Random Informative Advertising with Vertically Differentiated Products

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Random Informative Advertising with Vertically Differentiated Products*

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Abstract

We study a simple model in which two vertically differentiated firms compete in prices and mass advertising on an initially uninformed market. Consumers differ in their preference for quality. There is an upper bound on prices since consumers cannot spend more on the good than a fixed amount (say their income). Depending on this income and on the ratio between advertising cost and quality differential (relative advertising cost), either there is no equilibrium in pure strategies or there exists one of the following three types: 1) an Interior equilibrium where both firms have positive natural markets and charge prices lower than the consumer’s income; 2) a Constrained interior equilibrium where both firms have positive natural markets and the high-quality firm charges the consumer’s income or 3) a Corner equilibrium where the low-quality firm has no natural market selling only to uninformed customers. We show that no corner equilibrium exists in which the high-quality firm would have a null natural market. At an equilibrium (whenever there exists one), the high-quality firm always advertises more, charges a higher price and makes a higher profit than the low-quality one. As the relative advertising cost goes to infinity, prices become equal and the advertising intensities converge to zero as well as the profits. Finally, the advertising intensities are, at least globally, increasing with the quality differential.

Keywords: random advertising, Advertising cost, Vertical differentiation

JEL classification: D83, L13, M37.

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1 Introduction

The global advertising and communication market today weighs more than 1,370 billion dollars (i.e. approximately 1.5 per cent of global GDP, in 2019) and continues growing faster than the world GDP. Advertising appears to be a key factor in competition between firms. In France, for instance commercial communication "expenditure" strictly speaking (excluding human resources in particular) weighs 31 billion, which is nearly the equivalent of private investment in R&D (32 billion).

The question of differentiation and quality is a natural part of the debate. Advertising has been studied mainly in the case of horizontally differentiated markets. Only a few papers deal with the case of advertising in vertically differentiated markets, leaving pending important questions. We aim at filling the gap by studying random (or mass) advertising in a vertical differentiation model with price competition. We aim at determining whether the advertisement increases with the quality sold and whether an increase of the advertising cost has a differential or a similar impact on the firms’ prices, advertising intensities and profits.

Advertising is generally considered either as persuasive or informative. In the first case it does not provide any actual information on the product but tries to appeal to consumers’ desires and drives to have them buy the good. In the second one it provides information on the existence of the product, the price, the characteristics of the good and so on. Advertising is also often considered as a quality signal if higher quality products are more advertised than lower quality ones, in which case it is indirectly informative.

In this paper we deal with directly informative advertising in the framework of oligopolistic competition. We analyze a simple vertical differentiation duopoly model with a low-quality firm and a high-quality one where consumers differ in their preference for quality. The consumers are initially uninformed of the existence of the firms. When receiving an ad from one firm, they learn its existence, its price and its product quality.\footnote{We assume that this information, notably the one on product quality, can be trusted, for instance because false advertising is banned or because quality amounts to some verifiable characteristics.} We focus on random (or mass) advertising. Each firm chooses its price and its advertising intensity. We assume that consumers cannot spend on the good more than their (identical) income (or possibly a predetermined share of it), which puts an-upper bound on prices\footnote{While here this limit comes from the fact that prices cannot exceed consumers’ income, it could alternatively be derived from the existence of a maximum utility level from consuming the good.}

In the model we consider, consumers differ w.r.t. some intrinsic characteristic that we call “intensity of preference for quality” and that measures how strong a consumer is sensitive to quality, thus how much a priori s/he is willing to pay to acquire a better quality. Moreover we suppose consumers to be limited by a budget constraint or equivalently, that prices are upper-bounded by an exogenous limit. In doing so, we are supposing that the heterogeneity in intensity of preference for quality is not equivalent to heterogeneity in income. An abundant literature in marketing and psychology may found this hypothesis, mainly using motivation theory (Reeve, 2017). Any purchase occurs, as any behavior, to satisfy physical (hunger, thirst) or psychological needs (recognition, esteem, belonging). When the need is activated, the consumer experiences a state of tension driving the consumer to try to satisfy the need. The strength of the tension determines the intensity with which the individual is going to seek for the satisfaction of his/her need. Suppose the quality refers to environmental attributes, i.e. measures the effort made by the firm to respect the environment in all the process. The consumers differing in terms of individual and familial histories, physical and intellectual ca-
pabilities, cultural backgrounds, reaction and sensitivity to marketing, they would necessarily differ intrinsically w.r.t. the efforts they are willing to make to be friendly to the environment, possibly independently from their budget constraint or income. The intensity of preference for quality ($\theta$ in the model) representing the motives we have just described, is different in nature from the income ($y$) which represents what extrinsically limits the expenses. Considering both in the same model gives rise to interesting results we were not able to observe, have we considered only one of them.

First we characterize the firms’ choices at equilibrium. We show that depending on the consumer’s income and the ratio between advertising cost and quality differential (relative advertising cost), either there is no equilibrium in pure strategies or, one of three possible types of equilibrium holds. 1) Interior Equilibrium (IE) where both firms have positive “natural markets” and charge prices lower than the consumer’s income; 2) Constrained Interior Equilibrium (CIE) where both firms have positive natural markets and the high-quality firm charges the consumer’s income; 3) Corner equilibrium (COR) where the low-quality firm has no natural market. The necessity for the existence of an interior equilibrium of an upper-limit on prices is a first contribution to the existing literature. Once this upper limit is introduced, it plays an explicit role in the existence and the nature of the equilibrium, i.e., in particular it entails the possible existence of a corner equilibrium and of a constrained interior equilibrium. This is a second contribution of this paper. All these features have indeed been overlooked in the literature (see for instance Grossman and Shapiro, 1984, Tirole, 1988) and thus correspond to a specific contribution of this paper.

In the second place, we provide several comparative statics results at equilibrium, studying how the outcome at equilibrium varies with the relative advertising cost, for some given consumer’s income. Depending on this income, when this relative advertising cost varies, we may go through two or three regimes, and even go through a hole (with no equilibrium). Interestingly, contrary to what has been supposed by existing literature that considered only the interior equilibrium overlooking the problem of existence and the possible existence of equilibria of different types, the equilibrium may never be an interior one (for sufficiently low consumer’s income). For sufficiently high consumer’s income, the equilibrium is an interior equilibrium for low enough values of the relative advertising cost, but the type of equilibrium necessarily changes as the relative advertising cost goes beyond some threshold, and for still higher consumer’s income, we may come up against an existence problem. Moreover, the higher the consumer’s income, the larger the segment of relative advertising cost for which there is no equilibrium. Hence looking for all the possible types of equilibrium and investigating properly the existence problem, are not superfluous mathematical exercises.

Very intuitively, prices are increasing (in a broad sense) with the relative advertising cost. Beyond some critical threshold of this relative cost, both prices become equal to the consumer’s income.

