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New Constructions of Two-Dimensional Binary Z-Complementary Array Sets with Large Zero Correlation Zone Width

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Abstract
The one-dimensional (1-D) Z-complementary sequence set (ZCSS) has been widely employed in communications engineering. The concept of 1-D ZCSS can be extended to the two-dimensional (2-D) Z-complementary array set (ZCAS) where the 2-D aperiodic autocorrelations of constituent arrays sum to zero for the 2-D certain shifts. 2-D ZCASs play an important role in radar, 2-D multi-carrier code-division multiple access (MC-CDMA) systems, multiple-input and multiple-output (MIMO) systems, ultra wideband communication systems, etc. In this paper, we focus on designing new 2-D ZCASs by using Boolean functions and concatenation. We first present a direct construction of \((2^k, 2^k, 2^n, 14 \cdot 2^m, 2^n, 12 \cdot 2^m)\)-ZCASs by employing Boolean functions. The constructed ZCASs have the largest known zero correlation zone (ZCZ) width and low peak to-mean envelope power (PMEPR). Next, we provide a class of 2-D ZCASs with ZCZ around the
end-shift position by using concatenation, which have new lengths and large ZCZ widths.

**Keywords:** Two-Dimension (2-D), Z-complementary array set (ZCAS), Boolean function, zero correlation zone (ZCZ)

## 1 Introduction

One-dimensional Golay complementary pairs (1-D GCPs) are sequences with zero aperiodic auto-correlation sums at each non-zero time-shift \([1]\). Davis et al. \([2]\) directly constructed 1-D GCPs from Boolean functions. The construction based on Boolean functions has algebraic structures and hence can be very efficient for hardware. By extending 1-D GCPs, Tseng et al. proposed the concept of 1-D Golay complementary sequence set (GCSS) which consists of two or more constituent sequences with zero aperiodic correlation sums at each time-shift \([3]\). If the set size equals the number of constituent sequences, 1-D GCSSs are called 1-D complete complementary codes (CCCs) \([4]\). 1-D GCPs and GCSSs have found many engineering applications, such as peak-to-mean envelope power ratio (PMEPR) in orthogonal frequency division multiplexing (OFDM) communication systems, suppressing the multiple access interference (MAI), synchronization, increasing system capacity, channel estimation and so on.

It is well-known that the existing binary 1-D GCPs are limited by the length \(2^\alpha \cdot 10^\beta \cdot 26^\gamma\), where \(\alpha, \beta, \gamma \in \{0, 1, 2, \cdots\}\) \([5]\). The length of binary 1-D CCCs based on Boolean functions can only be power-of-two. In order to obtain more flexible lengths, the 1-D Z-complementary pairs (ZCPs) and 1-D Z-complementary sequence sets (ZCSSs) were proposed such that aperiodic correlation is zero over the time-shift range \([6]\). This time-shift is said to be zero correlation zone (ZCZ) width \([7]\). According to the relaxation of the autocorrelation constraint, 1-D ZCPs was shown to exist for all lengths. Since then, there have been a large number of literature studying the 1-D ZCPs and 1-D ZCSSs \([8–14]\).

The concepts of 1D ZCPs and ZCSSs can be extended to two-dimensional (2-D) array pairs and sets. Such array pairs and sets are called 2-D Z-complementary array pairs (ZCAPs) and 2-D Z-complementary array sets (ZCASs) with ZCZ width. When the ZCZ width equals size of the array, we refer to the array pairs and sets as a 2-D Golay complementary array pair (GCAP) \([15]\) and 2-D Golay complementary array set (GCAS)\([16]\), respectively. 2-D arrays can be applied as precoding matrices for omnidirectional transmission in a massive MIMO system \([17–19]\). Owing to their good autocorrelation properties, they have applications in radar, synchronization. In 2003, Zhang et al. proposed an application of 2D-GCASs in ultra wide-band (UWB) multi-carrier code division multiple access (MC-CDMA) system \([20]\).

As the length is limited in 2-D GCAPs, Zeng et al. studied the constructions of 2-D ZCAPs \([21]\). Later, Li et al. constructed periodic 2-D ZCAPs by using interleaving technology \([22]\). Pai et al. proposed 2-D ZCAPs based on 2-D generalized Boolean functions with large ZCZ width \([23][24]\). They also obtained 2-D GCAPs with low row and column sequence peak-to-average power-ratios (PAPRs) \([25]\). 2-D ZCAPs can
be constructed by employing 1-D ZCPs or using concatenation \cite{26,27}. Roy et al. presented a class of q-ary 2-D ZCAPs based on 2-D generalized Boolean functions \cite{28}. Zhang et al. obtained a class of 2-D ZCAPs with large ZCZ width based on 2-D generalized Boolean functions \cite{29}.

There has been several construction of 2-D ZCASs in the literature. Recently, Liu et al. provided 2-D GCASs by using 2-D multivariable functions \cite{30}. Pai et al. obtained 2-D GCASs by using Boolean functions with low PAPRs \cite{25}. Shibsankar et al. proposed a class of 2-D ZCASs by employing generating polynomials \cite{31}. Motivated by this, in this paper, novel constructions of 2-D ZCASs based on Boolean functions are proposed. Our obtained 2-D ZCASs are direct constructions, which yield a class of 2-D ZCASs with new length $14 \cdot 2^m$ and the largest ZCZ width $Z = 12 \cdot 2^m$. The constructed ZCASs have low row and column PMEPR. Through concatenation construction, we also obtain a class of 2-D ZCASs with ZCZ around the end-shift position of $12 \cdot 2^m$ and $14 \cdot 2^m$ lengths, which have the largest 2-D $ZCZ_{\text{ratio}} = \frac{14}{12}$ and $ZCZ_{\text{ratio}} = \frac{14}{14}$, respectively. Here the $ZCZ_{\text{ratio}}$ is defined as the ratio of the ZCZ width over the array size.

This paper is organized as follows. In Section 2, several useful notations and definitions are given. In Section 3, we present a construction of ZCAS based on Boolean functions. In Section 4, we propose a construction of ZCAS with ZCZ around the end-shift position by using concatenation. Finally, we conclude this paper in Section 5.

2 Preliminaries

2.1 One-dimensional sequences

Let $a = (a_0, a_1, \ldots, a_{N-1})$ and $b = (b_0, b_1, \ldots, b_{N-1})$ be two binary sequences, the aperiodic correlation function of sequences $a$ and $b$ at shift $\tau$ is given as follows:

$$R_{a,b}(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} a_ib_{i+\tau}, & 0 \leq |\tau| \leq N - 1 \\ 0, & |\tau| \geq N. \end{cases}$$ (1)

$R_{a,b}(\tau)$ is called aperiodic cross-correlation function (ACCF) if $a \neq b$; otherwise, it is called the aperiodic auto-correlation function (AACF), denoted by $R_a(\tau)$ for simplicity.

**Definition 1.** Let $a_1$ and $b_1$ be two sequences with length $N$. If $R_{a_1}(\tau) + R_{b_1}(\tau) = 0$, $0 < |\tau| < N$, then the sequence $(a_1, b_1)$ is called a Golay complementary pair (GCP). If $R_{a_1, a_2}(\tau) + R_{b_1, b_2}(\tau) = 0$, $0 \leq |\tau| < N$, then the GCP $(a_2, b_2)$ is a mate of $(a_1, b_1)$.

