Global Fixed-Time Stabilization for Chained Nonholonomic Systems via Output Feedback Control

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Abstract This paper studies the fixed-time output feedback stabilization control problem for chained nonholonomic systems. By means of switching control and *bi*-limit homogeneous techniques, it is firstly constructed two fixed-time state feedback stabilizing controllers for the considered systems. Then, a new state observer with a formalized switching law is proposed to fixed-time estimate system states, where high-order terms are applied to get uniform convergence regardless of initial conditions and low-order terms are aimed to the exact convergence in finite time. Finally, based on *bi*-limit homogeneous technique and Lyapunov stability theorem, fixed-time output feedback stabilizing controllers, one of which is discontinuous with a specific switching control law and the other is continuous, are constructed and the fixed-time output feedback stabilization of the considered systems is thus guaranteed. An example is presented to show the feasibility of the proposed fixed-time output feedback stabilization control strategy.

Keywords Nonholonomic systems · Fixed-time stabilization · Output-feedback control · *bi*-limit homogeneous technique · Switching control

1 Introduction

Fixed-time control, whose stability guaranteed the settling time irrelevant to initial conditions firstly introduced in [24], has drawn an increasing attention in recent years [18, 34]. Fixed-time stabilization problems have been studied in many control systems, see [11, 13, 26, 31, 33] and the references therein. For example, through introducing implicit Lyapunov theorems for fixed-time stability analysis of nonlinear systems, fixed-time state feedback stabilizing controllers were designed in [26] for a chain of integrators. By backstepping method, [13] constructed decentralized fixed-time state feedback controllers to stabilize nonlinear interconnected systems. By means of recursive approach, a state feedback controller was proposed in [33] to study the fixed-time almost disturbance decoupling problem for a class of nonlinear systems. [11] investigated the global fixed-time state feedback stabilization of switched nonlinear systems with general powers. However, all the works mentioned above are based on designing state feedback controllers to achieve the fixed-time stabilization control.

Observer design, a classical control problem of control systems when part of states are available, has gotten a great variety of solutions [1,8,21,28]. As a straightforward idea, how to design fixed-time observers for control systems has also attracted extensive attention [20, 22, 25, 27]. Recently, a hybrid observer was proposed in [27] to fixed-time state estimation for a kind of linear systems. The *bi*-limit homogeneous technique introduced in [2] has shown that a fixed-time stable system is asymptotically stable firstly, and is also homogeneous with negative degree in 0-limit and homogeneous with positive degree in ∞-limit. By means of this technique, [22] studied fixed-time observer design problems for many kinds of nonlinear systems including linearizable systems up to input-output injection and uniformly observable systems. Moreover, through introducing implicit Lyapunov function approach, nonlinear dynamic observers were proposed in [20] to deal with fixed-time observation of a class of linear systems. Through proposing distributed observers, [7] investigated the distributed fixed-time consensus problem for

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heterogeneous multi-agent systems. However, the aforementioned results on fixed-time observer design have great challenges in being applied to construct fixed-time output feedback stabilizing controllers of nonlinear systems. To our knowledge, few works have achieved the fixed-time output feedback stabilization of control systems based on designing fixed-time observer. By means of *bi*-limit homogeneous technique, [29] proposed a fixed-time observer to achieve the fixed-time output feedback stabilization of double integrator systems. [19] studied the global fixed-time output feedback stabilization and estimation of a chain of integrators.

On the other hand, chained nonholonomic systems are one of the most fundamental systems in control theory and have many applications in practice, such as wheeled mobile robots and the knife edge [17, 23]. And for fixed-time control design, by means of switching control technique, [30] studied the fixed-time state feedback stabilization problem for chained nonholonomic systems. [5] dealt with the fixed-time state feedback stabilization problem through sliding mode control for chained nonholonomic systems. Through proposing distributed observers, the consensus tracking problem for multi-agent systems with chained nonholonomic dynamics was studied in [23]. However, no work has been done for the fixed-time output feedback stabilization control of chained nonholonomic systems, and partly inspired by [10, 23, 29], this paper focuses on this problem by means of switching control and *bi*-limit homogeneous technique.

The main contributions of this paper are as follows. Firstly, by synthesizing switching control and *bi*-limit homogeneous techniques, the fixed-time output feedback stabilization problem of chained nonholonomic systems is studied for the first time. Secondly, a unified framework for constructing two coupled fixed-time output feedback stabilizing controllers is proposed. In detail, apparently different from [29] only designing one controller, two coupled fixed-time output feedback stabilizing controllers, one of which is discontinuous combining with a switching law, and the other is continuous, are constructed simultaneously, which is more challenging. Moreover, the switching control law has a more complex structure and requires more sophisticated design skills than that of in [10]. Thirdly, by the aid of the discontinuous controller and the designed switching control law, a fixed-time observer is proposed to estimate system states in a fixed time, and then based on this observer, the control objective is thus achievement.

The rest of this paper is organized as follows. Problem preliminaries are given in Section 2. Main theoretical results on constructing a fixed-time observer and fixed-time output feedback stabilizing controllers are stated in Section 3. Convincing simulation is shown in Section 4. Finally, Section 5 concludes this paper.

2 Preliminaries

2.1 Notations

Throughout the paper, the argument of the functions will be omitted or simplified whenever no confusion arises from the context. For example, we denote $x(t)$, $\phi(x):\ \mathbb{R}^n \to \mathbb{R}$ and $\Phi(x):\ \mathbb{R}^n \to \mathbb{R}^n$ by x, $\phi(x)$ and $\Phi(x)$, respectively. $\mathbb{R}$ is the set of real numbers. For any given non-negative real number ϱ, $s \mapsto \lceil s \rfloor^{\varrho}$ is a function defined as $\lceil s \rfloor^{\varrho} = |s|^{\varrho}\mathrm{sign}(s)$, $\forall s \in \mathbb{R}$, where $\mathrm{sign}(s)$ denotes the standard sign function ([4,6]). From the definition, $\frac{d\lceil s \rfloor^{\varrho}}{ds} = \varrho|s|^{\varrho-1}$, $\lceil s \rfloor^0 = \mathrm{sign}(s)$, $\lceil s \rfloor = s$ and $\lceil s \rfloor^2 = s|s|$ hold.

2.2 Definitions and lemmas

Consider the nonlinear system

$$\dot{x}(t) = \Phi(x(t)), \quad t \geq t_0, \quad x(t_0) = x_0, \tag{1}$$

with $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ being system states and $\Phi(x)$ denoting a possibly discontinuous vector field. If $\Phi(x)$ is discontinuous, the solution of (1) is understood in the sense of Filippov ([9]), and in this paper, the origin is assumed as an equilibrium point of system (1).

Let $r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n$ be a weight vector with $r_i > 0,\ i = 1, \ldots, n$ and $\Lambda_\lambda^r(x) = (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)$ be a dilation mapping with $\lambda > 0$.

Definition 1 ([3]) A function $\phi(x)$ is r-homogeneous with degree $k \in \mathbb{R}$ if $\phi(\Lambda_\lambda^r(x)) = \lambda^k\phi(x)$, $\forall x \in \mathbb{R}^n$, $\forall \lambda > 0$. Similarly, a vector field $\Phi(x)$ is r-homogeneous with degree k if its component $\Phi_i(x)$ is r-homogeneous of degree $k + r_i,\ i = 1, \ldots, n$.

Definition 2 ([2]) A function $\phi(x)$ is homogeneous in the p-limit ($p = 0$ or ∞) with (r_p, k_p, ϕ_p), in which $r_p = (r_{p,1}, \ldots, r_{p,n})^T \in \mathbb{R}^n$ is the weight vector, k_p is the degree, and ϕ_p is the approximating function. If $\phi(x)$ is continuous, ϕ_p is continuous and not identically zero, and hence, for each compact set $C \in \mathbb{R}^n\backslash\{0\}$, $\lim_{\lambda\to p}\max_{x\in C}|\lambda^{-k_p}\phi(\Lambda_\lambda^{r_p}(x)) - \phi_p(x)| = 0$ holds.

