SUPPLEMENTARY MATERIAL

Proof of Theorem 2:

(I)-(II) Straight forward.

(II) We have from Definition 9,

$\lambda(\tilde{y}_{\wp}^{(1)}(\mho)\,\tilde{\oplus}\,\tilde{y}_{\wp}^{(2)}(\mho))$

$$=\lambda\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda\left(\frac{1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\sum_{j=1}^{2}\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\right.$$

$$\left.\tilde{g}^{-1}\left(\left(1+\left\{\lambda\left(\frac{1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda\sum_{j=1}^{2}\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\lambda\sum_{j=1}^{2}\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

On the other hand, $(\lambda\tilde{y}_{\wp}^{(1)}(\mho))\,\tilde{\oplus}\,(\lambda\tilde{y}_{\wp}^{(2)}(\mho))$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda\left(\frac{\tilde{g}_{\xi}^{1}}{1-\tilde{g}_{\xi}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\lambda\left(\frac{1-\tilde{g}_{\vartheta}^{1}}{\tilde{g}_{\vartheta}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$\tilde{\oplus}\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda\left(\frac{\tilde{g}_{\xi}^{2}}{1-\tilde{g}_{\xi}^{2}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}), \tilde{g}^{-1}\left(\left(1+\left\{\lambda\left(\frac{1-\tilde{g}_{\vartheta}^{2}}{\tilde{g}_{\vartheta}^{2}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\lambda\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\lambda\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\right.$$

$$\left.\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\lambda\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\lambda\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda\sum_{j=1}^{2}\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}), \tilde{g}^{-1}\left(\left(1+\left\{\lambda\sum_{j=1}^{2}\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

Thus, $\lambda(\tilde{y}_{\wp}^{(1)}(\mho)\,\tilde{\oplus}\,\tilde{y}_{\wp}^{(2)}(\mho))=(\lambda\tilde{y}_{\wp}^{(2)}(\mho))\,\tilde{\oplus}\,(\lambda\tilde{y}_{\wp}^{(1)}(\mho))$.

(IV) Similar to (III).

(V) We have, $(\lambda_1+\lambda_2)\tilde{y}_{\wp}^{(1)}(\mho)$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{(\lambda_1+\lambda_2)\left(\frac{\tilde{g}_{\xi}^{1}}{1-\tilde{g}_{\xi}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}), \tilde{g}^{-1}\left(\left(1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\tilde{g}_{\vartheta}^{1}}{\tilde{g}_{\vartheta}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

On the other hand, $(\lambda_1\tilde{y}_{\wp}^{(1)}(\mho))\,\tilde{\oplus}\,(\lambda_2\tilde{y}_{\wp}^{(1)}(\mho))$

$$
=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda_1\left(\frac{\tilde{g}_\xi^1}{1-\tilde{g}_\xi^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\lambda_1\left(\frac{1-\tilde{g}_\vartheta^1}{\tilde{g}_\vartheta^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle
$$

$$
\tilde{\oplus}\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\lambda_2\left(\frac{\tilde{g}_\xi^1}{1-\tilde{g}_\xi^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\lambda_2\left(\frac{1-\tilde{g}_\vartheta^1}{\tilde{g}_\vartheta^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle
$$

$$
=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\lambda_j\left(\frac{\tilde{g}_\xi^1}{1-\tilde{g}_\xi^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\lambda_j\left(\frac{\tilde{g}_\xi^1}{1-\tilde{g}_\xi^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\right.
$$

$$
\left.\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\lambda_j\left(\frac{1-\tilde{g}_\vartheta^1}{\tilde{g}_\vartheta^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\lambda_j\left(\frac{1-\tilde{g}_\vartheta^1}{\tilde{g}_\vartheta^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle
$$

$$
=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{(\lambda_1+\lambda_2)\left(\frac{\tilde{g}_\xi^1}{1-\tilde{g}_\xi^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{(\lambda_1+\lambda_2)\left(\frac{1-\tilde{g}_\vartheta^1}{\tilde{g}_\vartheta^1}\right)^\phi\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle
$$

(VI) Similar to (V).

<u>Proof of Theorem 3:</u>

1st part easily follows from definition 10. To prove the rest part, let us use the principle of induction on '*n*'. For *n*=1, the result is obvious. By Definition 10, for *n*=2, we obtain,

$DPLDPWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho))$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\Theta_1\left(\frac{\tilde{g}_{\xi}^{1}}{1-\tilde{g}_{\xi}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\Theta_1\left(\frac{1-\tilde{g}_{\vartheta}^{1}}{\tilde{g}_{\vartheta}^{1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$\tilde{\oplus}\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\Theta_2\left(\frac{\tilde{g}_{\xi}^{2}}{1-\tilde{g}_{\xi}^{2}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\Theta_2\left(\frac{1-\tilde{g}_{\vartheta}^{2}}{\tilde{g}_{\vartheta}^{2}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\right.$$

$$\left.\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{2}\left(\frac{1-\left(1+\left\{\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{2}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{2}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

Thus, Eq. (10) holds good for n=2. Let us assume that Eq. (10) holds good for n=l. Then
$DPLDPWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),........,\tilde{y}_{\wp}^{(l)}(\mho))$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