**Definition 2.** Let $a$ and $b$ be two sequences with length $N$. $(a, b)$ is called a Z-complementary pair (ZCP) with the zero correlation zone (ZCZ) $Z$, if

$$R_a(\tau) + R_b(\tau) = \begin{cases} 0, & 1 \leq |\tau| \leq Z - 1, Z < N, \\ 2N, & \tau = 0. \end{cases}$$ (2)
In the case of $Z = N$, the pair of sequences $(a, b)$ is also said to be a complementary pair. For sequences $a_1$ and $a_2$ with length $N$, the concatenation of $a_1$ and $a_2$, denoted by $a = a_1||a_2$, is a sequence with length $2N$ such that the upper half part of sequence $a$ corresponds to $a_1$ and the lower part to $a_2$, respectively.

Consider a collection of $K$ sequence sets $\mathcal{A} = \{ A_i \mid i = 0, 1, \cdots, K - 1 \}$, where each sequence set $A_i$ contains $M$ constituent sequences $A_i = \{ a_{i,1}, a_{i,2}, \cdots, a_{i,M} \}$. The length of a sequence $a_{i,m}$ is $N$, $1 \leq m \leq M$.

**Definition 3.** The code set $\mathcal{A}$ is called a $(K, M, N, Z)$-ZCSS, if

$$R_{A_i, A_j}(\tau) = \sum_{m=0}^{M-1} R_{a_{i,m}, a_{j,m}}(\tau)$$

where $Z$ denotes the ZCZ width.

### 2.2 Two-dimensional sequences

Denote a binary array $C$ of size $L_1 \times L_2$ by

$$C = (C_{i,j}), \quad 0 \leq i \leq L_1 - 1, 0 \leq j \leq L_2 - 1,$$

where $C_{i,j} = (-1)^{c_{i,j}}$ and $a_{i,j} \in \{0, 1\}$.

**Definition 4.** The 2-D aperiodic cross-correlation function (ACCF) of two arrays $C$ and $D$ at time-shifts $(\tau_1, \tau_2)$ is defined as

$$R_{C,D}(\tau_1, \tau_2) = \sum_{i=0}^{L_1-1} \sum_{j=0}^{L_2-1} C_{i,j} D_{i+\tau_1,j+\tau_2}$$

$$= \sum_{i=0}^{L_1-1} \sum_{j=0}^{L_2-1} (-1)^{c_{i,j}+d_{i+\tau_1,j+\tau_2}},$$

for $-L_1 + 1 \leq \tau_1 \leq L_1 - 1$, $-L_2 + 1 \leq \tau_2 \leq L_2 - 1$. When $C = D$, we call $R_{C,D}(\tau_1, \tau_2)$ the aperiodic auto-correlation function (AACF) of array $C$ which is denoted by $R_C(\tau_1, \tau_2)$.

**Definition 5.** The array pair $(C, D)$ of size $L_1 \times L_2$ is said to be an $(L_1, L_2, Z_1, Z_2)$-ZCAP if

$$R_C(\tau_1, \tau_2) + R_D(\tau_1, \tau_2)$$

$$= \begin{cases} 2L_1L_2, & \text{for } (\tau_1, \tau_2) = (0, 0), \\ 0, & \text{for } 0 \leq |\tau_1| < Z_1, 0 \leq |\tau_2| < Z_2, \\ \text{and } (\tau_1, \tau_2) \neq (0, 0), \end{cases}$$

for $0 \leq |\tau_1| < Z_1, 0 \leq |\tau_2| < Z_2$. When $C = D$, we call $R_C(\tau_1, \tau_2)$ the aperiodic auto-correlation function (AACF) of array $C$ which is denoted by $R_C(\tau_1, \tau_2)$. When $C = D$, we call $R_C(\tau_1, \tau_2)$ the aperiodic auto-correlation function (AACF) of array $C$ which is denoted by $R_C(\tau_1, \tau_2)$.
where \((Z_1, Z_2)\) is ZCZ width.

We are concentrated on the 2D aperiodic correlation sums within the rectangular zone \(Z_1 \times Z_2\). Consider a 2-D array set \(\mathcal{A} = \{A_i, i = 0, 1, \cdots, K-1\}\), where each array set \(A_i\) contains \(M\) constituent arrays \(A_{i,m} = \{A_{i,0}, A_{i,1}, \cdots, A_{i,M-1}\}\). Each 2-D array \(A_{i,m}\) is of size \(L_1 \times L_2\), \(0 \leq m \leq M - 1\).

**Definition 6.** The array set \(\mathcal{A}\) is referred to as a \((K, M, L_1, L_2, Z_1, Z_2)\)-ZCAS, if

\[
R_{A_i, A_j} (\tau_1, \tau_2) = \sum_{m=0}^{M-1} R_{A_{i,m}, A_{j,m}} (\tau_1, \tau_2)
\]

\[
= \begin{cases} 
ML_1L_2, & \text{for } (\tau_1, \tau_2) = (0,0), \\
0, & \text{for } 0 \leq |\tau_1| < Z_1, \\
0 \leq |\tau_2| < Z_2, & \text{and } (\tau_1, \tau_2) \neq (0,0).
\end{cases}
\]

When \((Z_1, Z_2) = (L_1, L_2)\), the array set \(\mathcal{A}\) is called a GCAS.

**Definition 7.** The 2-D ZCZ ratio of a \((K, M, L_1, L_2, Z_1, Z_2)\)-ZCAS is the ratio of the rectangular ZCZ given by

\[
\text{ZCZ ratio} = \frac{Z_1Z_2}{L_1L_2}.
\]

Here, we will give the concept of 2-D Boolean function. A 2-D Boolean function is a mapping \(f : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2\) defined on \(n + m\) variables \((u_n, u_{n-1}, \cdots, u_1, v_m, v_{m-1}, \cdots, v_1) \in \mathbb{F}_2^{n+m}\). We denote the associated array by

\[
f = \begin{pmatrix} 
& f_{0,0} & f_{0,1} & \cdots & f_{0,2^{m-1}} \\
f_{1,0} & f_{1,1} & \cdots & f_{1,2^{m-1}} \\
& \vdots & \vdots & \ddots & \vdots \\
f_{2^{n-1},0} & f_{2^{n-1},1} & \cdots & f_{2^{n-1},2^{m-1}} 
\end{pmatrix}_{2^n \times 2^m},
\]

where \(f_{u,v} = f(u_n, u_{n-1}, \cdots, u_1, v_m, v_{m-1}, \cdots, v_1)\), \((u_n, u_{n-1}, \cdots, u_1)\) and \((v_m, v_{m-1}, \cdots, v_1)\) are binary representations of integers \(u = \sum_{i=0}^{n-1} u_i2^{i-1}\) and \(v = \sum_{j=1}^{m} v_j2^{j-1}\), respectively.