Definition 3 ([2]) A vector field $\Phi(x)$ is homogeneous in the p-limit ($p = 0$ or ∞) with (r_p, k_p, Φ_p), in which $r_p = (r_{p,1}, \ldots, r_{p,n})^T \in \mathbb{R}^n$ is the weight vector, k_p is the degree and Φ_p is the approximating vector field, if, for each i, $k_p + r_{p,i} > 0$ and $\Phi_i(x)$ is homogeneous in the p-limit with $(r_p, k_p + r_{p,i}, \Phi_{p,i})$.

Definition 4 ([2]) A function (or vector field) is homogeneous in the *bi*-limit if it is homogeneous in the 0-limit and homogeneous in the ∞-limit simultaneously.

Definition 5 ([16]) The origin of system (1) is globally finite-time stable if it is globally asymptotically stable and there exists $T(x_0) > 0,\ \forall x_0 \in \mathbb{R}^n$, such that the trajectory stabilizes at the origin in $T(x_0)$.

Definition 6 ([24]) The origin of system (1) is fixed-time stable if it is globally finite-time stable and there is a constant $T_{\max} > 0$, such that $T(x_0) \leq T_{\max},\ \forall x_0 \in \mathbb{R}^n$.

Lemma 1 *([2]) For system (1), suppose that $\Phi(x)$ is a homogeneous vector field in the bi-limit with (r_0, k_0, Φ_0) and $(r_\infty, k_\infty, \Phi_\infty)$. If the origins of systems $\dot{x} = \Phi(x)$, $\dot{x}_0 = \Phi_0(x)$, $\dot{x}_\infty = \Phi_\infty(x)$ are globally asymptotically stable, then we have*

1) The origin of system (1) is fixed-time stable as long as $k_\infty > 0 > k_0$ holds;
2) Let d_{W_0} and d_{W_∞} be real numbers satisfying $d_{W_0} > \max_{1\leq i\leq n} r_{0,i}$ and $d_{W_\infty} > \max_{1\leq i\leq n} r_{\infty,i}$. There exists a continuous, positive definite and proper function $W(x)$ such that $\frac{\partial W}{\partial x_i}$ is homogeneous in the bi-limit with $\left(r_0, d_{W_0} - r_{0,i}, \frac{\partial W_0}{\partial x_i}\right)$ and $\left(r_\infty, d_{W_\infty} - r_{\infty,i}, \frac{\partial W_\infty}{\partial x_i}\right)$ and $\frac{\partial W}{\partial x}\Phi(x)$ is negative definite.

3 Main Results

In this section, we first explicitly construct fixed-time state feedback stabilizing controllers by *bi*-limit homogeneous technique. Then, we propose an fixed-time state observer. Finally, the unmeasurable states in state feedback controllers is replaced with the estimate recovered from the observer to fixed-time stabilize the considered systems.

3.1 Fixed-time state feedback control

Throughout this paper, we consider a unicycle-type mobile robot given in Fig 1, whose dynamic equations is

$$\dot{x}_l = v\cos\theta,\quad \dot{y}_l = v\sin\theta,\quad \dot{\theta} = w, \tag{2}$$

with (x_l, y_l) denoting the position of the center of mass, θ being the heading angle, and v and w standing for the forward velocity and the angular velocity of the robot, respectively. Similar to [10], introducing the state transformation

$$x_0 = x_l,\quad x_1 = y_l,\quad x_2 = \tan\theta,$$
$$u_0 = v\cos\theta,\quad u_1 = w\sec^2\theta,$$

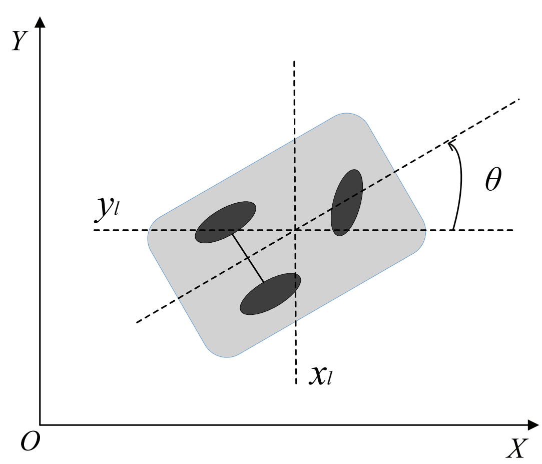

Fig. 1 The planar graph of a unicycle-type mobile robot.

system (2) is converted into the chained form

$$\dot{x}_0 = u_0,\quad \dot{x}_1 = u_0 x_2,\quad \dot{x}_2 = u_1, \tag{3}$$

with $\left(x_0, x^T\right)^T = (x_0, x_1, x_2)^T \in \mathbb{R}^3$ being system states, and $u_0,\ u_1 \in \mathbb{R}$, $y = (x_0, x_1) \in \mathbb{R}^2$ being inputs and outputs, respectively. Without loss of generality, the initial conditions not equal to zero can be viewed as $x_0(0)$ and $x(0)$.

This paper is aimed to propose fixed-time output feedback stabilizing controllers of the form

$$\begin{cases} \dot{z} = \Theta_o(y, z),\quad z \in \mathbb{R}^2, \\ u_0 = \Theta_1(x_0, z), \\ u_1 = \Theta_2(y, z), \end{cases} \tag{4}$$

such that system (3) under (4) is fixed-time stable. It should be noted that two output feedback controllers need to be constructed simultaneously, which brings challenges for choosing u_0 to fixed-time stabilize x_0-subsystem and u_1 to fixed-time stabilize x-subsystem, respectively.

Firstly, we formalize the form of the state feedback controller

$$u_0 = \mathrm{sign}(x_0(0)\|x\|)c + (1 - \mathrm{sign}(\|x\|))\varpi_2(x_0) \tag{5}$$

with c being a small positive constant and $\varpi_2(x_0)$ being continuous function determined later.

Remark 1 It should be noted that the initial condition $x(0)$ of x-subsystem is not equal to zero, which indicates switching controller $u_0 = \mathrm{sign}(x_0(0)\|x\|)c$ being implemented until $\|x(t)\| = 0,\ t > 0$, that is, before achieving the fixed-time stabilization of x-subsystem, u_0 is a constant controller same as the sign of the initial value $x_0(0)$. Furthermore, after achieving the fixed-time stabilization of x-subsystem, switching controller $u_0 = \varpi_2(x_0)$ is used to fixed-time stabilize x_0-subsystem.

Then, we reconsider x-subsystem

$$\dot{x}_1 = u_0 x_2, \quad \dot{x}_2 = u_1 \tag{6}$$

based on switching controller (5).

To achieve the fixed-time stabilization of system (6), state feedback controller u_1 is constructed as

$$\begin{aligned} u_1(x) = &- \left(k_{11}\lceil x_1\rfloor^{\gamma_1} + k_{12}\lceil x_1\rfloor + k_{13}\lceil x_1\rfloor^{\gamma_1'}\right) \\ &- \left(k_{21}\lceil x_2\rfloor^{\gamma_2} + k_{22}\lceil x_2\rfloor + k_{23}\lceil x_2\rfloor^{\gamma_2'}\right), \end{aligned} \tag{7}$$

where parameters $k_{ij} > 0,\ i = 1, 2,\ j = 1, 2, 3$ and $\gamma_i,\ \gamma_i',\ i = 1, 2$ are chosen as

$$\gamma_1 = \frac{\gamma}{2-\gamma},\ \gamma_2 = \gamma,\ \gamma_1' = \frac{4-3\gamma}{2-\gamma},\ \gamma_2' = \frac{4-3\gamma}{3-2\gamma} \tag{8}$$

with $\gamma \in (0, 1)$. Then, denoting $\Phi_s(x) = (u_0 x_2, u_1)^T$ the vector field of system (6) under (7) and defining