Concerning the advertising intensities, both are decreasing (in a broad sense) in the cases of intermediate and high consumer’s income, as it can be predicted intuitively. But in the case of low consumer’s income, the advertising intensity of the low-quality firm is increasing on a

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3 and also in terms of “perceived consumer effectiveness” (PCE), i.e. their perceived belief in that her/his purchase will prove to have an actual effect (Nurse et al., 2012).

4 the natural market of a firm is composed of consumers who, when informed of both products, will choose the firm’s product.
range of intermediate levels of the relative cost.

Regarding the profits, both go through 3 phases (not synchronized): they are decreasing at Phase 1, increasing at Phase 2 and then decreasing at phase 3, converging both to zero as the ratio goes to infinity.

As for the profits ratio, equal to the high-quality firm’s profit over the low-quality firm’s one, it is always larger than 1, meaning that it always pays to be the high-quality firm. The variation of this ratio is however not simple. It goes through an increasing phase for a range of relative advertising cost close to zero and it is decreasing for sufficiently high levels of this relative cost, converging to 1, as the relative cost advertising goes to infinity, meaning that it pays less and less to be the high-quality firm as the relative cost increases unboundedly.

**Literature review.**

There is an important literature in this field originating with the works of Butters (1977) in the case of homogenous goods (see also the more recent contribution of Roy, 2000). Grossman and Shapiro (1984) and Tirole (1988) launched the basis for the case of horizontally differentiated markets, distinguishing between mass (random) advertising (where there is no correlation between advertising intensities and consumers’ types) and targeted advertising (when the firms advertise more their more interesting consumers). More recent papers consider the question of advertising within the horizontal differentiation framework, such as Celik (2007), Ben Elhadj-Ben Brahim et al. (2011), Esteban and Hernandez (2014). Simbanegavi (2009) also considers advertising within a horizontally differentiated market, but deals with the different question of cooperation between the firms on advertising or prices. The horizontal differentiation papers generally deal with ex ante symmetric firms, contrary to vertical differentiation ones.

A strand of literature considers questions relative to advertising targeting supposing an exogenous segmentation of consumers: Esteves and Resende (2016, 2019), Iyer et al. (2005), Esteban and Hernandez (2016), Zhang and He (2019) and Zhang, Cao and Yue (2018), Galeotti and Moraga-Gonzalez (2003). Even if the models in the cited papers may reflect some differentiation, to the extent that consumers do not react in the same way to a price differential between firms, it is definitely not vertical differentiation, as there is no unanimity on the ranking of the products by well-informed consumers.

Concerning asymmetry, all these papers consider symmetric firms except Zhang and He who consider exogenous cost asymmetry between firms, while in our model, this asymmetry is inherent to vertical differentiation and cost asymmetry emerges endogenously through the asymmetry in the choice of firms in terms of advertising intensities.

Colombo and Lambertini (2003) is one of the few papers we identified dealing with advertising in a vertically differentiated market. But our work is different in several respects. First, they consider persuasive advertising, while advertising is informative in our paper. Second, they deal with the endogenous interplay between advertising and product quality, while we consider exogenous qualities. Third, they consider a vertical differentiation model different from the one we consider. Tremblay and Martin-Filho (2001) and Tremblay and Polasky (2002) consider advertising within a vertically differentiated market but with persuasive advertising. Esteban and Hernandez (2007) and Esteban and Hernandez (2018) consider a vertical differentiation model but consider mass advertising as the distribution of ads to all the market with no choice of advertising intensities.

Loosely related to our paper, Shen and Villas-Boas (2018) deal with behavioral-based advertising but in the case of monopoly in a two-period model where the valuation of consumers of the product in the second period may be correlated with her valuation in the first period. Johnson (2013) deals with targeting within a model where a continuum of firms choose the
advertising amount while consumers have the possibility to block ads, with no competition among firms.

The paper is organized as follows. Section 2 describes the model. Section 3 provides the results. Section 4 provides some comparative statics. Section 5 concludes.

All proofs are given in Appendix B.

2 The Model

Two firms produce two vertically differentiated products with exogenous qualities. Firms 1 and 2 are respectively the high and the low-quality firm. The quality differential is denoted by $\Delta = q_1 - q_2$. We assume that marginal costs are zero and that Firms 1 and 2 compete in uniform prices, respectively $p_1$ and $p_2$. Firms invest in advertising to inform consumers about their existence, products’ characteristics and prices.

There is a unit mass of consumers. Consumers are initially totally unaware of the existence of the firms and become informed only through advertisements. Consumers informed only of the existence of Firm $i$ buy one unit of $i$’s good, provided that its price is not greater than their income $Y$. Consumers informed of the existence of both firms buy the product which better fits their needs.$^5$ A type $\theta$-customer derives a gross utility $U + \theta q_i$ from consuming one unit of the quality $i$-good in a given period, hence the indirect utility $U + \theta q_i - p_i$. Characteristic $\theta$ is uniformly distributed over $[0, 1]$ with a density normalized to 1.

The prices are thus assumed to belong to $[0, Y]$. $U$ is assumed to satisfy $U > Y$, so that all consumers who are aware of the existence of at least one firm buy the good.

We define $\hat{\theta}$ as the marginal consumer, i.e. the consumer who, when informed of the existence of both firms, is indifferent between purchasing at Firm 1 or at Firm 2. That is:

$$\hat{\theta} = \frac{p_1 - p_2}{\Delta}. \quad (1)$$

When fully informed, consumers with types greater than $\hat{\theta}$ buy from Firm 1 while consumers of types smaller than $\hat{\theta}$ buy from Firm 2. From now on we call $[\hat{\theta}, 1]$ Firm 1’s "natural market" and $[0, \hat{\theta}]$ Firm 2’s "natural market". When $p_1 \geq \Delta + p_2$, Firm 2’s natural market is the whole market and Firm 1 has a null natural market share. When $p_2 < p_1 < \Delta + p_2$, both firms have a strictly positive natural market share. When $p_1 \leq p_2$, Firm 1’s natural market is the whole market and Firm 2 has a null natural market.

We consider here the case of mass advertising in which the advertising intensity $\Psi_i$ of Firm $i$, i.e. the proportion of consumers who are informed about product $i$, is uniform over all consumers’ types. This means that a fraction $\Psi_1\Psi_2$ of consumers are informed of the existence of both firms, a fraction $(1 - \Psi_1)(1 - \Psi_2)$ are informed of the existence of none of them, a fraction $\Psi_1(1 - \Psi_2)$ are only informed of the existence of Firm 1 (and buy from it), a fraction $\Psi_2(1 - \Psi_1)$ are only informed of the existence of Firm 1 (and buy from it). The cost of reaching a fraction of type $\theta$-consumers is simply

$$C(g_i(\theta)) = \frac{a_2}{2} \Psi_i^2. \quad (2)$$

For convenience, we define the relative prices $v_i = \frac{p_i}{\Delta}$, the relative income $y = \frac{Y}{\Delta}$ and the relative cost $\alpha = \frac{a}{\Delta}$.