**Example 1.** For \(n = 2\) and \(m = 3\), the associated array to the 2-D Boolean function \(f = x_1x_2 + y_1\) is given by

\[
f = \begin{pmatrix} 
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

For a 2-D \(n + m\) variables Boolean function \(f\), the size of the associated array \(f\) is \(2^n \times 2^m\). We denote the truncated array by \(f^{(L_1 \times L_2)}\) after deleting the last \(2^n - L_1\) rows and the last \(2^m - L_2\) columns from \(f\). For simplicity, we can disregard the superscript \(L_1 \times L_2\) of \(f^{(L_1 \times L_2)}\) when \(L_1 \times L_2\) can be obtained from the context.
Consider OFDM systems with \( N \) subcarriers, \( \Delta f \) denotes the subcarrier spacing and \( f_c \) denotes the base frequency. For binary array \( C \) of size \( L_1 \times L_2, C = (C_{i,j}) \), \( 0 \leq i \leq L_1 \), and \( 0 \leq j \leq L_2 - 1 \). We define \( C_i \) and \( C_j^T \) as the \( i \)-row sequence and the \( j \)-column sequence, respectively, where
\[
C_i = (c_{i,0}, c_{i,1}, \cdots, c_{i,L_2-1}), \quad C_j = (c_{0,j}, c_{1,j}, \cdots, c_{L_1-1,j})^T.
\]
For the row sequence \( C_i \), The transmitted signal can be modeled as the real part of
\[
s_{C_i}(t) = \sum_{l=0}^{L_2-1} (-1)^{c_{i,l}} \exp(\sqrt{-1}2\pi(f_c + l\Delta f)t).
\]
In [2], it is shown that
\[
|s_{C_i}(t)|^2 = R_{C_i}(0) + 2 \sum_{\tau=1}^{L_2-1} \text{Re}(R_{C_i}(\tau)\exp(\sqrt{-1}2\pi\Delta f\tau t)),
\]
where \( \text{Re}(x) \) denotes the real part of the complex-valued data \( x \). For a sequence \( C_i \), the PMEPR is determined by
\[
PMEPR(C_i) = \frac{1}{L_2} \sup_{0 \leq t \leq 1/\Delta f} |s_{C_i}(t)|^2. \tag{5}
\]
The column sequence PAPR is also similarly given as
\[
PMEPR(C_j^T) = \frac{1}{L_1} \sup_{0 \leq t \leq 1/\Delta f} |s_{C_j^T}(t)|^2. \tag{6}
\]
Lemma 1. [2] For \( x = (x_m, x_{m-1}, \cdots, x_1) \in \mathbb{F}_2^m \), let \( f_1(x) \) be an \( m \)-variable Boolean function denoted by
\[
f_1(x) = \sum_{k=1}^{m-1} x_{\pi(k)}x_{\pi(k+1)} + \sum_{k=1}^{m} c_k x_k + c, \tag{7}
\]
where \( \pi \) is a permutation of \( \{1, 2, \cdots, m\} \), and \( c_k, c \in \{0, 1\} \). We define another Boolean function \( f_2(x) = f_1(x) + x_{\pi(1)} + c', c' \in \{0, 1\} \). The sequences \( f_1 \) and \( f_2 \) correspond to \((1, -1)\)-sequences of Boolean functions \( f_1 \) and \( f_2 \), respectively. Then the sequence pair \((f_1, f_2)\) is a GCP of length \( 2^m \).
Lemma 2. [4] Let \( Q \) be an \( n \)-variable Boolean function and \( \overline{Q} \) be its reversal. For \( J = (j_0, j_1, \cdots, j_{k-1}) \) and \( x_J = (x_{j_0}, x_{j_1}, \cdots, x_{j_{k-1}}) \), if \( G(Q|_{x_J=c}) \) is a path, then...
Define 4-variable Boolean functions as follows:
\[ Q(x) = \sum_{a=0}^{n-k-2} x_{\pi(a)} x_{\pi(a+1)} + \sum_{b=0}^{n-1} u_b x_b + u, \]
where \( \pi \) is a permutation of \( \{0, 1, \ldots, n-k-1\} \), and \( u_b, u \in \{0, 1\}, 0 \leq b \leq n-k-1 \).
Let \( (e_{k-1}, e_{k-2}, \ldots, e_0) \) be the binary representation of integer \( e = \sum_{i=0}^{n-k-1} e_i 2^i \). Define sequence sets \( \mathcal{C}_c = \{e_0, e_1, \ldots, e_{2^k-1}\} \) by
\[
\left\{ Q(x) + \sum_{a=0}^{k-1} d_\alpha x_{j_a} + d x_\gamma \mid d, d_\alpha \in \{0, 1\} \right\},
\]
and \( \mathcal{C}_{2^k+c} = \{e_{2^k+c0}, e_{2^k+c1}, \ldots, e_{2^k+c2^k-1}\} \) by
\[
\left\{ Q(x) + \sum_{a=0}^{k-1} d_\alpha x_{j_a} + d x_\gamma \mid \overline{d}, d_\alpha \in \{0, 1\} \right\},
\]
where \( \gamma \) is the either end vertex in the path, and \( x = 1 + x \). The set \( \mathcal{C} = \{\mathcal{C}_c, \mathcal{C}_{2^k+c}\} \) generates a \((2^{k+1}, 2^{k+1}, 2^n)\)-CCC.

**Lemma 3.** Define 4-variable Boolean functions as follows:
\[
\begin{align*}
f_1(x_1) &= x_3 x_1 x_0 + x_3 x_0 + x_2 x_0 + x_2, \\
f_2(x_1) &= x_3 x_2 + x_3 x_1 + x_1 x_0 + x_0, \\
f_3(x_1) &= x_3 x_2 + x_1 x_0 + x_3 + x_2 + x_1 + x_0, \\
f_4(x_1) &= x_3 x_1 x_0 + x_3 x_0 + x_2 x_0 + x_3 x_1 + x_3 + x_1,
\end{align*}
\]
where \( x_1 = (x_3, x_2, x_1, x_0) \in \mathbb{F}_2^4 \). Let \( g(x_2), g(x_2) + x_{\pi(1)} \) denote the GCP given in Lemma 1, and \( (\mathfrak{f}(x_2)) + x_{\pi(1)}, \mathfrak{g}(x_2)) \) be the mate of \( (g(x_2), g(x_2) + x_{\pi(1)}) \), where \( x_2 \in \mathbb{F}_2^n \). Then the constructed sequence set \( \mathcal{T} = \{T_1, T_2, T_3, T_4\} \) is a \((4, 4, 14 \cdot 2^m, 12 \cdot 2^m)\)-ZCSS, where
\[
T_1 = \begin{pmatrix}
f_1(x_1) + g(x_2) \\
f_2(x_1) + g(x_2) \\
f_3(x_1) + g(x_2) + x_{\pi(1)} \\
f_4(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix},
\]
\[
T_2 = \begin{pmatrix}
f_3(x_1) + g(x_2) \\
f_4(x_1) + g(x_2) \\
f_5(x_1) + g(x_2) + x_{\pi(1)} \\
f_6(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix},
\]
\[
T_3 = \begin{pmatrix}
f_1(x_1) + g(x_2) \\
f_2(x_1) + g(x_2) \\
f_3(x_1) + g(x_2) + x_{\pi(1)} \\
f_4(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix},
\]
\[
T_4 = \begin{pmatrix}
f_3(x_1) + g(x_2) \\
f_4(x_1) + g(x_2) \\
f_5(x_1) + g(x_2) + x_{\pi(1)} \\
f_6(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix}.
\]
\[ T_3 = \begin{pmatrix}
    f_1(x_1) + g(x_2) + x_{\pi(1)} \\
    f_2(x_1) + g(x_2) + x_{\pi(1)} \\
    f_3(x_1) + g(x_2) + x_{\pi(1)} \\
    f_4(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix}, \]

\[ T_4 = \begin{pmatrix}
    f_3(x_1) + g(x_2) + x_{\pi(1)} \\
    f_4(x_1) + g(x_2) + x_{\pi(1)} \\
    f_5(x_1) + g(x_2) + x_{\pi(1)} \\
    f_6(x_1) + g(x_2) + x_{\pi(1)}
\end{pmatrix}. \]

\textit{Proof.} For 4-variable Boolean functions \( f_1(x_1) \) and \( f_2(x_1) \), the corresponding truth tables are \([000010100100111] \) and \([010001000111100] \) with 14 length, respectively. The corresponding truncated sequence of Boolean functions \( f_1 \) and \( f_2 \) are \( f_1 \) and \( f_2 \) with 14 length, respectively. We have \( f_1 = (+ + + + - - - - + + + + - -) \), and \( f_2 = (+ + + + - - - - + + + + - -) \).