$$\Phi_{s0}(x) = (u_0 x_2, -k_{11}\lceil x_1\rfloor^{\gamma_1} - k_{21}\lceil x_2\rfloor^{\gamma_2})^T, \tag{9}$$

$$\Phi_{s\infty}(x) = \left(u_0 x_2, -k_{13}\lceil x_1\rfloor^{\gamma_1'} - k_{23}\lceil x_2\rfloor^{\gamma_2'}\right)^T, \tag{10}$$

it is shown from (8) that $\forall\gamma \in (0, 1),\ 0 < \gamma_i < 1 < \gamma_i',\ i = 1, 2$ hold, which makes $\Phi_{s0}(x)$ and $\Phi_{s\infty}(x)$ be regarded as approximating functions for $\Phi_s(x)$ in 0-limit and ∞-limit, respectively. Furthermore, choosing $r_{s0} = \left(\frac{2-\gamma}{1-\gamma}, \frac{1}{1-\gamma}\right)^T$, the vector field $\Phi_{s0}(x)$ is r_{s0}-homogeneous of degree $k_{s0} = -1$. Similarly, the vector field $\Phi_{s\infty}(x)$ is $r_{s\infty}$-homogeneous of degree $k_{s\infty} = 1$ with $r_{s\infty} = \left(\frac{2-\gamma}{1-\gamma}, \frac{3-2\gamma}{1-\gamma}\right)^T$. Therefore, system (6) under (7) with parameters (8) is homogeneous in the *bi*-limit with $(r_{s0}, k_{s0}, \Phi_{s0}(x))$ and $(r_{s\infty}, k_{s\infty}, \Phi_{s\infty}(x))$.

Proposition 1 *Under controller (5), system (6) can be fixed-time stabilized by controller (7) with parameters given in (8),* $\gamma \in (0, 1)$ *and* $k_{ij} > 0,\ i = 1, 2,\ j = 1, 2, 3$.

Proof To prove the fixed-time stabilization of system (6), choose the Lyapunov function

$$\begin{aligned} W_s = &2k_{11}\left(\gamma_1' + 1\right)|x_1|^{\gamma_1+1} + k_{12}(\gamma_1 + 1)\left(\gamma_1' + 1\right)|x_1|^2 \\ &+ 2k_{13}(\gamma_1 + 1)|x_1|^{\gamma_1'+1} + c(\gamma_1 + 1)\left(\gamma_1' + 1\right)|x_2|^2. \end{aligned} \tag{11}$$

Obviously, W_s is continuously differentiable, positive definite and radially unbounded. The derivative of W_s can be derived by

$$\begin{aligned} \dot{W}_s|_{(6)} = &k_{11}\Xi_1\lceil x_1\rfloor^{\gamma_1}\dot{x}_1 + k_{12}\Xi_1\lceil x_1\rfloor\dot{x}_1 \\ &+ k_{13}\Xi_1\lceil x_1\rfloor^{\gamma_1'}\dot{x}_1 + c\Xi_1\lceil x_2\rfloor\dot{x}_2 \end{aligned} \tag{12}$$

with $\Xi_1 = 2(\gamma_1 + 1)\left(\gamma_1' + 1\right)$. Substituting (6) and (7) into (12) yields

$$\begin{aligned} \dot{W}_s|_{(6)} = &\Xi_1 u_0 x_2\left(k_{11}\lceil x_1\rfloor^{\gamma_1} + k_{12}\lceil x_1\rfloor + k_{13}\lceil x_1\rfloor^{\gamma_1'}\right) \\ &- c\Xi_1\lceil x_2\rfloor\Big(k_{11}\lceil x_1\rfloor^{\gamma_1} + k_{12}\lceil x_1\rfloor + k_{13}\lceil x_1\rfloor^{\gamma_1'} \\ &+ k_{21}\lceil x_2\rfloor^{\gamma_2} + k_{22}\lceil x_2\rfloor + k_{23}\lceil x_2\rfloor^{\gamma_2'}\Big). \end{aligned} \tag{13}$$

Since $\lceil x_2\rfloor = x_2$ and $\lceil x_2\rfloor\lceil x_2\rfloor^{\varrho} = |x_2||x_2|^{\varrho},\ \forall\varrho \geq 0$ hold, (13) can be rewritten as

$$\begin{aligned} \dot{W}_s|_{(6)} = &\Xi_1 u_0|x_2|\left(k_{11}|x_1|^{\gamma_1} + k_{12}|x_1| + k_{13}|x_1|^{\gamma_1'}\right) \\ &- c\Xi_1|x_2|\left(k_{11}|x_1|^{\gamma_1} + k_{12}|x_1| + k_{13}|x_1|^{\gamma_1'}\right) \\ &- c\Xi_1|x_2|\left(k_{21}|x_2|^{\gamma_2} + k_{22}|x_2| + k_{23}|x_2|^{\gamma_2'}\right). \end{aligned}$$

From Remark 1, we have $u_0 = \text{sign}(x_0(0)\|x\|)c$, which further indicates

$$\dot{W}_s|_{(6)} \leq -c\Xi_1|x_2|\left(k_{21}|x_2|^{\gamma_2} + k_{22}|x_2| + k_{23}|x_2|^{\gamma_2'}\right). \tag{14}$$

It can be concluded that $\dot{W}_s$ in (14) is semi-negative definite. Moreover, $\dot{W}_s|_{(6)} = 0$ implies that $x_2 = 0$. By LaSalle's invariance principle ([15]), all the trajectories will converge to the invariant set $\Pi_1 = \{(x_1, x_2)|x_2 = 0\}$. Together with system (6) with controller (7), we can conclude $x_1 = 0$ in Π_1. Therefore, the closed-loop system is asymptotically stable.

For $\dot{x} = \Phi_{s0}(x)$ with $\Phi_{s0}(x)$ given in (9), if the Lyapunov function is chosen as

$$W_{s0} = k_{11}|x_1|^{\gamma_1+1} + 0.5c(\gamma_1 + 1)|x_2|^2,$$

and it can be easily calculated that

$$\dot{W}_{s0} \leq -ck_{21}(\gamma_1 + 1)|x_2|^{1+\gamma_2}.$$

Similarly, if we choose

$$W_{s\infty} = k_{13}|x_1|^{\gamma_1'+1} + 0.5c\left(\gamma_1' + 1\right)|x_2|^2$$

for $\dot{x} = \Phi_{s\infty}(x)$ with $\Phi_{s\infty}(x)$ given in (10), it can be seen that

$$\dot{W}_{s\infty} \leq -ck_{23}\left(\gamma_1' + 1\right)|x_2|^{\gamma_2'+1}$$

holds. By LaSalle's invariance principle, $\dot{x} = \Phi_{s0}(x)$ and $\dot{x} = \Phi_{s\infty}(x)$ are asymptotically stable. Thus, from 1) of Lemma 1, we have proven Proposition 1.

3.2 Fixed-time state observer design

In this section, for the case when only the system output signals x_0 and x_1 can be captured, the previous state controller (7) is not implementable since the unavailability of state information x_2. Thus, a new fixed-time observer for system (6) is proposed as follows

$$\begin{aligned}\dot{z}_1 &= z_2 + \bar{k}_{11}\lceil\zeta_1\rfloor^{\bar{\gamma}_1} + \bar{k}_{12}\lceil\zeta_1\rfloor + \bar{k}_{13}\lceil\zeta_1\rfloor^{\bar{\gamma}_1^{'}},\\ \dot{z}_2 &= u_0u_1 + \bar{k}_{21}\lceil\zeta_1\rfloor^{\bar{\gamma}_2} + \bar{k}_{22}\lceil\zeta_1\rfloor + \bar{k}_{23}\lceil\zeta_1\rfloor^{\bar{\gamma}_2^{'}}\end{aligned} \tag{15}$$

with $\zeta_1 = x_1 - z_1,\ \bar{k}_{ij} > 0,\ i = 1,2,\ j = 1,2,3$ and parameters $\bar{\gamma}_i,\ \bar{\gamma}_i^{'},\ i = 1,2$ are given by

$$\bar{\gamma}_1 = \bar{\gamma},\ \ \bar{\gamma}_2 = 2\bar{\gamma} - 1,\ \ \bar{\gamma}_1^{'} = 2 - \bar{\gamma},\ \ \bar{\gamma}_2^{'} = 3 - 2\bar{\gamma}, \tag{16}$$

with $\bar{\gamma} \in \left(\frac{1}{2}, 1\right)$. From the structure of controller (5), it should be noted that controller (5) cannot be applied to the design of observer (15) directly. Thus, to solve this challenging problem, the controller u_0 is further reconstructed as

$$u_0 = \mathrm{sign}(x_0(0)\varpi_1(z))c + (1 - \mathrm{sign}(\varpi_1(z)))\varpi_2(x_0), \tag{17}$$

where $\varpi_1(z)$ with $z = (z_1, z_2)^T$ and $\varpi_2(x_0)$ are continuous functions to be determined later.