$^5$The same proviso obviously applies
The game: the firms choose simultaneously prices $p_i$ in $[0, Y]$ (or equivalently the relative prices $v_i$ in $[0, y]$) and advertising intensities $\Psi_i \in [0, 1]$.

3 The equilibrium outcomes

At equilibrium, whenever there exists any, three cases are possible. 1) the two firms have positive natural markets and charge prices lower than the consumer’s revenue (Interior Equilibrium); 2) the two firms have positive natural markets with the high-quality firm charging a price equal to the consumer’s revenue (Constrained Interior Equilibrium); 3) only one of the firms has a positive natural market (CORner equilibrium). When no equilibrium candidate among the three described is an equilibrium, the game admits no pure-strategy equilibrium.

From the definitions, the profits of the two firms are respectively:

$$
\Pi_1 = \begin{cases} 
  p_1\Psi_1 (1 - \Psi_2) - \frac{a}{2}\Psi_1^2 & \text{if } p_1 \geq \Delta + p_2, \\
  p_1\Psi_1 ((1 - \hat{\theta}) + \hat{\theta}(1 - \Psi_2)) - \frac{a}{2}\Psi_1^2 & \text{if } p_2 < p_1 < \Delta + p_2, \\
  p_1\Psi_1 - \frac{a}{2}\Psi_1^2 & \text{if } p_1 \leq p_2.
\end{cases}
$$

(3)

$$
\Pi_2 = \begin{cases} 
  p_2\Psi_2 (1 - \Psi_1) - \frac{a}{2}\Psi_2^2 & \text{if } p_1 \leq p_2, \\
  p_2\Psi_2 ((1 - \hat{\theta})(1 - \Psi_1)) - \frac{a}{2}\Psi_2^2 & \text{if } p_2 < p_1 < \Delta + p_2, \\
  p_2\Psi_2 - \frac{a}{2}\Psi_2^2 & \text{if } p_1 \geq \Delta + p_2.
\end{cases}
$$

(4)

The profits are thus defined above in the three possible price configurations: (i) when $p_1 \geq \Delta + p_2$, Firm 1 has no natural market and sells only to consumers unaware of the existence of its rival but informed of its own existence while Firm 2 can sell to all customers informed of its existence; (ii) $p_2 < p_1 < \Delta + p_2$, both firms have a positive natural market and sell to both consumers in their natural market and to consumers unaware of the existence of their rival, provided they are informed of their existence and (iii) if $p_1 \leq p_2$, Firm 2 has no natural market and sells only to consumers unaware of the existence of its rival but informed of its own existence while Firm 1 can sell to all customers informed of its existence.

We define the relative profits to be $\pi_i = \frac{\Pi_i}{\Delta}$.

A pure-strategy Nash equilibrium of this game is a quadruple $(v_i^*, \Psi_1^*, v_j^*, \Psi_j^*)$ such that $(v_i^*, \Psi_i^*) \in [0, y] \times [0, 1]$ is a best reply to $(v_j^*, \Psi_j^*)$ for each $i, j = 1, 2$, and $i \neq j$. Proposition 1 characterizes the equilibrium whenever it exists in the space $(\alpha, \gamma)$ and Figure 1 pictures in this space the areas corresponding to the different types of equilibria and to the non existence of a pure-strategy equilibrium.

Proposition 1 (Equilibrium). Depending on the position of $(\alpha, \gamma)$ relative to the zones depicted in Figure 1 and defined analytically in Appendix A, there are 4 main cases in terms of existence and type of equilibrium.

1. Zone IE (Interior Equilibrium): both firms have positive natural markets and charge prices lower than the revenue of the consumers. This zone is divided into 3 sub-zones, depending on whether or not the firms reach all consumers.
IE(i) Both firms reach all consumers ($\Psi^*_1 = \Psi^*_2 = 1$) and relative prices are $v^*_1 = 2/3$, $v^*_2 = 1/3$.

IE(ii) The high quality firm reaches all consumers while the low quality firm reaches only a fraction of them: $\Psi^*_1 = 1$, $\Psi^*_2 = (\frac{1}{\alpha})^{1/3}$; and the equilibrium relative prices are:

$$v^*_1 = 2\left(\frac{\alpha}{3}\right)^{1/3}, \quad v^*_2 = \left(\frac{\alpha}{3}\right)^{1/3}.$$ 

IE(iii) Both firms reach only a fraction of consumers.  

2. Zone CIE (Constrained Interior Equilibrium): Both firms have positive natural markets but the high quality firm charges the revenue of consumers. This zone is also divided into 3 sub-zones, depending on whether or not firms reach all consumers.

CIE(i) Both firms reach all consumers: $\Psi^*_1 = \Psi^*_2 = 1$, and charge the relative prices $v^*_1 = y$, $v^*_2 = y/2$.

CIE(ii) The high-quality firm reaches all customers ($\Psi^*_1 = 1$), the low quality firm only a fraction $\Psi^*_2 = y^2/4\alpha$ of them and relative prices are given by: $v^*_1 = y$, $v^*_2 = y/2$;

CIE(iii) Both firms reach only a fraction of customers, the high-quality firm charges the relative price $v^*_1 = y$ and the low quality firm a lower price.  

3. Zone COR (CORner equilibrium): the low-quality firm has a zero natural market, with the following relative prices and advertising intensities:

$$v^*_1 = v^*_2 = y; \quad \Psi^*_1 = \frac{y}{\alpha}, \quad \Psi^*_2 = \frac{y}{\alpha}(1 - \frac{y}{\alpha}).$$ 

4. Zone NE: There is no equilibrium in pure strategies.

It appears clearly from Figure 1 that a pure strategy equilibrium exists whatever the value of $y$ when the relative advertising cost is small enough (smaller than $8/9$). The intuition why is quite clear. When the advertising cost is small and/or the quality differential is high, Firm 1 informs all customers, which leaves no possibility for Firm 2 to serve uninformed consumers at a high price. Another feature is that the range of values of $y$ for which an interior equilibrium exists shrinks when the relative advertising cost increases. This is because, as this relative cost increases, less and less consumers are informed so that a deviation toward serving at a high price only consumers unaware of the existence of one’s rival becomes more and more profitable, which prevents the equilibrium candidate to be an equilibrium.

To prove Proposition 1, we deal consecutively with each possible case. For the first case (Interior Equilibrium), we write the first order conditions for the associated Lagrangian supposing that each firm has a positive natural market. After eliminating the trivial solution with null prices and advertising rates, we examine the 4 possible cases: (i) both firms reach all customers ($\Psi_1 = \Psi_2 = 1$); (ii) Firm 1 reaches all customers, Firm 2 only a fraction of them ($\Psi_1 = 1$, $\Psi_2 < 1$); (iii) Both firms reach only a fraction of their customers ($\Psi_1 = 1$, $\Psi_2 < 1$); (iv) Firm 1 reaches only a fraction of its customers and Firm 2 its whole natural market ($\Psi_1 < 1$, $\Psi_2 = 1$).  