For \( 1 \leq \tau \leq 11 \), we have \( R_{f_1}(\tau) + R_{f_2}(\tau) = 0 \). For \( \tau = 12 \), we have \( |R_{f_1}(\tau) + R_{f_2}(\tau)| = 4 \).

For sequence set \( T_1 \), the AACF of \( T_1 \) is \( R_{T_1}(\tau) = R_{T_1}(2^m \tau_1 + \tau_2) \), where \( \tau = 2^m \tau_1 + \tau_2, 0 \leq \tau_1 \leq 13, \) and \( 0 \leq \tau_2 \leq 2^m - 1 \).

\[ R_{T_1}(\tau) = R_{f_1(x_1) + g(x_2) + x_{\pi(1)}}(\tau) + R_{f_2(x_1) + g(x_2) + x_{\pi(1)}}(\tau) + R_{f_3(x_1) + g(x_2) + x_{\pi(1)}}(\tau) + R_{f_4(x_1) + g(x_2) + x_{\pi(1)}}(\tau) + R_{f_5(x_1) + g(x_2) + x_{\pi(1)}}(\tau) + R_{f_6(x_1) + g(x_2) + x_{\pi(1)}}(\tau).
\]

Since \((g(x_2), g(x_2) + x_{\pi(1)})\) is a GCP according to Lemma 1, we have \( R_{g(x_2)}(\tau_2) + R_{g(x_2) + x_{\pi(1)}}(\tau_2) = 0 \), for \( 1 \leq \tau_2 \leq 2^m - 1 \). We can deduce \( R_{T_1}(\tau) = 0 \), for \( 1 \leq \tau \leq 12 \cdot 2^m - 1 \). For other cases, we can similarly deduce \( R_{T_1}(\tau) = 0 \), and \( R_{T_2, T_3}(\tau) = 0 \), where \( 1 \leq \tau \leq 12 \cdot 2^m - 1 \). Then we have the sequence set \( T \) is a is a \( (4, 4, 14 \cdot 2^m, 12 \cdot 2^m) \)-ZCSS.

\section{A Construction of Two-Dimensional Z-complementary Array Sets}

In this section, we first construct a class of 2-D ZCASs with non-power-of-two lengths based on 2-D Boolean functions. We focus on binary 2-D ZCASs with new lengths and large ZCZ widths throughout this paper. Then we give a comparison of our proposed constructions with previous work. 2-D ZCASs can have potential applications in 2-D MC-CDMA system.
3.1 Proposed Construction of 2-D ZCAS

Theorem 1. Let $Q(y)$ be an $n$-variable Boolean function, which satisfies

$$Q|_{y=c} = \sum_{a=0}^{n-k-2} x_{\pi(a)} x_{\pi(a+1)} + \sum_{b=0}^{n-1} u_b x_b,$$

where $y \in \mathbb{F}_2^n$, and $y = \{y_1, y_{n-1}\}$. We define a Boolean function $g_{t,l}(y)$, $0 \leq t \leq 2^{k+1} - 1$ and $0 \leq l \leq 2^{k+1} - 1$, i.e.,

$$g_{t,l}(y) = \begin{cases}
Q(y) + \sum_{b=0}^{n-1} u_b y_b + \\
\sum_{\alpha=0}^{k-1} (d_{\alpha} + l_{\alpha}) y_{\alpha} + dy_t, & \text{for } 0 \leq t \leq 2^k - 1,
\end{cases}$$

$$\begin{cases}
Q(y) + \sum_{b=0}^{n-1} u_b y_b + \\
\sum_{\alpha=0}^{k-1} (d_{\alpha} + l_{\alpha}) y_{\alpha} + dy_t, & \text{for } 2^k \leq t < 2^{k+1}.
\end{cases}$$

Let $x = (x_1, x_2) = (x_{m+4}, x_{m+3}, \ldots, x_1) \in \mathbb{F}_2^{m+4}$, where $x_1 = (x_{m+4}, x_{m+3}, x_{m+2}, x_{m+1}) \in \mathbb{F}_2^4$, and $x_2 = (x_m, x_{m-1}, \ldots, x_1) \in \mathbb{F}_2^m$. We define an $m$-variable Boolean function $f(x) = \sum_{k=1}^{m} x_{\pi(k)} x_{\pi(k+1)} + \sum_{k=1}^{m} c_k x_k + c$, where $\pi$ is a permutation of $\{1, 2, \cdots, m\}$, and 4-variable Boolean functions given by

$$f_1(x_1) = x_{m+4} x_{m+2} x_{m+1} + x_{m+4} x_{m+1} + x_{m+3} x_{m+1} + x_{m+3},$$

$$f_2(x_1) = x_{m+4} x_{m+3} + x_{m+4} x_{m+2} + x_{m+2} x_{m+1} + x_{m+1},$$

$$f_3(x_1) = x_{m+4} x_{m+3} + x_{m+4} x_{m+2} + x_{m+4} + x_{m+3} + x_{m+2} + x_{m+1},$$

$$f_4(x_1) = x_{m+4} x_{m+2} x_{m+1} + x_{m+4} x_{m+1} + x_{m+3} x_{m+1} + x_{m+4} x_{m+2} + x_{m+4} + x_{m+2}.$$

The array sets are represented as follows

$$S_t = \{s_{t,0}, s_{t,1}, \cdots, s_{t,2^{k+3}-1}\}$$

$$= \{g_{t,l}(y) + f_1(x_1) + f(x_2),$$

$$g_{t,l}(y) + f_2(x_1) + f(x_2),$$

$$g_{t,l}(y) + f_1(x_1) + f(x_2) + x_{\pi(1)},$$

$$g_{t,l}(y) + f_2(x_1) + f(x_2) + x_{\pi(1)},$$

$$g_{t+2^{k+1},l}(y) + f_1(x_1) + f(x_2),$$

$$g_{t+2^{k+1},l}(y) + f_2(x_1) + f(x_2),$$

$$g_{t+2^{k+1},l}(y) + f_3(x_1) + f(x_2),$$

$$g_{t+2^{k+1},l}(y) + f_4(x_1) + f(x_2),$$

$$g_{t+2^{k+1},l}(y) + f_1(x_1) + f(x_2),$$

$$g_{t+2^{k+1},l}(y) + f_2(x_1) + f(x_2).$$
where $0 \leq \text{each constituent array set } A$

Each array set $S_1$ contains $2^{k+3}$ arrays, where $0 \leq i \leq 2^{k+3} - 1$. Define the array set $A = \{A_0, A_1, \cdots, A_{2^{k+3} - 1}\} = \{S_0^{(2^n \times 14 \cdot 2^m)}, S_1^{(2^n \times 14 \cdot 2^m)}, \cdots, S_{2^{k+3} - 1}^{(2^n \times 14 \cdot 2^m)}\}$, where each constituent array set $A_i = \{a_i \mid 0 \leq i \leq 2^{k+3} - 1\} = \{s_i^{(2^n \times 14 \cdot 2^m)} \mid 0 \leq i \leq 2^{k+3} - 1\}$. Then the constructed array set $A$ forms a 2-D $(2^{k+3}, 2^{2k+3}, 14 \cdot 2^m, 2^n, 12 \cdot 2^m)$-ZCAS.

Proof. Without loss of generality, we determine the ACF of constituent array set $A_0$, and ACCF between array set $A_0$ and array set $A_1$.

We define $(m + 4)$-variable Boolean functions

$$h_1(x) = f_1(x_1) + f(x_2), \quad h_2(x) = f_2(x_1) + f(x_2),$$
$$h_3(x) = h_1(x) + x_\pi(1), \quad h_4(x) = h_2(x) + x_\pi(1),$$

where $x = (x_1, x_2) \in \mathbb{F}_2^{m+4}$.