To achieve to the fixed-time output feedback stabilization control objective, an appropriate function $\varpi_1(z)$ is chosen yielding the following proposition holding, whose rigorous proof can be seen in Appendix 6.

Proposition 2 *Under an appropriate $\varpi_1(z)$, switching controller (17) is a constant before achieving the fixed-time output feedback stabilization of system (6).*

Then, based on Proposition 2, subtracting (15) from (6) yields

$$\begin{aligned}\dot{\zeta}_1 &= -\bar{k}_{11}\lceil\zeta_1\rfloor^{\bar{\gamma}_1} - \bar{k}_{12}\lceil\zeta_1\rfloor - \bar{k}_{13}\lceil\zeta_1\rfloor^{\bar{\gamma}_1^{'}} + \zeta_2,\\ \dot{\zeta}_2 &= -\bar{k}_{21}\lceil\zeta_1\rfloor^{\bar{\gamma}_2} - \bar{k}_{22}\lceil\zeta_1\rfloor - \bar{k}_{23}\lceil\zeta_1\rfloor^{\bar{\gamma}_2^{'}}\end{aligned} \tag{18}$$

with $\zeta_2 = u_0x_2 - z_2$. The parameters in (16) are aimed to make system (18) be *bi*-limit homogeneous. To demonstrate this, let $\zeta = (\zeta_1, \zeta_2)^T$ and denote $\Phi_p(\zeta)$ be the vector field of system (18), which makes system (18) can be recorded as $\dot{\zeta} = \Phi_p(\zeta)$. Then, defining

$$\Phi_{p0}(\zeta) = \left(\bar{k}_{11}\lceil\zeta_1\rfloor^{\bar{\gamma}_1} + \zeta_2, -\bar{k}_{21}\lceil\zeta_1\rfloor^{\bar{\gamma}_2}\right)^T, \tag{19}$$

$$\Phi_{p\infty}(\zeta) = \left(\bar{k}_{13}\lceil\zeta_1\rfloor^{\bar{\gamma}_1^{'}} + \zeta_2, -\bar{k}_{23}\lceil\zeta_1\rfloor^{\bar{\gamma}_2^{'}}\right)^T, \tag{20}$$

it can be seen that $\Phi_{p0}(\zeta)$ is homogeneous with $k_{p0} = -1$ and $r_{p0} = \left(\frac{1}{1-\bar{\gamma}}, \frac{\bar{\gamma}}{1-\bar{\gamma}}\right)^T$, and $\Phi_{p\infty}(\zeta)$ is homogeneous with $k_{p\infty} = 1$ and $r_{p\infty} = \left(\frac{1}{1-\bar{\gamma}}, \frac{2-\bar{\gamma}}{1-\bar{\gamma}}\right)^T$. Since $\bar{\gamma} \in \left(\frac{1}{2}, 1\right)$, $0 < \bar{\gamma}_i < 1 < \bar{\gamma}_i^{'},\ i = 1,2$ hold, the vector fields $\Phi_{p0}(\zeta)$ and $\Phi_{p\infty}(\zeta)$ can be regarded as as approximating homogeneous functions for $\Phi_p(\zeta)$ in 0-limit and ∞-limit, respectively. Therefore, the following Proposition can be summarized.

Proposition 3 *Considering observer (15) with parameters given in (16), system (18) is fixed-time stabilized if $\bar{k}_{ij} > 0,\ i = 1,2,\ j = 1,2,3$ with $\bar{\gamma} \in \left(\frac{1}{2}, 1\right)$.*

Proof Choose a Lyapunov function

$$\begin{aligned}W_o = &\bar{k}_{21}(2 - \bar{\gamma})|\zeta_1|^{2\bar{\gamma}} + \bar{k}_{22}\bar{\gamma}(2 - \bar{\gamma})|\zeta_1|^2\\ &+ \bar{k}_{23}\bar{\gamma}|\zeta_1|^{4-2\bar{\gamma}} + \bar{\gamma}(2 - \bar{\gamma})|\zeta_2|^2.\end{aligned} \tag{21}$$

Obviously, W_o in (21) is continuously differentiable, positive definite and radially unbounded for any $\bar{\gamma} \in \left(\frac{1}{2}, 1\right)$. The derivative of W_o can be derived by

$$\dot{W}_o|_{(18)} = \Xi_2\left(\bar{k}_{21}\lceil\zeta_1\rfloor^{2\bar{\gamma}-1} + \bar{k}_{22}\lceil\zeta_1\rfloor + \bar{k}_{23}\lceil\zeta_1\rfloor^{3-2\bar{\gamma}}\right)\dot{\zeta}_1 \tag{22}$$

with $\Xi_2 = 2\bar{\gamma}(2 - \bar{\gamma})$. Substituting (18) into (22) results in

$$\begin{aligned}&\dot{W}_o|_{(18)}\\ &= -\Xi_2\bar{k}_{21}\left(\bar{k}_{11}|\zeta_1|^{3\bar{\gamma}-1} + \bar{k}_{12}|\zeta_1|^{2\bar{\gamma}} + \bar{k}_{13}|\zeta_1|^{\bar{\gamma}+1}\right)\\ &\quad - \Xi_2\bar{k}_{22}\left(\bar{k}_{11}|\zeta_1|^{1+\bar{\gamma}} + \bar{k}_{12}|\zeta_1|^2 + \bar{k}_{13}|\zeta_1|^{3-\bar{\gamma}}\right)\\ &\quad - \Xi_2\bar{k}_{23}\left(\bar{k}_{11}|\zeta_1|^{3-\bar{\gamma}} + \bar{k}_{12}|\zeta_1|^{4-2\bar{\gamma}} + \bar{k}_{13}|\zeta_1|^{5-3\bar{\gamma}}\right).\end{aligned} \tag{23}$$

It can be concluded that $\dot{W}_o$ in (21) is semi-negative definite. Moreover, $\dot{W}_o|_{(18)} = 0$ implies that $\zeta_1 = 0$. By LaSalle's invariance principle, all the trajectories will converge to the invariant set $\Pi_2 = \{(\zeta_1, \zeta_2)|\zeta_1 = 0\}$. Together with system (18), we can conclude $\zeta_2 = 0$ in Π_2. Therefore, system (18) is asymptotically stable.

Choosing $\bar{k}_{ij} = 0,\ i = 1,2,\ j = 2,3$ (or $\bar{k}_{ij} = 0,\ i,j = 1,2$) in (18) and (21)-(23), system $\dot{\zeta} = \Phi_{p0}(\zeta)$ (or $\dot{\zeta} = \Phi_{p\infty}(\zeta)$) can be asymptotically stabilized with $\Phi_{p0}(\zeta)$ (or $\Phi_{p\infty}(\zeta)$) given in (19) (or (20)).

Considering the *bi*-limit homogeneity of system (18) and by means of Lemma 1, system (18) is fixed-time stable.

