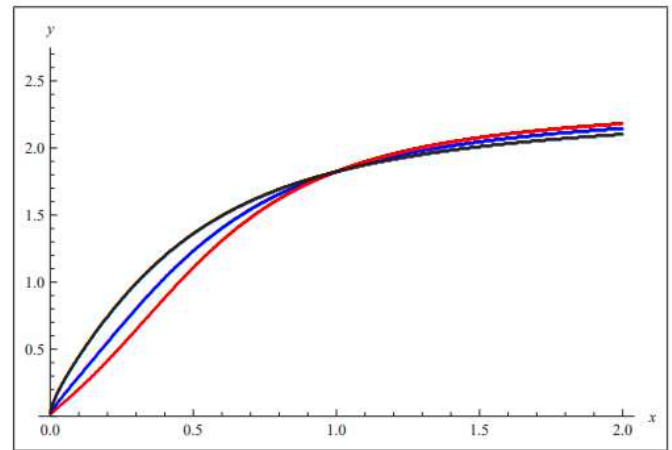
(a)  $\alpha = 1$



(b)  $\alpha = 0.9$



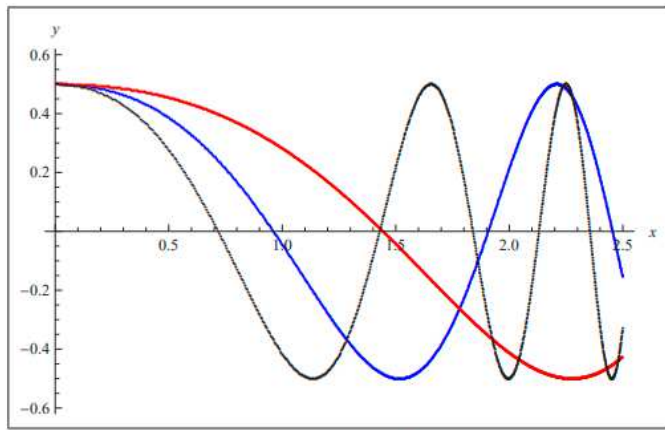
(c)  $\alpha = 0.8$



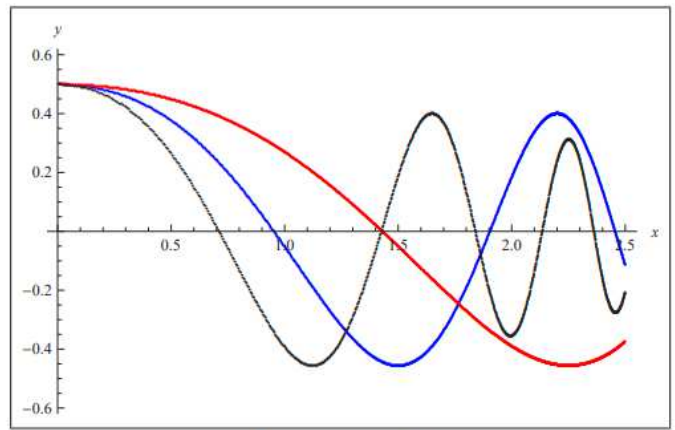
(d)  $\alpha = 0.7$

**Figure 1**

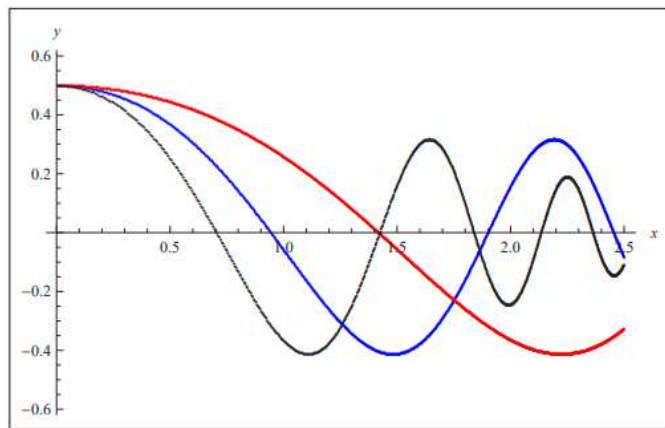
Please see the Manuscript PDF file for the complete figure caption



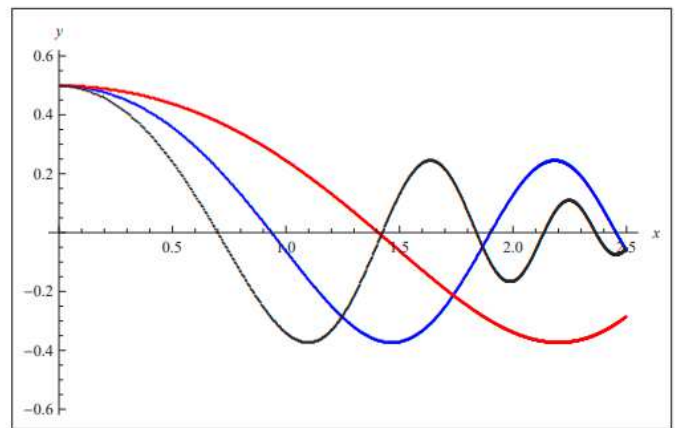
(a)  $\alpha = 2$



(b)  $\alpha = 1.95$



(c)  $\alpha = 1.9$



(d)  $\alpha = 1.85$

**Figure 2**

Please see the Manuscript PDF file for the complete figure caption