6Appendix A provides more details on the equilibrium outcome.  
7More details may be found in Appendix A.  
8The only one identified in the literature.
For sub-cases (i), (ii) and (iii), we calculate the equilibrium candidates and determine necessary and sufficient conditions for each candidate to correspond to a maximum for the set of prices such that both firms have positive natural markets. As for case (iv), it turns out that it can never correspond to an equilibrium.

The reasoning above eliminates the deviations such that each firm has a positive natural market, but not deviations such that one of them has no natural market. Look at the profit of Firm $i$ when it has no natural market share and its competitor does not reach all the market ($\Psi_j < 1$). We see easily that this profit may increase unboundedly with the price, hence may become higher than the profit at the equilibrium candidate and thus constitute a profitable deviation. Therefore, if the upper-bound on prices is too high, the equilibrium candidate cannot be an equilibrium. In other words, to ensure that the identified candidate is an equilibrium, the price must not be allowed to be too high, so that the best possible deviation is not profitable. In each sub-case of case 1) of Proposition 1, we write conditions on $y$ and $\alpha$ such that, on the one hand, the profit at the best possible deviation is lower than the profit at the equilibrium candidate; and on the other hand, the price candidates are less than $Y$.

Regarding Case 2 (Constrained Interior Equilibrium), we proceed exactly in the same way as for Case 1, except that we take the constraint on prices into account in the Lagrangian.

As for Case 3 (Corner equilibrium), we identify the corner equilibrium in each considered situation (either Firm 1 or Firm 2 has a null natural market). Then we consider possible deviations.

There is an asymmetry between firms regarding the existence of corner equilibria. While a corner equilibrium with a null natural market for the low-quality firm may exist, there is never an equilibrium with a null natural market for the high-quality firm. Indeed the low-quality firm is the one which has less incentives to have customers who would buy the product when they know its "true value". Thus it may have interest to rely completely on uninformed customers.
We are now going to provide some comparative statics at equilibrium whenever it exists. There are qualitatively 3 cases depending on the position of the consumer’s income relative to the two critical values (2/3 and approximately 2.0477) as depicted in Figure 1. We refer to the three cases as low, intermediate and high consumer’s income in a self-explaining way. We are going to study, in the three cases, consecutively, the prices/advertising intensities, the profits and the profits ratio (high-quality firm’s profit over low-quality firm’s profit), as function of \( \alpha \), the relative advertising cost. This amounts to moving along a horizontal line on Figure 1. Doing so, we may go through multiple regions characterized by different types of equilibrium. For instance, for \( y < 2/3 \), increasing \( \alpha \) starting from zero, we go through CIE(i), then CIE(ii), then CIE(iii) and finally COR and remain there.

Corollary 1 provides the comparative statics for prices and advertising intensities. Figures 2, 3 and 4 depict the relative equilibrium prices as function of the relative advertising cost, respectively in the low, intermediate and high consumer’s income. Figures 5, 6 and 7 depict the equilibrium advertising intensities as function of the relative advertising cost, respectively in the three cases in the same order.

**Corollary 1 (Prices and advertising intensities).** At equilibrium (whenever it exists), the high-quality firm charges a higher price and advertises more (in a broad sense) than the low quality one.

The proof corresponds to the representation of the relative prices \( v_i^* \) and advertising intensities \( \Psi_i^* \) given in Proposition 1 in each case (low, intermediate and high consumer’s income), as function of \( \alpha \).

The variation of prices with \( \alpha \) is as expected. The higher \( \alpha \), the higher the relative advertising cost, the higher they have to set prices in order to cover their costs.

Also as expected, the equilibrium price of the high quality firm is always larger (in a broad sense) than the equilibrium price of the low quality one and the high-quality firm advertises more (in a broad sense) than the low-quality one.
Figure 3: Comparative statics for prices: the case of intermediate consumer’s income.

Figure 4: Comparative statics for prices: the case of high consumer’s income.
Figure 5: Comparative statics for advertising intensities: the case of low consumer’s income.

Figure 6: Comparative statics for advertising intensities: the case of intermediate consumer’s income.
Interestingly, for high enough $\alpha$, both prices become equal (to the consumer’s income), while Firm 1 invests more in advertising than its rival. Hence for high enough $\alpha$ (high advertising cost and/or low quality differential), the price cannot signal the quality, whereas the advertising intensity may do so. On the contrary, for low enough $\alpha$ (low advertising cost and/or high quality differential), the advertising intensities are both equal to 1, while prices are distinct, Firm 1 charging the highest price. Hence, for low enough $\alpha$, the advertising intensities cannot serve as a signal of quality, whereas prices may do so.

As for advertising intensities, they are decreasing with $\alpha$, that is decreasing with the advertising cost but increasing with the quality differential\(^9\), except for Firm 2 in the case of low consumer’s income under CIE(iii) and a part of COR (Figure 5). Indeed, increasing $\alpha$ has two contradictory effects on advertising intensities. It has a direct negative effect as relative advertising costs are higher. As it thus discourages the competitor’s investment in advertising, it has an indirect positive effect: the less the rival firm invests in advertising, the more a firm is encouraged to do so. It appears that the direct negative effect is dominant except for Firm 2 in the case of low consumer’s income for some range of $\alpha$.

Corollary 2 provides the comparative statics for the firms’ profits. Figures 8, 9 and 10 depict the relative profits at equilibrium respectively in the cases of low, intermediate and high consumer’s income.

Corollary 2 (Profits). The high-quality firm makes a strictly larger profit than the low-quality one. Both profits converge to zero as $\alpha$ goes to infinity.

To prove the result, we just represent the two firms’ profits as function of $\alpha$ in the 3 cases of consumer’s income.

Note that even if the firms’ prices become equal beyond some critical value of $\alpha$, this is not profit destructive: profits are always positive. In fact when firms charge the same price, they maintain differentiation through distinct advertising intensities.

\(^9\)In this sense one can say that more differentiated industries advertise more.
Figure 8: Comparative statics for profits: the case of low consumer’s income.

Figure 9: Comparative statics for profits: the case of intermediate consumer’s income.
The variation of the profit functions with the cost parameter is qualitatively the same across firms. The profit of each firm goes through 3 phases: at Phase 1, it is decreasing, Phase 2, increasing, and Phase 3, decreasing. Both profits converge to zero as $\alpha$ goes to infinity.