3.3 Fixed-time output feedback control

Thus, it is time to construct fixed-time output feedback stabilizing controllers u_1 for system (6). Replacing x_2 in (7) as z_2/u_0 yields

$$\begin{aligned}u_1(z) = &-\left(k_{11}\lceil x_1\rfloor^{\gamma_1} + k_{12}\lceil x_1\rfloor + k_{13}\lceil x_1\rfloor^{\gamma_1^{'}}\right)\\ &-\left(k_{21}\left\lceil\frac{z_2}{u_0}\right\rfloor^{\gamma_2} + k_{22}\left\lceil\frac{z_2}{u_0}\right\rfloor + k_{23}\left\lceil\frac{z_2}{u_0}\right\rfloor^{\gamma_2^{'}}\right),\end{aligned} \tag{24}$$

where x_0 and x_1 are known outputs. Thus, the main results can be described in the following theorem.

Theorem 1 *Under switching controller (17) determined by (28) and (32), and controller (24) with z_2 updated by (15), the closed-loop system (3), (15), (17) and (24) is fixed-time stable if $k_{ij} > 0$, $\bar{k}_{ij} > 0$, $i = 1,2$, $j = 1,2,3$ and γ_i, $\gamma_i^{'}$, $\bar{\gamma}_i$, $\bar{\gamma}_i^{'}$, $i = 1,2$ are chosen as in Propositions 1 and 3, respectively.*

Proof Firstly, it follows from Proposition 2 that before achieving the fixed-time output feedback stabilization of system (6), the controller $u_0 = \text{sign}(x_0(0)\varpi_1(z))c$ and then follows from Proposition 3 that there is a time $T_1 > 0$, independently of initial estimation error $\zeta_1(0)$ and $\zeta_2(0)$ such that $z_2(t) = u_0(t)x_2(t)$, $\forall t \geq T_1$. Hence, controller (24) coincides with controller (7) for all $t \geq T_1$.

Moreover, following Proposition 1, if all states of system (3) under (17) and (24) do not escape during $t \in [0, T_1]$, there must be a time $T_2 > 0$, independently of $x_1(T_1)$, $x_2(T_1)$ to achieve the fixed-time stabilization of system (6). Therefore, it is sufficient to prove Theorem 1 from the closed-loop system (3), (15) under (17) and (24) not escaping in finite time.

To achieve the proof, we reconsider the Lyapunov function W_s in (11) whose derivative along the trajectory (6) under (17) and (24) is given by

$$\dot{W}_s|_{(6)} \leq -c\Xi_1 x_2 \left(k_{21} \left\lceil \frac{z_2}{u_0} \right\rfloor^{\gamma_2} + k_{22} \left\lceil \frac{z_2}{u_0} \right\rfloor + k_{23} \left\lceil \frac{z_2}{u_0} \right\rfloor^{\gamma_2^{'}} \right) \tag{25}$$

with $\Xi_1 = 2(\gamma_1+1)\left(\gamma_1^{'}+1\right)$. By means of $z_2 = u_0 x_2 - \zeta_2$, $x_2 \left\lceil \frac{z_2}{u_0} \right\rfloor^{\varrho} = x_2 \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\varrho} \text{sign}\left(x_2 - \frac{\zeta_2}{u_0} \right)$, $\forall \varrho > 0$ can be easily obtained, yielding the following two different cases.

Case 1: If $|x_2| > \left| \frac{\zeta_2}{u_0} \right|$, $\text{sign}\left(x_2 - \frac{\zeta_2}{u_0} \right) = \text{sign}(x_2)$ holds. Then, one has $x_2 \left\lceil \frac{z_2}{u_0} \right\rfloor^{\varrho} = |x_2| \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\varrho}$, $\forall \varrho > 0$ and $\dot{W}_s$ in (25) can be rewritten as

$$\begin{aligned} \dot{W}_s|_{(6)} \leq& -c\Xi_1 |x_2| \Big(k_{21} \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\gamma_2} + k_{22} \left| x_2 - \frac{\zeta_2}{u_0} \right| \\ &+ k_{23} \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\gamma_2^{'}} \Big). \end{aligned} \tag{26}$$

Case 2: If $|x_2| \leq \left| \frac{\zeta_2}{u_0} \right|$, it is also easy to obtain $-x_2 \left\lceil \frac{z_2}{u_0} \right\rfloor^{\varrho} = |x_2| \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\varrho}$, $\forall \varrho > 0$, which indicates $\dot{W}_s$ in (25) satisfies

$$\begin{aligned} \dot{W}_s|_{(6)} \leq& c\Xi_1 |x_2| \Big(k_{21} \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\gamma_2} + k_{22} \left| x_2 - \frac{\zeta_2}{u_0} \right| \\ &+ k_{23} \left| x_2 - \frac{\zeta_2}{u_0} \right|^{\gamma_2^{'}} \Big). \end{aligned} \tag{27}$$

Then, we will confirm that for above two cases, there exists a constant $\Delta > 0$ such that $\dot{W}_s|_{(6)} \leq \Delta$ in time interval $[0, T_1]$. Firstly, from Proposition 3, the fixed-time convergence of ζ_2 can be gotten, which indicates ζ_2 is bounded. Denoting $\Delta = \sup \Big\{ \Xi_1 \big(\frac{k_{21} 2^{\gamma_2}}{c^{1+\gamma_2}} |\zeta_2|^{\gamma_2+1} + \frac{2k_{22}}{c} |\zeta_2| + \frac{k_{23} 2^{\gamma_2^{'}}}{1+\gamma_2^{'}} |\zeta_2|^{\gamma_2^{'}+1} \big) \Big\}$, Δ can be viewed as an upper bound of the right-hand side of (26) and (27), yielding $\dot{W}_s|_{(6)} \leq \Delta$. Therefore, W_s and system states x_1 and x_2 cannot escape in $[0, T_1]$. Moreover, from Proposition 3, observer states z_1 and z_2 also cannot escape in $[0, T_1]$.

Furthermore, since $u_0(t) = \text{sign}(x_0(0)\varpi_1(z))c$, $t \in [0, T_1]$, we have $x_0(t) \leq x_0(0) + ct$, $t \in [0, T_1]$, which indicates the state x_0 cannot escape in $[0, T_1]$. Thus, the closed-loop system (6), (15) under (17) and (24) does not escape in $[0, T_1]$, which has proven that the closed-loop system (6), (15) under (17) and (24) is fixed-time stable.

Now, the fixed-time output feedback stabilization of x_0-subsystem is given below. Since the fixed-time output feedback stabilization of x-subsystem has been obtained, we have $x(t) = z(t) = u_1(t) = 0$, $t \geq T_2$, which indicates the controller $u_0(t) = \varpi_2(x_0)$, for $t \geq T_2 + \tau$. Therefore, from [24] we construct

$$\varpi_2(x_0) = -k_{11} x_0^{\gamma_1} - k_{13} x_0^{\gamma_1^{'}} \tag{28}$$

to fixed-time stabilize x_0-subsystem with bounded initial condition $x_0(T_2 + \tau)$.

Remark 2 If the initial condition $x_0(0) = 0$, the switching controller (17) just needs to be rewritten into $u_0 = \text{sign}(\varpi_1(z))c + (1 - \text{sign}(\varpi_1(z)))\varpi_2(x_0)$, and following Propositions 1 and 3, and Theorem 1, the fixed-time output feedback stabilization of system (3) can also be obtained, which indicates the proposed control strategy is global sense.

Corollary 1 *The output feedback control strategy proposed in Propositions 1, 3 and Theorem 1 can fixed-time output feedback stabilize the following system*

$$\begin{aligned} \dot{x}_0 &= \left(1 - \frac{\varepsilon^2}{2} \right) u_0, \\ \dot{x}_1 &= u_0 x_2, \\ \dot{x}_2 &= u_1, \end{aligned} \tag{29}$$

where ε is an unknown small bias in orientation, and satisfies $|\varepsilon| \leq \iota < 1$ with ι being a known constant.