By the definition of the array set $A_0$,

$$A_0 = \{a_0, a_1, \cdots, a_{2^{k+3} - 1}\}$$
$$= \{g_0, g_1, g_2, g_3, g_4\},$$

where $0 \leq l \leq 2^{k+1} - 1$. 

$$g_{l,1}(y) + f_3(x_1) + f(x_2) + x_\pi(1),$$
$$g_{l,1}(y) + f_4(x_1) + f(x_2) + x_\pi(1).$$
For $0 \leq |\tau_1| \leq 2^n - 1$ and $0 \leq |\tau_2| \leq 14 \cdot 2^m - 1$, we have

$$R_{A_0}(\tau_1, \tau_2) = \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_1(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_2(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_3(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_4(x)}(\tau_2).$$

According to Lemma 3, for $0 < |\tau_2| \leq 12 \cdot 2^m - 1$, we have $R_{h_1(x)}(\tau_2) + R_{h_2(x)}(\tau_2) + R_{h_3(x)}(\tau_2) + R_{h_4(x)}(\tau_2) = 0$. Since $g_0(l)$ is the constituent sequence of CCC, we have $\sum_{l=0}^{2^{k+1}-1} R_{g_0(l)g_1}(\tau_1) = 0$, for $0 < |\tau_1| \leq 2^n - 1$. For $0 \leq |\tau_1| \leq 2^n - 1$, $0 \leq |\tau_2| \leq 12 \cdot 2^m - 1$, and $(\tau_1, \tau_2) \neq (0, 0)$, we can deduce $R_{A_0}(\tau_1, \tau_2) = 0$.

By the definition of the array set $A_1$,

$$A_1 = \{a_{1,0}, a_{1,1}, \cdots, a_{1,2^{k+1} - 1}\}$$

$$= \{g_1(x) + h_1(x), g_1(x) + h_2(x), g_1(x) + h_3(x), g_1(x) + h_4(x)\},$$

where $0 \leq l \leq 2^{k+1} - 1$. Then we have

$$R_{A_0,A_1}(\tau_1, \tau_2) = \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_1(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_2(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_3(x)}(\tau_2) + \sum_{l=0}^{2^{k+1} - 1} R_{g_0(l)g_1}(\tau_1) \cdot R_{h_4(x)}(\tau_2).$$

According to Lemma 3, for $0 < |\tau_2| \leq 12 \cdot 2^m - 1$, $R_{h_1(x)}(\tau_2) + R_{h_2(x)}(\tau_2) + R_{h_3(x)}(\tau_2) + R_{h_4(x)}(\tau_2) = 0$ holds. Since $g_0(l)$ and $g_1(x)$ are the constituent sequences of CCC, we have $\sum_{l=0}^{2^{k+1}-1} R_{g_0(l)g_1}(\tau_1) = 0$, for $0 \leq |\tau_1| \leq 2^n - 1$. Then for $0 \leq |\tau_1| \leq 2^n - 1$, and $0 \leq |\tau_2| \leq 14 \cdot 2^m - 1$, we deduce $R_{A_0,A_1}(\tau_1, \tau_2) = 0$. $\square$
Corollary 1. For the constructed 2-D \((2^{k+3}, 2^{k+3}, 2^n, 14 \cdot 2^m, 12 \cdot 2^m)\)-ZCAS in Theorem 1, the row sequences of array set \(A\) have PMEPR at most 2.57.

Proof. Without loss of generality, for the array set \(A\), we consider the constituent array \(A_0\). The row sequences of array \(A_0\) consist of \(f_1(x_1) + f(x_2), f_2(x_1) + f(x_2), f_1(x_1) + f(x_2) + x_{\pi(1)}, \) and \(f_2(x_1) + f(x_2) + x_{\pi(1)}\). The sequence pairs \((f_1(x_1) + f(x_2), f_1(x_1) + f(x_2) + x_{\pi(1)})\) and \((f_2(x_1) + f(x_2), f_2(x_1) + f(x_2) + x_{\pi(1)})\) form \((14 \cdot 2^m, 12 \cdot 2^m)\)-ZCPs, respectively.

\[
|R_{f_1(x_1)+f(x_2),f_1(x_1)+f(x_2)+x_{\pi(1)}}(\tau)| = \begin{cases} 0, & \text{for } 1 \leq \tau \leq 12 \cdot 2^m - 1, \\ 4 \cdot 2^m, & \text{for } \tau = 12 \cdot 2^m, \\ 0, & \text{for } 12 \cdot 2^m + 1 \leq \tau \leq 14 \cdot 2^m - 1. \end{cases}
\]

According to the Equation 5, we can deduce the row sequence PMEPR of array \(A_0\) is shown as follows,

\[
PMEPR(A_0) = \frac{1}{14 \cdot 2^m} \sup_{0 \leq t \leq 1/\Delta f} |s_{A_0}(t)|^2 \leq \frac{1}{14 \cdot 2^m} (14 \cdot 2^m + 14 \cdot 2^m + 8 \cdot 2^m) = 2.57.
\]

For other array \(A_i\), we also have the PMEPR at most 2.57. We obtain the row sequences of array set \(A\) have PMEPR at most 2.57. \(\square\)

Corollary 2. For the constructed 2-D \((2^{k+3}, 2^{k+3}, 2^n, 14 \cdot 2^m, 12 \cdot 2^m)\)-ZCAS in Theorem 1, the column sequences of array set \(A\) have PMEPR at most \(2^{k+1}\).

The proof of Corollary 2 is similar to that of Corollary 1 and we omit it.

Theorem 2. Let \(Q(y)\) be an \(n\)-variable Boolean function, which satisfies

\[
Q|_{y_j=c} = \sum_{a=0}^{n-2-k} x_{\pi(a)}x_{\pi(a+1)} + \sum_{b=0}^{n-1} u_b x_b,
\]

where \(y \in \mathbb{F}_2^n\) and \(y = \{y_1, y_{n-1}\}\). We define a Boolean function \(g_{t,l}(y)\), \(0 \leq t \leq 2^{k+1} - 1\) and \(0 \leq l \leq 2^{k+1} - 1\), i.e.,

\[
g_{t,l}(y) = \begin{cases} (Q(y) + \sum_{b=0}^{n-1} u_b y_b + \sum_{a=0}^{k-1} (d_a + t_a) y_a, + dy_\gamma, & \text{for } 0 \leq t \leq 2^k - 1, \\ (\overline{Q}(y) + \sum_{b=0}^{n-1} u_b \overline{y}_b + \sum_{a=0}^{k-1} (d_a + t_a) \overline{y}_a, + d\overline{y}_\gamma, & \text{for } 2^k \leq t < 2^{k+1}. \end{cases}
\]
where \((d_0, d_1, \ldots, d_{k-1}, \bar{d})\) is the binary representation of integer \(l\), \(0 \leq l \leq 2^{k+1} - 1\). Define 4-variable Boolean functions by

\[
\begin{align*}
    f_1(x) &= x_3x_1x_0 + x_3x_0 + x_2x_0 + x_2, \\
    f_2(x) &= x_3x_2 + x_3x_1 + x_1x_0 + x_0, \\
    f_3(x) &= x_3x_2 + x_1x_0 + x_3 + x_2 + x_1 + x_0, \\
    f_4(x) &= x_3x_1x_0 + x_3x_0 + x_2x_0 + x_3x_1 + x_3 + x_1.
\end{align*}
\]