An increase in the relative advertising cost parameter has a direct negative effect on the firms’ profits, which equals $-\Psi_i^2/2$ for Firm $i$, $i = 1, 2$. This is the only one over the range of $\alpha$ over which they both reach all the customers and the equilibrium prices and advertising intensities do not depend on $\alpha$. Outside this range, there is in addition a positive strategic effect: an increase in the advertising cost raises the rival’s equilibrium price (Figures 2, 3 and 4) and generally lowers its equilibrium advertising intensity (Figures 5, 6 and 7). In the exceptional case where the advertising intensity of Firm 2 is increasing (over a range of $\alpha$ in the case of low consumer’s income, Figure 5), the mild increasingness of $\psi_2^*$ has a negative strategic effect on Firm 1’s profit. As a result, Firm 1’s profit is strongly decreasing on that range (consider jointly Figure 5 and 8).

Corollary 3 is about the profits ratio, i.e. Firm 1’s profit over Firm 2’s profit. Figures 11, 12 and 13 depict the profits ratio as function of the relative advertising cost, respectively in the low, intermediate and high consumer’s income.

**Corollary 3 (Profits ratio).** In all cases of consumer’s income, the profits ratio $(\pi_1^*/\pi_2^*)$ is increasing for low enough $\alpha$ and decreasing above some critical level of $\alpha$, converging to 1, as $\alpha$ goes to infinity.

The proof consists once more in representing the ratio as function of $\alpha$ in each case in terms of consumer’s income.

It appears that the profits ratio in the three cases goes first (for low $\alpha$) through a strongly increasing phase, during which, the higher $\alpha$ the more it pays to be the high-quality firm. Indeed for low $\alpha$, both firms reach all consumers who are thus aware of the existence of both products, which is to the advantage of the high-quality firm (Firm 1) that thus has the possibility to charge a much higher price than its competitor. As $\alpha$ increases remaining in a range close to zero, the profits ratio equals $d/(b - (d/\alpha/2))$. In the case of low consumers’ income, it occurs under CIE(i), with $d = 0.42$ and $b = 0.09$; and in the two cases of intermediate and high consumers’ income, it occurs under IE(i) for which $d = 4/9$ and $b = 1/9$. In all cases, the strong
Figure 11: Comparative statics for the profits ratio: the case of low consumer’s income.

Figure 12: Comparative statics for the profits ratio: the case of intermediate consumer’s income.
increasingness of the profits ratio with $\alpha$ is the result of a purely mathematical mechanism.

Now as $\alpha$ increases more, in the case of low consumers’ income, we enter zone CIE(ii) and the profits ratio remains increasing, and in the two other cases, we go into IE(ii), and the profits ratio becomes decreasing. We explain this contrasted effect as follows. In zone CIE(ii), while Firm 1’s price and advertising intensity remain constant, Firm 2’s advertising intensity is strongly decreasing. Firm 2’s profit is affected only by a direct negative effect, while Firm 1’s profit benefits from an indirect positive effect through the decreasingness of $\Psi_2^*$ resulting in an information advantage of the high-quality firm. In zone IE(ii) however, Firm 2 benefits from a positive indirect effect through the increasingness of Firm 1’s price which outweighs the negative direct effect.

In the three cases, when $\alpha$ goes beyond some critical value, we go into zone CIE(iii), and upon entering this zone, the profits ratio becomes decreasing in the case of low consumers’ income, while it becomes slightly increasing in the two other cases. This contrasted effect is explained as follows. In the case of low consumers’ income, the advertising intensity of Firm 2 becomes increasing in zone CIE(iii), while Firm 1’s advertising intensity becomes decreasing, which reduces the informational disadvantage of Firm 2 and allows it to charge a higher price ($p_2^*$ increasing), resulting in a decreasing profits ratio. In the case of intermediate and high consumers’ income, both advertising intensities are decreasing, but only Firm 2’s price is increasing, while Firm 1’s price is constant equal to the maximal value. Both profits bear direct negative effects of increasing $\alpha$. Firm 1 benefits from a positive indirect effect through the increasingness of Firm 2’s price and the decreasingness of its advertising intensity. It turns out that this is to the advantage of Firm 1, which results in an increasing profits ratio. But this increasingness is only slight as the effects on both firms are comparable.

As $\alpha$ goes beyond some critical level, in all cases, the profits ratio decreasingly converges to one. Indeed, for high enough $\alpha$, both prices become equal and both advertising intensities converge to zero. Hence, as $\alpha$ increases beyond a critical level, less and less consumers are aware of the existence of the products and thus of the advantage of the high-quality firm. It pays less and less to be the high-quality firm, as it is anyway constrained by the consumer’s income in terms of pricing and is able less and less to benefit from the high-quality status be-
cause of the lack of information.

5 Conclusion

In this paper, we studied mass advertising for vertically differentiated products, whereas it has been studied only for horizontally differentiated products. This has allowed us to analyze the way the different variables (prices, advertising intensities, profits) vary with the key parameters, possibly in different ways for the high and the low quality firms. We showed in particular that the high-quality firm has larger prices, advertising intensities and profits than the low-quality one but that the gap eventually shrinks as the relative advertising cost increases. We showed that the proportion of consumers who receive advertisements is increasing with the quality differential. We qualified the result put forward by Tirole (1988) according to which equilibrium profits first decrease with the advertising cost, due to the direct effect, before increasing, due to the strategic one. We indeed found that there are three phases rather than two: profits are first decreasing, then increasing but finally decreasing again with the relative advertising cost. Besides the specific comparative statics results following from addressing the vertical differentiation case, there is a methodological contribution of this paper. We indeed showed that without an upper limit on prices, there is no pure strategy equilibrium for a relative advertising cost above some critical level, because firms would benefit from deviating by posting very high prices and selling only to customers uninformed of the existence of their rival. Introducing accordingly an upper bound on prices then led us to discover the existence of two other types of equilibria, constrained interior and corner ones, which were overlooked by the existing literature. This paper has assumed from the beginning that the firms use random (mass) advertising. An interesting, if not necessary, extension will be to consider the alternative case of targeted advertising and to see if our results are robust. It is an ongoing research.

References


Such that the high-quality firm reaches all customers for values below the critical one


18


Appendix A: Definition of zones

Zone (IE)

(i) $IE(i) = \{(\alpha, y) \in \mathbb{R}^+ \times \mathbb{R}^+, \text{ such that } \alpha \leq 1/9 \text{ and } y \geq (2/3)\}.$

(ii) $IE(ii) = \{(\alpha, y) \in \mathbb{R}^+ \times \mathbb{R}^+, \text{ such that } \alpha \in (1/9, 8/9] \text{ and } y \geq 2(\alpha/3)^{1/3}\}.$

(iii) Let $\Psi^*_1(\alpha)$ be the equilibrium value of $\Psi_1$ as a function of $\alpha$, write the equilibrium profit of Firm 2 as $\Delta \pi_2^*(\alpha)$.

$IE(iii) = \{(\alpha, y) \in \mathbb{R}^+ \times \mathbb{R}^+, \text{ such that } \alpha > 8/9 \text{ and } y \leq \frac{\sqrt{2\alpha \pi_2^*(\alpha)}}{1 - \Psi_1^*(\alpha)}\}.$
Proof of Proposition 1.