4 An illustrative example

Consider the bilinear model of a mobile robot with small-angle measurement error [12, 14, 23], describing as

$$\begin{aligned} \dot{x}_l &= \left(1 - \tfrac{\varepsilon^2}{2}\right) v, \\ \dot{y}_l &= \theta_l v + \varepsilon v, \\ \dot{\theta}_l &= \omega, \end{aligned} \tag{30}$$

with $\varepsilon \leq 1$ being a small bias in orientation, $(x_l, y_l, \theta_l)^T$ standing for the state of the locally approximate model, v and ω being two inputs to denote the linear velocity and angular velocity, respectively. Taking the state transformation $x_0 = x_l,\ x_1 = y_l,\ x_2 = \theta_l + \varepsilon,\ u_0 = v$ and $u_1 = \omega$, system (30) is transformed into system (29), where $(x_0, x_1, x_2)^T$ is state, u_0 and u_1 are inputs and $y = (x_0, x_1)^T$ is output, respectively.

Firstly, by virtue of Propositions 1 and 3, and Theorem 1, fixed-time output feedback stabilizing controllers are constructed as

$$\begin{cases} \begin{cases} \dot{z}_1 = z_2 + \lceil \zeta_1 \rfloor^{\bar{\gamma}_1} + \lceil \zeta_1 \rfloor + \lceil \zeta_1 \rfloor^{\bar{\gamma}_1'}, \\ \dot{z}_2 = u_0 u_1 + \lceil \zeta_1 \rfloor^{\bar{\gamma}_2} + \lceil \zeta_1 \rfloor + \lceil \zeta_1 \rfloor^{\bar{\gamma}_2'}, \end{cases} \\ u_0 = \mathrm{sign}(x_0(0)\varpi_1(z))c + (1 - \mathrm{sign}(\varpi_1(z)))\varpi_2(x_0), \\ u_1 = -\left(\lceil x_1 \rfloor^{\gamma_1} + \lceil x_1 \rfloor + \lceil x_1 \rfloor^{\gamma_1'}\right) \\ \qquad - \left(\left\lceil \frac{z_2}{u_0} \right\rfloor^{\gamma_2} + \left\lceil \frac{z_2}{u_0} \right\rfloor + \left\lceil \frac{z_2}{u_0} \right\rfloor^{\gamma_2'}\right), \end{cases} \tag{31}$$

where $\zeta_1 = x_1 - z_1$, $\gamma = 0.6$ and $\bar{\gamma} = 0.6$, functions $\varpi_1(z)$ and $\varpi_2(x_0)$ are determined by (32) and (28), respectively, and initial conditions $x_0(0) = 15$, $x(0) = (20, -10)^T$ and $z(s) = (5, 5)^T,\ s \in [-\tau, 0]$ with $c = \tau = 0.5$. The numerical simulation result of the closed-loop system is shown in Fig 2.

Remark 3 Due to $\mathrm{sign}(\cdot)$ being used in constructing u_0, a threshold of small value instead of absolute zero is applied, that is, let $\varpi_1(z) \approx 0$, when $\varpi_1(z) \leq 10^{-5}$ ([32]).

5 Conclusion

In this paper, a novel design approach for constructing fixed-time output feedback stabilizing controllers has been proposed for chained nonholonomic systems. Remarkably, the systems in question require two controllers to be constructed simultaneously, which are considerably different from those in the closely related literatures. In our design, by applying switching control technique, we first constructed a fixed-time output feedback stabilizing controller u_0 formally, and then based on this controller and by means of *bi*-limit homogeneous technique and the Lyapunov theorem, the fixed-time output feedback stabilizing controller u_1 based on fixed-time observer of x-subsystem are constructed. On the other hand, to obtain a fact, that is, before achieving the fixed-time output feedback stabilization of system (6), controller u_0 is a constant same as the sign of the initial condition of x_0-subsystem, a novel switching mechanism has been introduced, and then based on this mechanism, the fixed-time output feedback stabilizing controller u_0 is thus constructed precisely. Since the switching control strategy proposed for controller u_0 is hard to applied to chained nonholonomic systems with external disturbances, which is our one possible future study topic.

6 Appendix

In this section, we will give the detailed proof of Proposition 2. Firstly, from Remark 1 and Proposition 1, the fact that the achievement of fixed-time state feedback stabilization of system (6) implying $\|x(t)\| = 0$, and vice versa. Moreover, since the state of x_2 cannot be measured, controller (5) should be reconstructed to be applied to fixed-time output feedback stabilize system (6), and an intuitive idea is to change $\|x\|$ in (5) as $\|z\|$.

However, the fact given in Section 3.1 will no longer hold, that is, the fixed-time output feedback stabilization of system (6) cannot be obtained from $\|z(t)\| = 0$. In fact, it may occur the phenomenon that $\|z(t)\| = 0$ happens infinite times during fixed-time observing the states of system (6), which will leads to the finite time T_1 not existing. Therefore, Proposition 3 will no longer hold. To avoid this phenomenon, we reconstruct the controller u_0 in (5) as (17) where $\varpi_1(z)$ is updated by

$$\dot{\varpi}_1(z) = \|z(t)\|^2 - \|z(t-\tau)\|^2 \tag{32}$$

with the initial condition $z(s) \neq 0,\ s \in [-\tau, 0]$.

Remark 4 Obviously, $\varpi_1(z)$ contains the information of $z(t)$ from time $t - \tau$ to t, which is critical to avoid the phenomenon that $\|z(t)\| = 0$ happens infinite times during fixed-time observing the states of system (6).

Next, we will prove that under the choosing of $\varpi_1(z)$ in (32), the fact that before the achievement of fixed-time output feedback stabilization of system (6) implying $\varpi_1(z) \neq 0$.

For convenience, we divide the proof into three steps. In step 1, we will provide that $\|z(t)\|,\ t \in (0, +\infty)$ is a continuous function. In step 2, we will prove that the set Ω defined as $\Omega = \{t \mid \|z(t)\| = 0,\ t > 0\}$ is a discrete set. In step 3, we prove that $\varpi_1(z)$ exists and $\varpi_1(z(t)) > 0$ before the achievement of fixed-time output feedback stabilization of system (6).

Step 1: Continuity analysis of $\|z(t)\|$ in $t \in (0, +\infty)$.

Firstly, from the construction of observer (15) and controllers (17) and (24), we obtain that although $\dot{z}_i(t),\ t \in (0, \infty),\ i = 1, 2$ are discontinuous at some points(where controllers (17) or (24) is discontinuous at these points), the functions $z_i(t),\ i = 1, 2$ are continuous in $(0, \infty)$. Then, based on the continuity of $z_i(t),\ t \in (0, \infty),\ i = 1, 2$ and by means of the definition of vector norm, we can directly prove the continuity of $\|z(t)\|$ in $t \in (0, +\infty)$. Similarly, the continuity of $x_i(t),\ i = 0, 1, 2$ and $\|x(t)\|$ in $t \in (0, +\infty)$ can be also obtained.

Step 2: Proof of discrete set Ω.

Firstly, we give the definition of discrete set that is a set of points of a topological space such that each point in the set is an isolated point, i.e. a point that has a neighborhood that contains no other points of the set.

Next, the proof by contradiction method is applied to verify Ω is a discrete set. Firstly, we assume that there exists $t^* \in \Omega$(the first one) is not an isolated point. There must exist a time sequence $\{t_n\}$ with $t_n \in \Omega$ such that $t_n \to t^*$ as $n \to +\infty$ and without loss of generality, we assume that the time sequence $\{t_n\}$ tends to t^* from the left hand.

Since $\|z(t^*)\| = 0$, we have $z_1(t^*) = z_2(t^*) = 0$. Furthermore, $\|x(t^*)\| \neq 0$ can be also obtained, otherwise, the fixed-time output feedback stabilization of system (6) is already achieved, which contradicts the fact that before the achievement of fixed-time output feedback stabilization of system (6). Thus, based on $\|x(t^*)\| \neq 0$, we analyze the following three cases.