For \(0 \leq l \leq 2^{k+1} - 1\), the array sets are given as follows

\[
\begin{align*}
    S_l &= \{s_{l,0}, s_{l,1}, \ldots, s_{l,2^{k+2} - 1}\} \\
    &= \{g_{l,i}(y) + f_1(x), g_{l,i}(y) + f_2(x)\}, \\
    S_{l+2^{k+1}} &= \{s_{l+2^{k+1},0}, s_{l+2^{k+1},1}, \ldots, s_{l+2^{k+1,2^{k+2} - 1}}\} \\
    &= \{g_{l,i}(y) + f_3(x), g_{l,i}(y) + f_4(x)\}.
\end{align*}
\]

For \(0 \leq i \leq 2^{k+2} - 1\), each array set \(S_i\) is comprised of \(2^{k+2}\) constituent arrays. We define the 2-D array set by \(A = \{A_0, A_1, \ldots, A_{2^{k+2}-1}\} = \{S_0^{(2^{n+4})}, S_1^{(2^{n+4})}, \ldots, S_{2^{k+2}-1}^{(2^{n+4})}\}\), where each constituent array set \(A_r = \{a_{r,i}\} = \{s_r^{(2^{n+4})}\}\). Then the array set \(A\) is a 2-D \(2^{k+2}, 2^{k+2}, 2^n, 14, 2^n, 12\)-ZCAS.

The proof of Theorem 2 is similar to that of Theorem 1 and we omit it.

Corollary 3. For the constructed 2-D \(2^{k+2}, 2^{k+2}, 2^n, 14, 2^n, 12\)-ZCAS in Theorem 2, the row sequences of array set \(A\) have PMEPR at most 2.57, and the column sequences of array set \(A\) have PMEPR at most 2.57.

Example 2. Let \(n = 3\). For \(t = 0\), and \(l = 0\), define the Boolean function \(g_{0,0}(y) = y_0y_1 + y_0y_2 + y_1y_2\), where \(y = (y_0, y_1, y_2) \in \mathbb{F}_2^3\). The truth table of \(g_{0,0}(y)\) is \([00010111]\). The set \(C = \{C_0, C_1, C_2, C_3\}\) generates \((4, 4, 8)\)-CCC in Lemma 2 as follows:

\[
C_0 = \begin{bmatrix} + & + & + & + & + & + & + & - & + & + & + & + & + \end{bmatrix}, \quad C_1 = \begin{bmatrix} + & - & + & + & + & + & + & - \end{bmatrix},
\]

\[
C_2 = \begin{bmatrix} + & + & + & + & + & + & + & - \end{bmatrix}, \quad C_3 = \begin{bmatrix} + & - & - & - & - & - & - & + \end{bmatrix}.
\]

The array sets \(A = \{A_0, A_1, \ldots, A_7\} = \{S_0^{(8x14)}, S_1^{(8x14)}, \ldots, S_7^{(8x14)}\}\), where each constituent array set \(A_r = \{a_{r,i}\} = \{s_r^{(8x14)}\}\), \(0 \leq n \leq 7\). For simplicity, we denote \(s_r^{(8x14)}\) by \(s_{n,i}\). For \(n = 0\), the array set \(A_0 = \{s_{0,0}, s_{0,1}, s_{0,2}, s_{0,3}, s_{0,4}, s_{0,5}, s_{0,6}, s_{0,7}\}\).
The constituent array $s_{0,i}$ is shown as follows,

\[
\begin{array}{c}
s_{0,0} = \\
s_{0,1} = \\
s_{0,2} = \\
s_{0,3} = \\
s_{0,4} = \\
s_{0,5} = \\
s_{0,6} = \\
s_{0,7} = \\
\end{array}
\]

Then we can deduce

\[
R_{A_0}(\tau_1, \tau_2) = \sum_{i=0}^{7} R_{s_{0,i}}(\tau_1, \tau_2)
\]

\[
= \begin{cases} 
896, & \text{for } \tau_1 = 0, \text{ and } \tau_2 = 0, \\
0, & \text{for } 0 \leq \tau_1 \leq 7, \text{ and } 0 \leq \tau_2 \leq 11, \text{ and } (\tau_1, \tau_2) \neq (0,0).
\end{cases}
\]
We have a similar representation of $A_0$ for all other array sets $A_i$. Then the array set $A = \{A_0, A_1, \ldots, A_7\}$ forms a 2-D $(8, 8, 14, 8, 12)$-ZCAS.

**Remark 1.** We directly construct 2-D ZCASs with large ZCZ width in Theorem 1 and Theorem 2, where $ZCZ_{ratio} = \frac{Z_1 \cdot Z_2}{L_1 \cdot L_2} = \frac{5}{7}$ with $Z_1 = L_1 = 2^n$, $Z_2 = 12 \cdot 2^m$, and $L_2 = 14 \cdot 2^m$. The direct construction of 2-D ZCAS has new length $14 \cdot 2^m$ and the largest ZCZ width. Note that the Boolean functions $g_{t,l}(y)$ in Theorem 1 and Theorem 2 correspond to the constituent sequence of CCC. The CCC with length $L_1 = 2^n$ is based on Boolean functions. We can replace $g_{t,l}(y)$ by the constituent sequence of ZCSSs which have more flexible lengths, such that the array length $L_1$ is non-power-of-two. We can obtain a new class of $(2^k, 2^k, L_1, L_2, Z_1, Z_2)$-ZCAS by using Boolean functions.

### 3.2 Comparison

In this subsection, we give a comparison of our proposed constructions with previous work in Table 1. In [30], Liu et al. proposed a class of $p$-ary GCAS, where $p$ is prime. In [25], Pai et al. obtained a class of GCAS with $2^m$ length by using Boolean functions. In [31], Shibsankar et al. presented a class of ZCAS by indirect method. In this paper, the direct construction of ZCASs have $2^n \times 14 \cdot 2^m$ array size and large ZCZ $(2^n, 12 \cdot 2^m)$ by employing Boolean function. Our construction doesn’t require initial matrices or sequences and produces more flexible parameters.

<table>
<thead>
<tr>
<th>Ref.</th>
<th>Methods</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>[30]</td>
<td>Indirect</td>
<td>$(p^k, p^k, p^m, p^m, p^m)$, $p$ is prime</td>
</tr>
<tr>
<td>[25]</td>
<td>Direct</td>
<td>$(2^k, 2^k, 2^m, 2^m, 2^m, 2^m)$</td>
</tr>
<tr>
<td>[31]</td>
<td>Indirect</td>
<td>$(K, M, K, K, K)$, $K = MP$</td>
</tr>
<tr>
<td>Th. 1</td>
<td>Direct</td>
<td>$(2^k, 2^k, 2^m, 14 \cdot 2^m, 2^m, 12 \cdot 2^m)$</td>
</tr>
</tbody>
</table>

### 4 A Construction of Other ZCASs

In this section, we construct a new class of ZCASs with ZCZ around the end-shift position. The constructed ZCASs have large ZCZ width and more flexible lengths. 