Case 1 (Interior equilibrium)

Elimination of the trivial solution \( p_i = \Psi_i = 0 \).

More precisely, if \( p_1 = p_2 = 0 \), then necessarily \( \Psi_1 = \Psi_2 = 0 \). Indeed with a null price, a firm has no revenues and should not invest in advertising.

However \( p_1 = \Psi_1 = 0 \) does not correspond to an equilibrium. Indeed for \( p_2 = \psi_2 = 0 \), the profit of Firm 1 is given by: \( \pi_1 = p_1 \psi_1 - \frac{y}{2} \psi_1^2 \), which is not maximal at \( p_1 = \psi_1 = 0 \).
First Order Conditions with a positive natural market for each firm \((0 < \hat{\theta} < 1)\).

Given the expressions of the profits provided in Equations 3 and 4, the profit maximization by Firms 1 and 2 under the constraints \(\Psi_i \leq 1, i = 1, 2\), with the associated non-negative Lagrangian multipliers \(\mu_i, i = 1, 2\), yields the necessary conditions:

\[
\begin{align*}
\frac{\partial L_1}{\partial p_1} &= \Psi_1 \left(1 + \frac{\Psi_2(p_2 - 2p_1)}{\Delta}\right) = 0, \quad (6) \\
\frac{\partial L_2}{\partial p_2} &= \Psi_2 \left(1 + \frac{\Psi_1(p_1 - 2p_2 - \Delta)}{\Delta}\right) = 0, \quad (7) \\
\frac{\partial L_1}{\partial \Psi_1} &= p_1 - a\Psi_1 + \frac{p_1(-p_1 + p_2)\Psi_2}{\Delta} - \mu_1 = 0, \quad (8) \\
\frac{\partial L_2}{\partial \Psi_2} &= p_2 - a\Psi_2 - \frac{p_2(\Delta - p_1 + p_2)\Psi_1}{\Delta} - \mu_2 = 0. \quad (9)
\end{align*}
\]

At equilibrium, \(\Psi_i > 0\), for \(i = 1, 2\).

Indeed if one of the \(\Psi_i = 0\), then necessarily, by Equations 6 and 7, the second \(\Psi_j = 0\). Hence, by Equations 8 and 9, prices are \(p_i = \mu_i\). But \(\Psi_i = 0 < 1\), hence \(\mu_i = 0\), then \(p_i = 0\). But we have just proved that \(p_i = \Psi_i = 0\) does not correspond to an equilibrium.

Second Order Conditions: Any solution to Equations 6, 7, 8 and 9 such that \(\Psi_i > 0\), \(i = 1, 2\), corresponds for each firm to a profit maximum.

Indeed, as \(\Psi_1 > 0\), Equation 6 implies \(1 + \frac{\Psi_2(p_2 - 2p_1)}{\Delta} = 0\), which corresponds precisely to \(\frac{\partial^2 L_1}{\partial p_1^2} = 0\), which is thus equal to zero. We prove similarly that \(\frac{\partial^2 L_2}{\partial \Psi_2^2} = 0\).

The Hessian matrix for Firm 1 is given by:

\[
H_1 = \begin{pmatrix}
-2\Psi_1\Psi_2 & \frac{1}{\Delta} \left(1 + \frac{\Psi_2(p_2 - 2p_1)}{\Delta}\right) = 0 \\
(1 + \frac{\Psi_2(p_2 - 2p_1)}{\Delta}) = 0 & -a
\end{pmatrix}
\]

which is definite negative.

As for Firm 2,

\[
H_2 = \begin{pmatrix}
-2\Psi_1\Psi_2 & \frac{1}{\Delta} \left(1 + \frac{\Psi_1(p_1 - 2p_2 - \Delta)}{\Delta}\right) = 0 \\
(1 + \frac{\Psi_1(p_1 - 2p_2 - \Delta)}{\Delta}) = 0 & -a
\end{pmatrix}
\]

which is also definite negative.

We now deal with 4 possible cases, depending on whether the constraints are binding or not. One of them turns out to be never possible.

Case (i): Both firms reach all customers.

Both constraints are binding so that \(\Psi_i = 1, i = 1, 2\). There is a unique solution of Equations (6), (7), (8) and (9) which is \(p_1 = 2\Delta/3, p_2 = \Delta/3, \mu_1 = \frac{4}{9}\Delta - a, \mu_2 = \frac{1}{9}\Delta - a\). The
condition \( a \in [0, \Delta/9] \) is necessary and sufficient to ensure that both multipliers are indeed non-negative. Finally, as both \( \Psi_i > 0 \), the solution corresponds to a maximum for each firm.

**Case (ii):** Firm 1 reaches all customers, Firm 2 only a fraction of them.

We must then have \( \Psi_1 = 1 \) and \( \mu_2 = 0 \). Solving Equations 6, 7, 8 and 9, we obtain a unique solution which is \( p_1 = 2\left(\frac{a\Delta}{3}\right)^{1/3} \), \( p_2 = \left(\frac{a\Delta}{3}\right)^{1/3} \), \( \Psi_2 = \left(\frac{\Delta}{9a}\right)^{1/3} \) and \( \mu_1 = -a + \frac{4a^{1/3}\Delta^{2/3}}{3x^{3/3}} \). Notice that \( \mu_1 \) is non-negative iff \( a \in [0, 8\Delta/9] \) while \( \Psi_2 < 1 \) iff \( a > \Delta/9 \).

As both \( \Psi_i > 0 \), then the solution corresponds to a maximum for each firm. Consequently Case (ii) corresponds to an equilibrium iff \( a \in (\Delta/9, 8\Delta/9] \).

**Case (iii):** Both firms reach only a fraction of customers (interior equilibrium).

Here \( \mu_1 = \mu_2 = 0 \). Let us first solve for advertising rates as functions of prices. We obtain:

\[
\Psi_1 = \frac{\Delta p_1(\Delta - p_1 p_2 + p_2^2)}{a^2 \Delta^2 - p_1 (p_1 - p_2) p_2 (\Delta - p_1 + p_2)}, \tag{10}
\]

\[
\Psi_2 = \frac{\Delta p_2(\Delta + p_1^2 - p_1 (\Delta + p_2))}{a^2 \Delta^2 - p_1 (p_1 - p_2) p_2 (\Delta - p_1 + p_2)}. \tag{11}
\]

We decompose the reasoning into 3 steps in order to facilitate reading.

**Step 1:** We prove that there is no equilibrium such that \( p_i = 0 \) and/or \( \Psi_i = 0, i = 1, 2 \).

(a) Consider first \( p_j = 0 \). From (10) it follows that \( \Psi_j = 0 \). Now from Equations 8 and 9 \( \frac{\partial \Psi_j}{\partial p_j} = p_j - a \Psi_j = 0 \) so that \( \Psi_j = p_j/a \).