Case 1): $x_1(t^*) \neq 0$ and $x_2(t^*) = 0$. When this case occurs, from observer (15) and controller (24), we obtain that at least $\dot{z}_1(t^*) \neq 0$. By means of the continuity of $x_1(t)$ in $t \in [0, +\infty)$, and based on local inheriting order property, there must exist a constant $\delta_1 > 0$, such that $\dot{z}_1(t) \neq 0,\ t \in \Omega_1 = (t^* - \delta_1, t^* + \delta_1)$ and then we have $|z_1(t)| > 0,\ t \in \Omega_1 \backslash \{t^*\}$, which also indicates $\|z(t)\| > 0,\ t \in \Omega_1 \backslash \{t^*\}$.

On the other hand, since $t^* \in \Omega$ is not an isolated point and $t_n \to t^*$ as $n \to +\infty$, that is, for any constant $\delta_2 > 0$ choosing as $\delta_2 < \delta_1$, there exists $N > 0$, such that for all $n > N$, $t_n \in \Omega_2 = (t^* - \delta_2, t^*)$, which contradicts $\|z(t)\| > 0,\ t \in \Omega_1 \backslash \{t^*\}$ (based on $\Omega_2 \subseteq \Omega_1$). Thus, $t^* \in \Omega$ is an isolated point, which has proven that Ω is a discrete set.

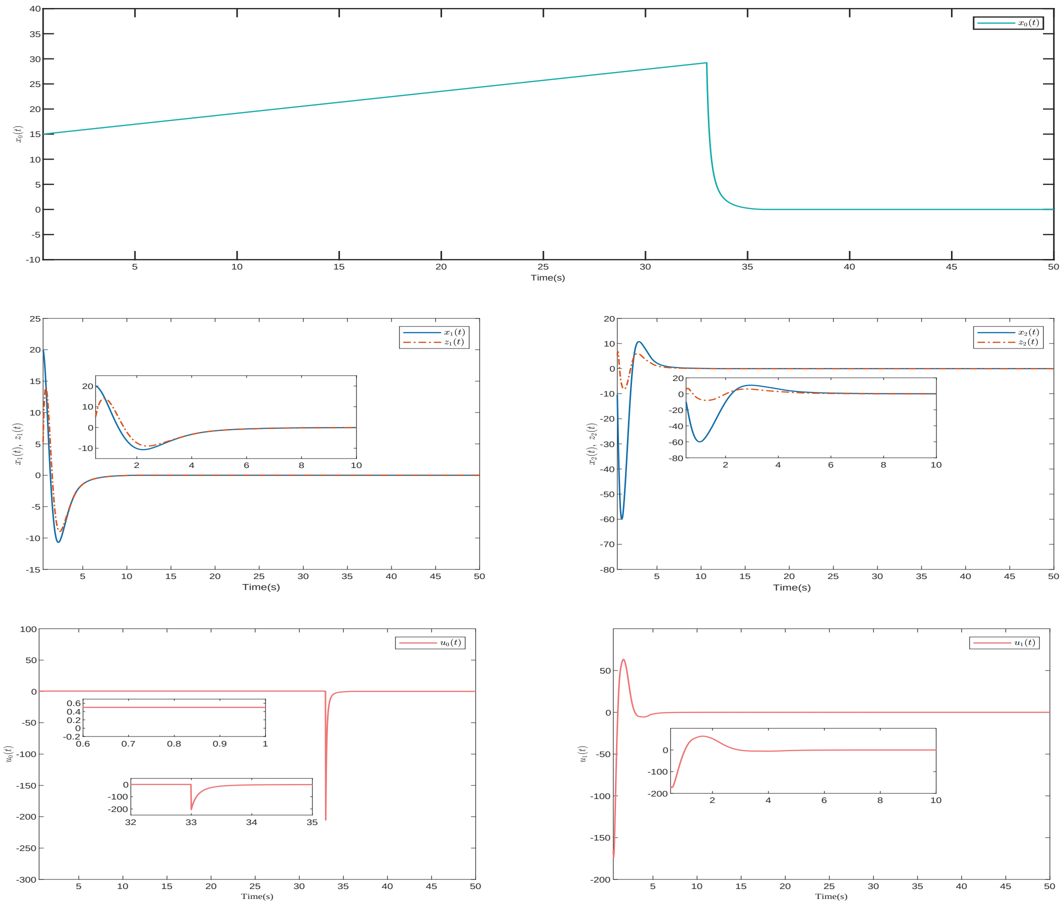

Fig. 2 Trajectories of the closed-loop system (28), (29), (31) and (32) with the small bias in orientation $\varepsilon = 0.5$.

Case 2): $x_2(t^*) \neq 0$ and $x_1(t^*) = 0$. When this case occurs, we will firstly prove the state of $x_0(t^*) \neq 0$.

Since $t^* \in \Omega$ (the first one) is not an isolated point, thus, for any constant $\delta_3 > 0$, there must be finite points recorded as $t_1, \ldots, t_M \in \Omega_3 = [0, t^* - \delta_3]$ such that $\|z(t_k)\| = 0,\ k = 1, \ldots, M$. Then, by means of the continuity of $\|z(t)\|,\ t \in \Omega_3$ and the initial condition $z(s) \neq 0,\ s \in [-\tau, 0]$, and from (32), we have

$$\varpi_1(z(t)) = \int_{t-\tau}^{t} \|z(s)\|^2 ds = \int_{t-\tau}^{t-t_1} \|z(s)\|^2 ds + \cdots + \int_{t-t_M}^{t} \|z(s)\|^2 ds > 0,$$

which indicates the controller $u_0(t) = \mathrm{sign}(x_0(0)\varpi_1(z))c$ and $|x_0(t)| \geq |x_0(0)|$ in $t \in \Omega_3$. Therefore, for any δ_3, if $x_0(t^*) = 0$, there exists $\varepsilon_3 = \frac{|x_0(0)|}{2}$, such that when $|t^* - (t^* - \delta_3)| \leq \delta_3$, $|x_0(t^*) - x_0(t^* - \delta_3)| \geq \varepsilon_3 = \frac{|x_0(0)|}{2}$ holds, which contradicts the continuity of $x_0(t)$ in $t \in [0, +\infty)$. Thus, we have proven $x_0(t^*) \neq 0$, and similarly, $x_0(t) \neq 0,\ t \in \Omega_4 = (t^* - \delta_3, t^* + \delta_3)$ can also be obtained, which also indicates the controller $u_0(t) \neq 0,\ t \in \Omega_4$.

Next, from system (6) with $u_0(t) \neq 0,\ t \in \Omega_4$ and $x_2(t^*) \neq 0$, we have $\dot{x}_1(t) \neq 0,\ t \in \Omega_4$ and based on local inheriting order property, there is a constant $\delta_4 > 0$ such that $x_1(t) \neq 0,\ t \in \Omega_5 = (t^* - \delta_4, t^* + \delta_4) \backslash \{t^*\}$. Then, from Case 1), we have proven that $t^* \in \Omega$ is an isolated point and Ω is a discrete set.

Case 3): $x_1(t^*) \neq 0$ and $x_2(t^*) \neq 0$. When this case occurs, combining with Case 1) and Case 2), we have proven that $t^* \in \Omega$ is an isolated point and Ω is a discrete set. This completes the proof.

Step 3. Proof of $\varpi_1(z(t)) > 0$.

