A new perspective of countable and uncountable infinite sets on Georg Cantor's definition in set theory

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Abstract

Georg Cantor defined countable and uncountable sets for infinite sets. Natural number set is defined as a countable set, and real number set is proven as an uncountable set by Cantor's diagonal method. However, in this paper, natural number set will be proven as an uncountable set using Cantor's diagonal method, and real number set will be proven as a countable set by Cantor's definition. The process of argumentation provides us new perspectives to consider about the size of infinite sets.

Introduction

"Infinite" is an unclear concept, and many scholars try to describe or define it. In set theory, the sets with infinite members are concerned and debated. Georg Cantor defined countable and uncountable sets for infinite sets. The main concepts of Cantor's definition for countable sets are:

Definition 1: If the infinite members in a set can be listed by order, then the infinite set is a countable set.

Definition 2: According to the definition given above, the natural number set is a countable set.

Definition 3: For any infinite set X, X is called countable if there is a bijection between X and the natural number set, or X is called uncountable.

The concepts are approved and applied by most scholars up to now. Under the concept and definition, Georg Cantor declared that it is impossible to construct a bijection between natural number set and real number set. Furthermore, real number set is proven as an uncountable set by Cantor's diagonal method (1, 2). The proof can be briefly described as follows:

StepA1: Assuming that real number set is a countable infinite set.

StepA2: Under the assumption, the members in real number set can be listed by order. Any part of the members in real number set can be listed by order. Real numbers between 0 and 1 can be listed by order.

StepA3: Each real number can be represented by infinite decimal. For example:

0.1 = 0.100000000……

0.25 = 0.250000000……

0.597 = 0.597000000……

StepA4: Each real number between 0 and 1 can be represented by infinite decimal and can be listed. Mark them as s1, s2, s3, ………, sn, ……

s1 = 0.**1**00000000……

s2 = 0.3**3**3333333……

s3 = 0.59**7**570255……

s4 = 0.627**8**98900……

s5 = 0.2555**5**5555……

s6 = 0.77777**7**777……

s7 = 0.101010**1**01……

s8 = 0.9766625**5**5……

s9 = 0.01010101**0**……

⋮

⋮

⋮

StepA5: When all real numbers between 0 and 1 are listed. We can construct a number S and let S differs from sn in its nth digit (Notice bold digits marked in StepA4):

1st digit of S cannot be 1

2nd digit of S cannot be 3

3rd digit of S cannot be 7

4th digit of S cannot be 8

⋮

⋮

StepA6: S is a real number. S is not any one real number listed above, since their nth digits differ.

StepA7: All real numbers between 0 and 1 are listed, so S should has been listed (Step A5). However, S is not any one real number in the list (Step A6). There is a contradiction under the assumption at Step A1.

StepA8: The assumption "real number set is a countable infinite set" is wrong. Thus, real number set is an uncountable infinite set.

However, natural number set will be proven as an uncountable set using Cantor's diagonal method, and real number set will be proven as a countable set by Cantor's definition in this paper. The results of argumentation will subversively change the concept of "infinite" in set theory, and the process of argumentation will provide us new perspectives to consider about the size of infinite sets.

Methods

Argument 1: Natural number set is uncountable

Natural number set could be proven as an uncountable set by the same demonstration program:

StepB1: We know that nature number could be represented as different formats:

1 = 01 = 001 = 0001 = ……01

2 = 02 = 002 = 0002 = ……02

3 = 03 = 002 = 0003 = ……03

:

:

StepB2: Rewrite all natural numbers by opposite left-right direction. For example:

50792 could be rewrite as 29705

However, 50792 = ……0000050792

So it could be rewrite as 2970500000……

StepB3: After rewriting, all natural numbers can be listed:

N1 = 100000000000000……

N2 = 200000000000000……

N3 = 300000000000000……

:

:

N50792 = 297050000000000……

:

:

StepB4: A rewritten natural number S can be constructed as:

S differs from Nn in its nth digit

StepB5: By the construction, S differs from each Nn, since their nth digits differ. According to the logic of Cantor's diagonal method, natural number set has been proven as an uncountable set.

Argument 2: Real number set is countable

Georg Cantor declared that it is impossible to construct a bijection between natural number set and real number set. However, I will construct a bijection between natural number set and real number set by following steps:

StepC1 ~ StepC3: Rewrite all natural numbers as the same method described at StepB1 ~ StepB3

StepC4: Rewrite all real numbers as the sequence:

real number		rewritten real numbers
1st digit on the left of decimal point	→	1st digit
1st digit on the right of decimal point	→	2nd digit
2nd digit on the left of decimal point	→	3rd digit
2nd digit on the right of decimal point	→	4th digit
:		
:		

For example:

…….00097531.24680000……… → 1234567890000000……….

…….00010267.57639000……… → 7567260319000000………..

Then we get a bijection between positive real number set and natural number set. Consider of positive number, negative number and zero, we get a bijection between real number set and integer set.

StepC5: According to the countable set theory, there is a bijection between integer and natural number set. So there is a bijection between real number set and natural number set. According to the definition of countable infinite set, real number set is countable.

Moreover, it is easy to see that there is a bijection between complex number set and the natural number set by similar demonstration process. Each complex number could be written as x + yi, and both x and y are real numbers. We could rewrite complex number by following rules:

x's 1st digit on the left of decimal point → 1st digit

y’s 1st digit on the left of decimal point → 2nd digit

x’s 1st digit on the right of decimal point → 3rd digit

y’s 1st digit on the right of decimal point → 4th digit

x’s 2nd digit on the left of decimal point → 5th digit

y’s 2nd digit on the left of decimal point → 6th digit

x’s 2nd digit on the right of decimal point → 7th digit

y’s 2nd digit on the right of decimal point → 8th digit

:

:

For example:

…051.370… + …062.480…I → 123456780000……….

Then we can finally get a bijection between complex number set and natural number set.

Discussion

Finding appropriate rewrite rules for any target numbers, and we can get one-to-one correspondence with the natural number set by similar demonstration process. I propose that we always can find some rewrite rules to get a bijection between any two infinite set. In these demonstration processes, we find that it is not necessary to distinguish countable or uncountable for any infinite set. It is nonsense to compare sizes of infinite sets because their members are all infinite. The size is defined to describe finite numbers.

There is a new perspective of the contradiction in the Cantor’s diagonal method. If we can list all real numbers, then we cannot construct the real numbers (StepA4 ~ A5) S and let S differs from sn in its nth digit. If we can construct the S, then we didn’t list all real numbers. The contradiction at StepA7 makes Georg Cantor reject the assumption at StepA1, but it could make us reject the StepA4 ~ A5. The construction of S is after that all infinite numbers are listed. There should be a rational feature of infinite numbers: infinite numbers cannot be listed of all, or they are not infinite. In my opinion, the S constructed at StepA5 proves that StepA4 is impossible to complete, but not proves that the assumption at StepA1 is wrong.

Declarations

Competing Interests

The authors declare no competing interests.

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