Now from (6), for \( j \neq i \), \( \frac{\partial \Psi_j}{\partial p_i} = p_j/a = 0 \), which implies \( p_j = 0 \) and then \( \Psi_j = 0 \).

(b) Consider then \( \Psi_i = 0 \). From (6), we obtain \( \Psi_j = 0 \) and then \( p_i = p_j = 0 \).

As shown above however we cannot have \( p_i = \Psi_i = 0, i = 1, 2 \), at equilibrium.

**Step 2: A necessary and sufficient condition for the solution.**

Substituting the expressions obtained in Equations 10 and 11, into Equations 6 and 7, and accounting for the fact that \( \Psi_i > 0 \) at equilibrium, one obtains the two equilibrium conditions which the equilibrium prices must satisfy:

\[
\left(\frac{a^2 \Delta^2 + p_2 \left(-2a \Delta p_1 - p_1^3 + a \Delta p_2 + p_2^2 (\Delta + p_2)\right)}{a^2 \Delta^2 - p_1 (p_1 - p_2) p_2 (\Delta - p_1 + p_2)}\right) = 0, \tag{12}
\]

and

\[
\left(\frac{a^2 \Delta^2 + p_1^2 (\Delta + p_2^2) - p_1 (\Delta^2 + 2a \Delta p_2 + p_2^3)}{a^2 \Delta^2 - p_1 (p_1 - p_2) p_2 (\Delta - p_1 + p_2)}\right) = 0. \tag{13}
\]

Subtracting (12) from (13) one obtains that the following condition must hold at equilibrium

\[-((\Delta - p_1)p_1 + p_2^2)(a \Delta + p_1 p_2) = 0 \]
We can then conclude that the equilibrium prices must satisfy

\[
p_1 = \frac{1}{2} \left( \Delta + \sqrt{\Delta^2 + 4p_2^2} \right),
\]

where \( p_1 \) is strictly greater than \( p_2 \) and strictly smaller than \( \Delta + p_2 \).

Substituting this value for \( p_1 \) in (13), we obtain that the equilibrium value of \( p_2 \) must satisfy:

\[
a^2\Delta^2 + p_2 (a\Delta p_2 - a\Delta (\Delta + \sqrt{\Delta^2 + 4p_2^2}) + \frac{1}{4} (\Delta + p_2) (\Delta + \sqrt{\Delta^2 + 4p_2^2})^2 - \frac{1}{8} (\Delta + \sqrt{\Delta^2 + 4p_2^2})^3) = 0.
\]

Let \( \alpha = a/\Delta \) and \( v_2 = p_2/\Delta \). The equilibrium condition (15) can be rewritten as:

\[
F(\alpha, v_2) = \frac{2\alpha^2 + 2v_2^2 - 2\alpha v_2(1 + \sqrt{1 + 4v_2^2}) - v_2^3(1 + \sqrt{1 + 4v_2^2}) + v_2^2(1 + 2\alpha + \sqrt{1 + 4v_2^2})}{2\alpha^2 + v_2(-2(1 + \sqrt{1 + 4v_2^2}) + v_2(-4v_2 + \sqrt{1 + 4v_2^2}))} = 0.
\]

**Step 3: Existence and unicity of the solution of Equation 16.**

Using the above change of variables and \( v_1 = p_1/\Delta \), we can write

\[
\Psi_1 = \frac{v_1(\alpha - v_1v_2 + v_2^2)}{\alpha^2 + v_1(\alpha - v_2)v_2(-1 + \alpha - v_2)}
\]

Let us then use the equilibrium relationship \( v_1 = \frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) \) to obtain

\[
\Psi_1(\alpha, v_2) = \frac{\frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) (\alpha - \frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) v_2 + v_2^2)}{\alpha^2 + \frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) \left( \frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) - v_2 \right) v_2 \left( -1 + \frac{1}{2} \left( 1 + \sqrt{1 + 4v_2^2} \right) - v_2 \right)},
\]

Equation 16 has two positive real solutions which are depicted in Figure 14 in the \((\alpha, v_2)\)-space using the *ContourPlot* function of Mathematica. In the same figure, using the *RegionPlot* function we have depicted in blue the area in this space where \( \Psi_1(\alpha, v_2) \leq 1 \). It turns out that only the smallest solution of 16 (corresponding to the expression of \( v_2^1 \) of Case IE(iii) of Proposition 1) is an equilibrium. The greatest one belongs to the white area where \( \Psi_1(\alpha, v_2) > 1 \).

Denoting by \( v_2(\alpha) \) the equilibrium value from Equation 16, we obtain that \( \Psi_1(8/9, v_2(8/9)) = 1 \) and \( \Psi_1(\alpha, v_2(\alpha)) < 1 \) for all \( \alpha > 8/9 \).

We define \( \Psi_2(\alpha, v_2(\alpha)) \) similarly to \( \psi_1(\alpha, v_2(\alpha)) \). We use Equation 11 giving \( \Psi_2 \) as a function of prices, then Equation 14 to eliminate \( p_1 \). We use the same change in variables to express \( \Psi_2 \) as function of \( \alpha \) and \( v_2 \) and finally use \( v_2(\alpha) \) the value of \( v_2 \) satisfying Equation 16.

Plotting \( \Psi_1(\alpha, v_2(\alpha)) \) and \( \Psi_2(\alpha, v_2(\alpha)) \) on the same figure 15 for all \( \alpha > 0 \), we obtain that (i) \( \Psi_1(\alpha, v_2(\alpha)) < 1 \) if and only if \( \alpha > 8/9 \), implying that the solution we just described is valid if and only if \( \alpha > 8/9 \); and (ii) \( \Psi_2 < \Psi_1 < 1 \).

---

11Implying that \( \bar{\theta} \in (0, 1) \).
Figure 14: Representation in the $(\alpha, \nu_2)$-space of Equation (16).

Figure 15: Representation of the expressions of $\Psi_1(\alpha, \nu_2(\alpha))$ and $\Psi_2(\alpha, \nu_2(\alpha))$ as function of $\alpha$. 
Finally the obtained $\Psi_i$ are both positive, thus the obtained solution corresponds to a maximum for each firm.

**Case (iv):** We prove that there is no equilibrium where Firm 2 reaches all consumers while Firm 1 reaches only part of them.