From Step 1, we have $\|z(t)\|$ is continuous in $t \in (0, +\infty)$, and since Ω is a discrete set, and then, for any given $t > 0$, there exist finite points recorded as $t_1, \ldots, t_N \in [t - \tau, t]$ such that $\|z(t_k)\| = 0,\ k = 1, \ldots, N$. Therefore, $\varpi_1(z(t))$ defined in (32) exists and satisfies

$$\varpi_1(z(t)) = \int_{t-\tau}^{t} \|z(s)\|^2 ds = \int_{t-\tau}^{t-t_1} \|z(s)\|^2 ds + \cdots + \int_{t-t_N}^{t} \|z(s)\|^2 ds > 0.$$

Thus, by means of the structure of controller (17), we have proven (17) is a constant controller before the achievement of fixed-time output feedback stabilization of system (6). This completes the proof.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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Figures

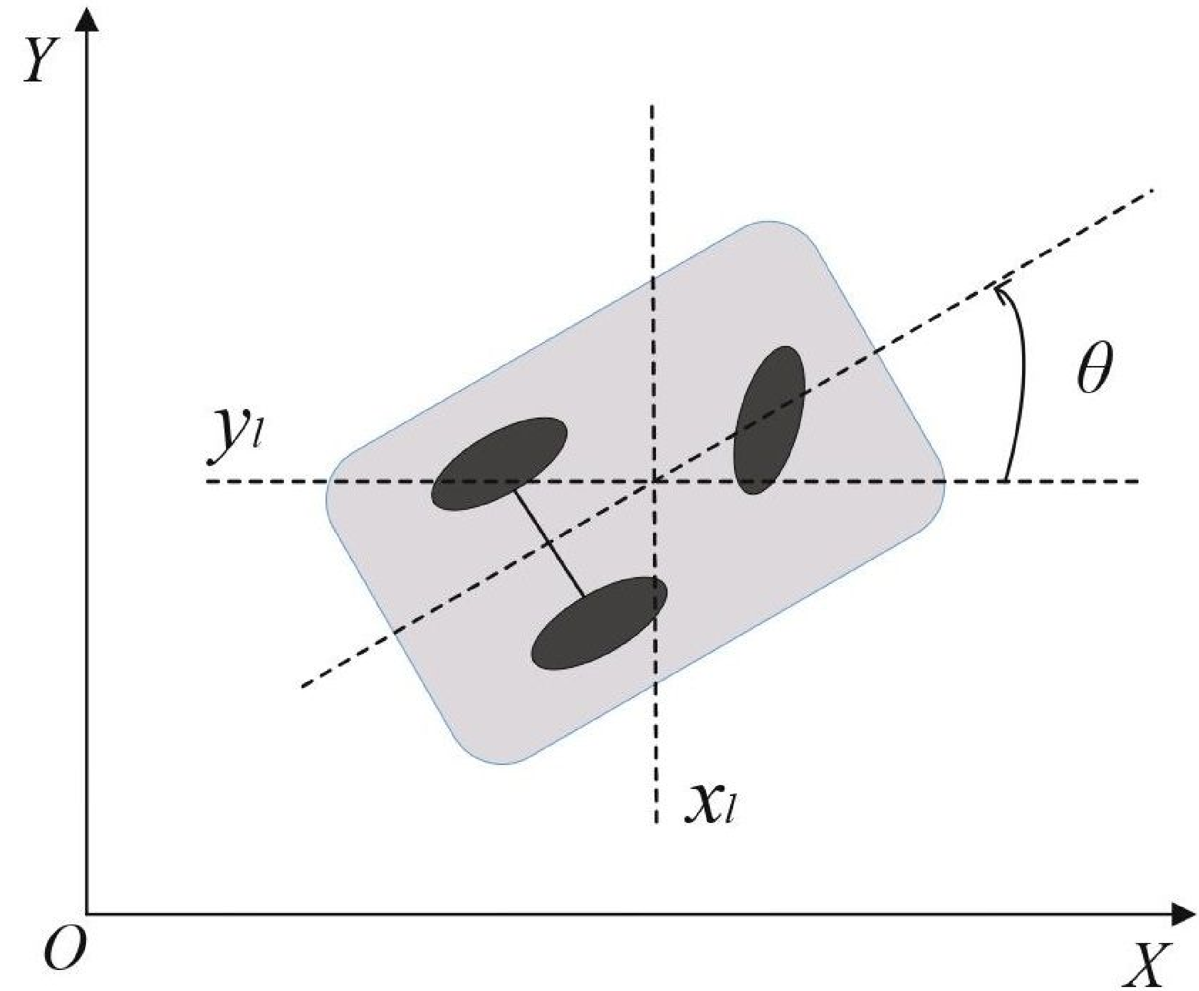

Figure 1

The planar graph of a unicycle-type mobile robot.

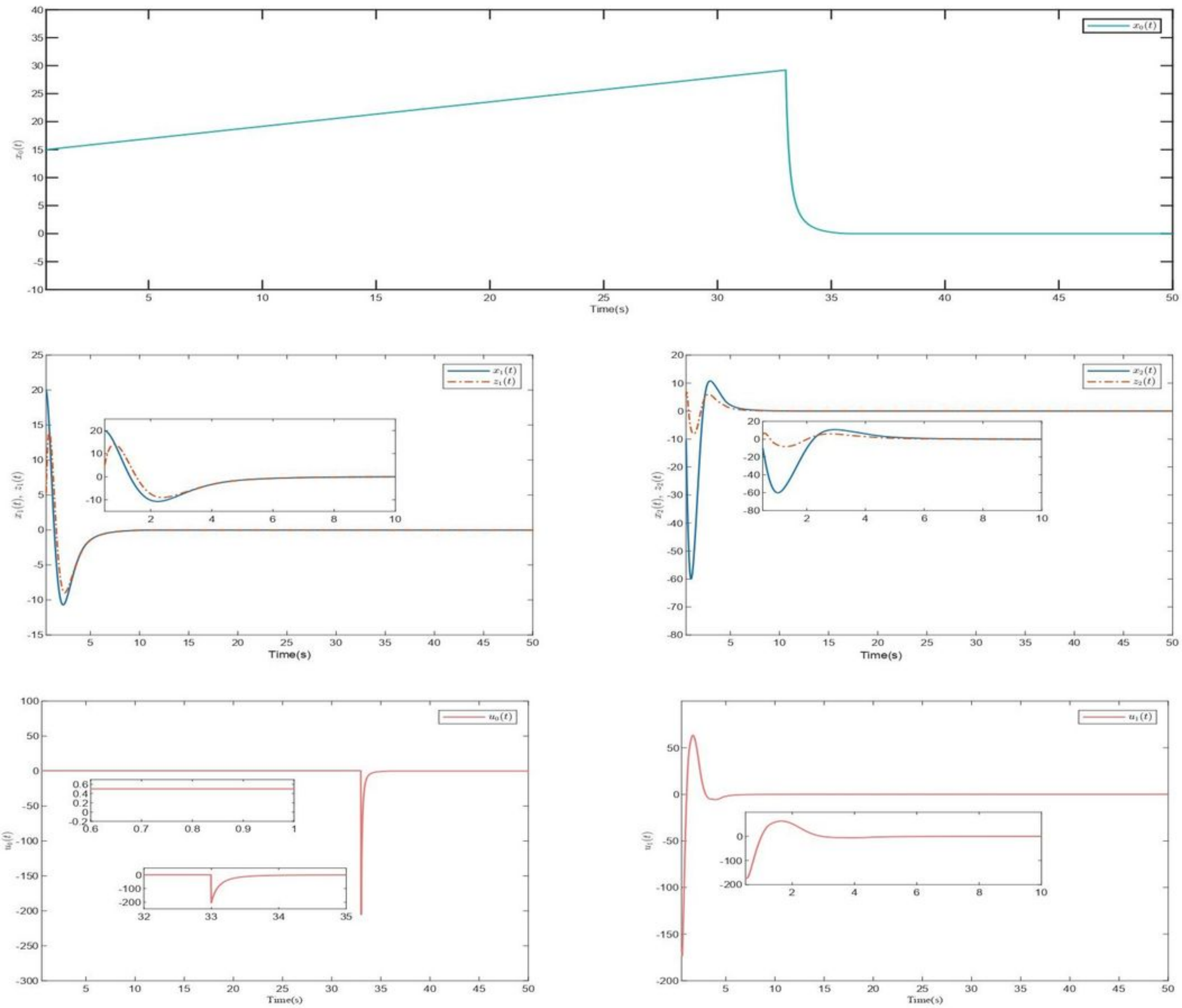

Figure 2

Trajectories of the closed-loop system (28), (29), (31) and (32) with the small bias in orientation ε = 0.5.