Now, for n=l+1, $DPLDPWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),........,\tilde{y}_{\wp}^{(l+1)}(\mho))$

$$DPLDWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),........,\tilde{y}_{\wp}^{(l)}(\mho))\tilde{\oplus}(w_{l+1}\tilde{y}_{\wp}^{(l+1)}(\mho))$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$\tilde{\oplus}\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\Theta_{l+1}\left(\frac{\tilde{g}_{\xi}^{l+1}}{1-\tilde{g}_{\xi}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\Theta_{l+1}\left(\frac{1-\tilde{g}_{\vartheta}^{l+1}}{\tilde{g}_{\vartheta}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\left(\frac{1-\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}+\left(\frac{1-\left(1+\left\{\Theta_{l+1}\left(\frac{\tilde{g}_{\xi}^{l+1}}{1-\tilde{g}_{\xi}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\Theta_{l+1}\left(\frac{\tilde{g}_{\xi}^{l+1}}{1-\tilde{g}_{\xi}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\right.$$

$$\left.\tilde{g}^{-1}\left(\left(1+\left\{\left(\frac{1-\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\sum_{j=1}^{l}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}+\left(\frac{1-\left(1+\left\{\Theta_{l+1}\left(\frac{1-\tilde{g}_{\vartheta}^{l+1}}{\tilde{g}_{\vartheta}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}{\left(1+\left\{\Theta_{l+1}\left(\frac{1-\tilde{g}_{\vartheta}^{l+1}}{\tilde{g}_{\vartheta}^{l+1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{l+1}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{l+1}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

Thus, Eq. (10) also holds good for n=l+1. Hence, by principle of induction, Eq. (10) is true for any natural number n.

Proof of Theorem 4:

By Theorem 3,

$$DPLDPWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),........,\tilde{y}_{\wp}^{(n)}(\mho))$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-(\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}_{\xi}^{L}}{1-\tilde{g}_{\xi}^{L}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{L}}{\tilde{g}_{\vartheta}^{L}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\left\{\left(\frac{\tilde{g}_{\xi}^{L}}{1-\tilde{g}_{\xi}^{L}}\right)^{\phi}\sum_{j=1}^{n}\Theta_j\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\left\{\left(\frac{1-\tilde{g}_{\vartheta}^{L}}{\tilde{g}_{\vartheta}^{L}}\right)^{\phi}\sum_{j=1}^{n}\Theta_j\right\}^{\frac{1}{\phi}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\frac{\tilde{g}_{\xi}^{L}}{1-\tilde{g}_{\xi}^{L}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\frac{1-\tilde{g}_{\vartheta}^{L}}{\tilde{g}_{\vartheta}^{L}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}\left(1-\left(1+\frac{\tilde{g}_{\xi}^{L}}{1-\tilde{g}_{\xi}^{L}}\right)^{-1}\right)(\mho^{(u)}),\tilde{g}^{-1}\left(\left(1+\frac{1-\tilde{g}_{\vartheta}^{L}}{\tilde{g}_{\vartheta}^{L}}\right)^{-1}\right)(\mho^{(v)})\right\rangle$$

$$=\left\langle \tilde{g}^{-1}(\tilde{g}_{\xi}^{L})(\mho^{(u)}),\tilde{g}^{-1}(\tilde{g}_{\vartheta}^{L})(\mho^{(v)})\right\rangle=\left\langle \wp_{\xi^{L(u)}}(\mho^{(u)}),\wp_{\vartheta^{L(v)}}(\mho^{(v)})\right\rangle=\tilde{y}_{\wp}^{(L)}(\mho)$$

Proof of Theorem 5:

Given, $\wp_{\xi^{j(u)}}\le\wp'_{\xi^{j(u)}}$ and $\wp_{\vartheta^{j(v)}}\ge\wp'_{\vartheta^{j(v)}}$ $\forall j$. Since each of $\tilde{g}$ and $\tilde{g}^{-1}$ is an increasing function, we have,

$$\wp_{\xi^{j(u)}}\le\wp'_{\xi^{j(u)}}$$

$$\Rightarrow \tilde{g}_{\xi}^{j}\le\tilde{g}'^{j}_{\xi} \qquad \text{[where } \tilde{g}_{\xi}^{j}=\tilde{g}(\wp_{\xi^{j(u)}}),\tilde{g}'^{j}_{\xi}=\tilde{g}(\wp'_{\xi^{j(u)}})]$$

$$\Rightarrow \sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\le\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}'^{j}_{\xi}}{1-\tilde{g}'^{j}_{\xi}}\right)^{\phi}$$

$$\Rightarrow 1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\le 1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}'^{j}_{\xi}}{1-\tilde{g}'^{j}_{\xi}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}$$

$$\Rightarrow \tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}_{\xi}^{j}}{1-\tilde{g}_{\xi}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)\le\tilde{g}^{-1}\left(1-\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{\tilde{g}'^{j}_{\xi}}{1-\tilde{g}'^{j}_{\xi}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)$$

Similarly, we can get, $\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{1-\tilde{g}_{\vartheta}^{j}}{\tilde{g}_{\vartheta}^{j}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)\geq\tilde{g}^{-1}\left(\left(1+\left\{\sum_{j=1}^{n}\Theta_j\left(\frac{1-\tilde{g}'^{j}_{\vartheta}}{\tilde{g}'^{j}_{\vartheta}}\right)^{\phi}\right\}^{\frac{1}{\phi}}\right)^{-1}\right)$.

Now, by applying the formula (3), we have $S(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),.....,\tilde{y}_{\wp}^{(n)}(\mho))\leq S(\tilde{y}'^{(1)}_{\wp}(\mho),\tilde{y}'^{(2)}_{\wp}(\mho),.....,\tilde{y}'^{(n)}_{\wp}(\mho))$. Consequently, by definition (4), $DPLDPWAA(\tilde{y}_{\wp}^{(1)}(\mho),\tilde{y}_{\wp}^{(2)}(\mho),.....,\tilde{y}_{\wp}^{(n)}(\mho))\prec PLqROFDPWAA(\tilde{y}'^{(1)}_{\wp}(\mho),\tilde{y}'^{(2)}_{\wp}(\mho),.....,\tilde{y}'^{(n)}_{\wp}(\mho))$.