**Definition 8.** Let sequences $c = (c_0, c_1, \ldots, c_{N-1})$ and $a = (a_0, a_1, \ldots, a_{L-1})$. The array $A$ can be denoted as follows

$$A = c \otimes a = \begin{bmatrix}
    c_0 \cdot a_0 & c_0 \cdot a_1 & \cdots & c_0 \cdot a_{L-1} \\
    c_1 \cdot a_0 & c_1 \cdot a_1 & \cdots & c_1 \cdot a_{L-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    c_{N-1} \cdot a_0 & c_{N-1} \cdot a_1 & \cdots & c_{N-1} \cdot a_{L-1}
\end{bmatrix},$$

where $\otimes$ denotes the Kronecker product.
Theorem 3. Let a sequence set $C = \{C_0, C_1, \ldots, C_{N-1}\}$ be an $(N, N, L_1)$-CCC, where for $0 \leq i \leq N-1$, each sequence set $C_i = \{c_{n,0}, c_{n,1}, \ldots, c_{n,N-1}\}$, and for $0 \leq n \leq N-1$, the constituent sequence of each sequence set $c_{n} = (c_{n,0}, c_{n,1}, \ldots, c_{n,N-1})$. Let $(a_1, b_1)$ be a $(L_2, Z)$-CZP with ZCZ around the end-shift position, where $a_1 = (a_0, a_1, \ldots, a_{L_2-1})$, and $b_1 = (b_0, b_1, \ldots, b_{L_2-1})$. The $(L_2, Z)$-CZP $(a_2, b_2)$ with ZCZ around the end-shift position is the mate of $(a_1, b_1)$, where $a_2 = (b_{L_2-1}, b_{L_2-2}, \ldots, b_0)$, and $b_2 = (-a_{L_2-1}, -a_{L_2-2}, \ldots, -a_0)$.

For array set $S = \{S_0, S_1, \ldots, S_{2N-1}\}$, where constituent array set $S_i = \{s_i^n \mid 0 \leq n \leq 2N-1\}$, and $0 \leq i \leq 2N-1$.

For $0 \leq n \leq N-1$ and $0 \leq i \leq N-1$, the constituent array $s_i^n$ of array set $S_i$ can be denoted as follows,

$$s_i^n = c_i^n \otimes a_i = \begin{bmatrix}
c_i^n,0a_0 & c_i^n,0a_1 & \cdots & c_i^n,0a_{L_2-1} \\
c_i^n,1a_0 & c_i^n,1a_1 & \cdots & c_i^n,1a_{L_2-1} \\
\cdots \cdots \\
c_i^n,n_{L_2-1}a_0 & c_i^n,n_{L_2-1}a_1 & \cdots & c_i^n,n_{L_2-1}a_{L_2-1}
\end{bmatrix}.$$

For $N \leq n \leq 2N-1$ and $0 \leq i \leq N-1$, the constituent array $s_i^n$ can be denoted as follows,

$$s_i^n = c_i^n \otimes b_i = \begin{bmatrix}
c_i^n-N,0b_0 & c_i^n-N,0b_1 & \cdots & c_i^n-N,0b_{L_2-1} \\
c_i^n-N,1b_0 & c_i^n-N,1b_1 & \cdots & c_i^n-N,1b_{L_2-1} \\
\cdots \cdots \\
c_i^n-N,n_{L_2-1}b_0 & c_i^n-N,n_{L_2-1}b_1 & \cdots & c_i^n-N,n_{L_2-1}b_{L_2-1}
\end{bmatrix}.$$

For $0 \leq n \leq N-1$ and $N \leq i \leq 2N-1$, the constituent array $s_i^n$ can be denoted as follows,

$$s_i^n = c_i^n \otimes a_2 = \begin{bmatrix}
c_i^{i-N},0b_{L_2-1} & c_i^{i-N},0b_{L_2-1} & \cdots & c_i^{i-N},0b_0 \\
c_i^{i-N},1b_{L_2-1} & c_i^{i-N},1b_{L_2-1} & \cdots & c_i^{i-N},1b_0 \\
\cdots \cdots \\
c_i^{i-N},n_{L_2-1}b_{L_2-1} & c_i^{i-N},n_{L_2-1}b_{L_2-1} & \cdots & c_i^{i-N},n_{L_2-1}b_0
\end{bmatrix}.$$

For $N \leq n \leq 2N-1$ and $N \leq i \leq 2N-1$, the constituent array $s_i^n$ can be denoted as follows,

$$s_i^n = c_i^n \otimes b_2 = \begin{bmatrix}
-c_i^{i-N},0a_{L_2-1} & -c_i^{i-N},0a_{L_2-1} & \cdots & -c_i^{i-N},0a_0 \\
-c_i^{i-N},1a_{L_2-1} & -c_i^{i-N},1a_{L_2-1} & \cdots & -c_i^{i-N},1a_0 \\
\cdots \cdots \\
-c_i^{i-N},n_{L_2-1}a_{L_2-1} & -c_i^{i-N},n_{L_2-1}a_{L_2-1} & \cdots & -c_i^{i-N},n_{L_2-1}a_0
\end{bmatrix}.$$

Then the array set $S$ is a $(2N, 2N, L_1, L_2, L_1, Z)$-ZCAS with ZCZ around the end-shift position.
Proof. Without loss of generality, we deduce the AACF of the array set \( S_0 \). For the \( S_0 \), we have
\[
R_{S_0}(\tau_1, \tau_2) = \sum_{n=0}^{2N-1} R_{S_0}^n(\tau_1, \tau_2)
\]
\[
= R_{a_1}(\tau_2) \sum_{n=0}^{N-1} c_{n,0}^0 a_{n,\tau_1} a_{\tau_2} + R_{b_1}(\tau_2) \sum_{n=N}^{2N-1} c_{n,0}^0 a_{n-\tau_1} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1}
\]
\[
= c_{n,0}^0 a_{n,\tau_1} a_{\tau_2} = R_{a_1}(\tau_2).
\]
Then for the array set \( S_0 \), we have
\[
R_{S_0}(\tau_1, \tau_2) = \sum_{n=0}^{2N-1} R_{S_0}^n(\tau_1, \tau_2)
\]
\[
= R_{a_1}(\tau_2) \sum_{n=0}^{N-1} c_{n,0}^0 a_{n,\tau_1} a_{\tau_2} + R_{b_1}(\tau_2) \sum_{n=N}^{2N-1} c_{n,0}^0 a_{n-\tau_1} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1} + \cdots + c_{-n,0}^0 a_{n-\tau_2} a_{\tau_2-1}
\]
\[
= R_{a_1}(\tau_2) R_{C_0}(\tau_1) + R_{b_1}(\tau_2) R_{C_0}(\tau_1)
\]
\[
= R_{C_0}(\tau_1)(R_{a_1}(\tau_2) + R_{b_1}(\tau_2)).
\]
Since \( C = \{ C_0, C_1, \cdots, C_{N-1} \} \) is an \((N, N, L_1)\)-CCC, we have \( R_{C_0}(\tau) = \sum_{n=0}^{N-1} R_{C_0}(\tau) = 0 \), where \( 1 \leq \tau \leq L_1 - 1 \). For \((a_1, b_1)\) being a ZCP with ZCZ around the end-shift position, we have \( R_{a_1}(\tau) + R_{b_1}(\tau) = 0 \), where \( L_2 - Z + 1 \leq \tau \leq L_2 - 1 \). Then we can deduce \( R_{S_0}(\tau_1, \tau_2) = 0 \), where \( 1 \leq \tau_1 \leq L_1 - 1 \), and \( L_2 - Z + 1 \leq \tau_2 \leq L_2 - 1 \).

For the ACCF of array sets \( S_0 \) and \( S_1 \), we have
\[
R_{S_0,S_1}(\tau_1, \tau_2) = \sum_{n=0}^{2N-1} R_{S_0,S_1}^n(\tau_1, \tau_2)
\]
\[
= R_{a}(\tau_2) \sum_{n=0}^{N-1} c_{n,0}^0 c_{n,\tau_1} c_{n,\tau_2} + R_{b}(\tau_2) \sum_{n=N}^{2N-1} c_{n,0}^0 c_{n-\tau_1} c_{n-\tau_2} + \cdots + c_{-n,0}^0 c_{-n,\tau_1} c_{-n,\tau_2} + \cdots + c_{-n,0}^0 c_{-n,\tau_1} c_{-n,\tau_2} + \cdots + c_{-n,0}^0 c_{-n,\tau_1} c_{-n,\tau_2}
\]
\[
= R_{a}(\tau_2) R_{C_0,c_1}(\tau_1) + R_{b}(\tau_2) R_{C_0,c_1}(\tau_1)
\]
\[
= R_{C_0,c_1}(\tau_1)(R_{a}(\tau_2) + R_{b}(\tau_2)).
\]
Since \( C = \{ C_0, C_1, \cdots, C_{N-1} \} \) is an \((N, N, L_1)\)-CCC, we have \( R_{C_0,c_1}(\tau) = \sum_{n=0}^{N-1} R_{C_0,c_1}(\tau) = 0 \), where \( 0 \leq \tau \leq L_1 - 1 \). Then we can deduce \( R_{S_0,S_1}(\tau_1, \tau_2) = 0 \), where \( 0 \leq \tau_1 \leq L_1 - 1 \), and \( 0 \leq \tau_2 \leq L_2 - 1 \).

Similarly, we can also obtain \( R_{S_0,S_1}(\tau_1, \tau_2) = 0 \), for \( 0 \leq \tau_1 \leq L_1 - 1 \), and \( L_2 - Z + 1 \leq \tau_2 \leq L_2 - 1 \).

Therefore, we can deduce that the array set \( S \) is a \((2N, 2N, L_1, L_2, L_1, Z)\)-ZCAS with ZCZ around the end-shift position. 

\[ \square \]
Example 3. Let $N = 4$, and $L_1 = 8$. We can obtain the sequence set $\mathbf{C} = \{C_0, C_1, C_2, C_3\}$ that is a $(4, 4, 8)$-CCC as follows:

\[
\]

\[
\]

Let a sequence pair $(a_1, b_1)$ be a $(14, 13)$-ZCP with ZCZ around the end-shift position, where $a_1 = (+++++++-+----)$, and $b_1 = (+++++++-+----)$. The sequence pair $(a_2, b_2)$ is the mate of $(a_1, b_1)$, where $a_2 = (+++++++-+----)$, and $b_2 = (+++++++-+----)$.

The array set $\mathbf{S} = \{S_0, S_1, \cdots, S_7\}$, where constituent array set $S_i = \{s_{i,0}, s_{i,1}, \cdots, s_{i,7}\}$. The array set $S_0$ is denoted as follows,

\[
s_{0,0} = \begin{bmatrix} + & + & + & + & - & - & - & - \\ + & + & + & - & - & - & + & + \\ + & + & - & + & - & + & + & + \\ - & + & - & + & + & + & - & - \end{bmatrix},
\]

\[
\]

\[
\]

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Proof. \end-shift position.

Theorem 4. We have a similar representation of \( S \). Then we can deduce

\[
R_{s_i}(\tau_1, \tau_2) = \sum_{i=0}^{7} R_{s_{0,i}}(\tau_1, \tau_2)
\]

\[
= \left\{
\begin{array}{ll}
896, & \text{for } \tau_1 = 0, \text{ and } \tau_2 = 0, \\
0, & \text{for } 0 \leq \tau_1 \leq 7, 2 \leq \tau_2 \leq 13.
\end{array}
\right.
\]

We have a similar representation of \( S_0 \) for all other \( S_i \). Then the array set \( S = \{S_0, S_1, \cdots, S_7\} \) forms a \((8,8,8,14,8,13)\)-ZCAS with ZCZ around the end-shift position.

Theorem 4. Let a sequence set \( C = \{C_0, C_1, \cdots, C_{N-1}\} \) be an \((N,N,L_1)\)-CCC, where for \( 0 \leq i \leq N - 1 \), each sequence set \( C_i = \{C_{0,i}, C_{1,i}, \cdots, C_{N-1,i}\} \), and for \( 0 \leq n \leq N - 1 \), the constituent sequence of each sequence set \( C_n = \{C_{n,0}, C_{n,1}, \cdots, C_{n,L_1-1}\} \).

Let \( A = \{A_0, A_1, \cdots, A_{M-1}\} \) be a \((M,M,L_2,Z)\)-ZCSS with ZCZ around the end-shift position, where for \( 0 \leq j \leq M - 1 \), each sequence set \( A_j = \{A_{0,j}, A_{1,j}, \cdots, A_{M-1,j}\} \), and for \( 0 \leq m \leq M - 1 \), the constituent sequence of each sequence set \( A_m = \{a_{m,0}, a_{m,1}, \cdots, a_{m,L_2-1}\} \).

For array set \( S = \{S_0, S_1, \cdots, S_{N,M-1}\} \), where \( S_u = \{s_{v,u} \mid 0 \leq v \leq N M - 1, \text{ and } 0 \leq u \leq N M - 1\} \), for \( u = N j + i, \text{ and } v = N m + n, \) we have

\[
s_{v,u} = C_n \otimes A_m
\]

\[
= \left[
\begin{array}{cccc}
{c_{i,0}} a_{m,0} & {c_{i,1}} a_{m,1} & \cdots & {c_{i,L_1-1}} a_{m,L_1-1} \\
{c_{i,0}} a_{m,1} & {c_{i,1}} a_{m,1} & \cdots & {c_{i,L_1-1}} a_{m,L_1-1} \\
\vdots & \vdots & \ddots & \vdots \\
{c_{i,0}} a_{m,L_1-1} & {c_{i,1}} a_{m,L_1-1} & \cdots & {c_{i,L_1-1}} a_{m,L_1-1}
\end{array}
\right].
\]

Then the array set \( S \) is an \((N M, N M, L_1, L_2, L_1, Z)\)-ZCAS with ZCZ around the end-shift position.

Proof. For the AACF of the array \( s_{v,u} \), we have

\[
R_{s_{v,u}}(\tau_1, \tau_2) = {c_{i,0}} a_{m,0} {c_{i,1}} a_{m,1} a_{m,\tau_2} + {c_{i,0}} a_{m,1} {c_{i,1}} a_{m,\tau_2+1} + \cdots + {c_{i,0}} a_{m,L_2-\tau_2-1} {c_{i,1}} a_{m,L_2-1} +
\]

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In this paper, we give the direct construction of 2-D ZCAS with length 14.

Conclusions

Remark 3. For $(M, M, L_2, Z)$-ZCSS with ZCZ around the end-shift position, we can obtain more flexible lengths, such as $12 \cdot 2^m$, $14 \cdot 2^m$, etc. For the constructed ZCAS of length $12 \cdot 2^m$ and $14 \cdot 2^m$ with ZCZ around the end-shift position, we have the largest $ZCZ_{ratio} = \frac{14}{12}$ and $ZCZ_{ratio} = \frac{14}{13}$, respectively. To the authors’ knowledge, both of the constructions have not been reported before.

5 Conclusions

In this paper, we give the direct construction of 2-D ZCAS with length $14 \cdot 2^m$ based on 2-D Boolean functions, which has largest $ZCZ_{ratio} = \frac{9}{5}$ in Theorem 1 and Theorem 2.

Next, we provide a construction of ZCASs with ZCZ around the end-shift position by...
using concatenation, which have largest $ZCZ_{ratio} = \frac{11}{12}$ and $ZCZ_{ratio} = \frac{13}{14}$. We have shown that our proposed ZCASs with new lengths and the large ZCZ width compared with existing constructions.

6 Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Chunlei Xie: Conceptualization, Methodology, Writing original draft. Chenrui Li: Writing review and editing. Yu Sun: Writing review and editing. Ziling Heng: Writing review and editing. Yang Ming: Writing review and editing. All authors reviewed the manuscript.

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