Suppose it is the case, we should then have $\Psi_2 = 1$ and $\mu_1 = 0$. Using the first order conditions w.r.t. prices (6), we would then obtain:

$$p_1 = \frac{\Delta + \Psi_1}{3\Psi_1},$$
$$p_2 = \frac{2 - \Psi_1}{3\Psi_1}.$$

Substituting for $p_1$ and $p_2$ the above values into Equation (8), we obtain ($\alpha = \frac{\mu_2}{\Delta}$):

$$1 + 2\Psi_1 + \Psi_1^2 - 9\alpha\Psi_1 = 0. \quad (18)$$

On the other hand from Equation (9), we obtain

$$\mu_2 = \frac{\Delta}{9} \left(-4 - 9\alpha + \frac{4}{\Psi_1} + \Psi_1 \right). \quad (19)$$

From (18) one obtains $\alpha = \frac{1+2\Psi_1+\Psi_1^2}{9\Psi_1}$ where the value of $\Psi_1$ is the candidate equilibrium one. Substituting this value for $\alpha$ in (19), we should have

$$\mu_2 = -4 - \frac{1 + 2\Psi_1 + \Psi_1^2}{\Psi_1} + \frac{4}{\Psi_1} + \Psi_1. \quad (20)$$

As pictured in Figure 16 (representing the expression given in Equation 20), the RHS is always **negative** for all values of $\Psi_1 \in [0, 1]$. Since the multiplier has to be positive, this does not correspond to an equilibrium.

**Deviations.** We deal with each sub-case of Case 1 to prove that no firm admits a profitable deviation.

**Case (i).** Here $\Psi_i^* = 1$, for $i = 1, 2$. As long as the price candidates are less than $Y$, no profitable deviations exist for firms. In fact each firm’s profit is always null outside of its natural market, as all consumers are informed on the competitor’s product, leaving no room to make profit on uninformed ones. Hence, for the equilibrium candidate to be an equilibrium, it suffices to have $Y \geq p_1^*$.  

**Case (ii).** Here there is no possible profitable deviation by Firm 2 since $\Psi_1 = 1$, if $p_2^* \leq Y$, for the same reason explained in Case (i).

As for Firm 1, as long as $p_1^* \leq Y$ only deviations toward prices $p_1^D > p_2^* + \Delta$ may potentially be profitable. But if $Y < p_2^* + \Delta$, such deviations do not exist at all.  

Moreover we know that $p_1^* < p_2^* + \Delta$ as $\hat{\theta} < 1$. Hence the interval $[p_1^*, p_2^* + \Delta]$ has a positive measure and for all $Y \in [p_1^*, p_2^* + \Delta]$, no deviation of Firm 1 giving it a null natural market is
possible.\footnote{This implies \( p_2^* \leq Y \) because \( p_2^* < p_1^* + \Delta \).}

**Case (iii).** Here we have to consider deviations by Firm 1 and by Firm 2.

Let us begin with Firm 1. We have to consider deviations to prices \( p_1^D \geq p_2^* + \Delta \) resulting in a null natural market for Firm 1.

We conduct the same reasoning as in Case (ii). Noting that \( p_1^* < p_2^* + \Delta \), for all \( Y \in [p_1^*, p_2^* + \Delta] \), such deviations are not possible.

Let us turn to Firm 2. The prices \( p_2^D \) such that Firm 2 has no natural market satisfy \( p_2^D \geq p_1^* \) and give the firm the profit:

\[
p_2 \Psi_2 (1 - \Psi_1^*) - a \Psi_2^2 / 2,
\]

which is maximal at \( p_2 = Y \).

The optimal value in terms of \( \Psi_2 \) is equal to \( \Psi_2^D = \frac{\Psi_1^*(1 - \Psi_1^*)}{a} \), which yields the profit:

\[
\pi_2^D (Y) = \frac{Y^2 (1 - \Psi_1^*)^2}{2a}.
\]

It has to be smaller than the candidate equilibrium profit which can be written as \( \Delta \pi_2^* (\alpha) \).

This is equivalent to

\[
y \leq \frac{\sqrt{2a \pi_2^* (\alpha)}}{1 - \Psi_1^* (\alpha)}.
\]

**Proof of Proposition 1, Case 2 (CIE).**

Given the expressions of the profits provided in Equations 3 and 4, the profit maximization by Firms 1 and 2 under the constraints \( \Psi_i \leq 1, i = 1, 2 \), and \( p_i \leq Y \) with respectively the associated non-negative Lagrangian multipliers \( \mu_i, i = 1, 2 \), and \( \lambda_i, i = 1, 2 \), yields the necessary conditions:

\[
\lambda_1 \geq \frac{\mu_1}{\mu_2} (\Psi_1^* - \Psi_1) \geq \frac{\mu_1}{\mu_2} (\Psi_2^* - \Psi_2).
\]
\[ \frac{\partial L_1}{\partial p_1} = \Psi_1 \left( 1 + \frac{\Psi_2 (p_2 - 2p_1)}{\Delta} \right) - \lambda_1 = 0, \quad (21) \]
\[ \frac{\partial L_2}{\partial p_2} = \Psi_2 \left( 1 + \frac{\Psi_1 (p_1 - 2p_2 - \Delta)}{\Delta} \right) - \lambda_2 = 0 \quad (22) \]
\[ \frac{\partial L_1}{\partial \Psi_1} = p_1 - a\Psi_1 + \frac{p_1(-p_1 + p_2)\Psi_2}{\Delta} - \mu_1 = 0, \quad (23) \]
\[ \frac{\partial L_2}{\partial \Psi_2} = p_2 - a\Psi_2 - \frac{p_2(\Delta - p_1 + p_2)\Psi_1}{\Delta} - \mu_2 = 0 \quad (24) \]

We are looking for an equilibrium such that \( p_1 = Y \) and each firm has a positive natural market. This implies necessarily \( p_2 < Y \), hence \( \lambda_2 = 0 \).

Now we consider one by one each possible case. We first write the first order conditions allowing to identify the candidate, then we check the second order conditions.

(i) For \( \Psi_1 = \Psi_2 = 1 \), from Equation (22), one obtains \( p_2 = Y/2 \). This corresponds to an interior equilibrium only if \( \hat{\theta} = Y/2\Delta < 1 \Leftrightarrow y < 2 \).

The F.O.C. with respect to \( p_1 \) (Equation 21), together with the condition \( \lambda_1 \geq 0 \) and the expressions of \( p_1 \) and \( p_2 \), imply \( y \leq 2/3 \).

On the other hand, from Condition 23 and the necessary condition \( \mu_1 \geq 0 \), we must have \( Y - a - \frac{1}{3} \frac{y^2}{\Delta} \geq 0 \Leftrightarrow \alpha \leq y - \frac{y^2}{2} \).

From Condition 24 and the necessary condition \( \mu_2 \geq 0 \), we must have \( \alpha \leq \frac{y^2}{4} \Leftrightarrow y \geq 2 \sqrt{\alpha} \).

Notice then that \( \frac{y^2}{4} < y - \frac{y^2}{2} \) whenever \( y < 8/3 \). Thus when \( y \leq 2/3 \), we have \( \frac{y^2}{4} < y - \frac{y^2}{2} \).

To sum up, only the conditions \( 2 \sqrt{\alpha} \leq y \leq 2/3 \) are necessary.

The second order conditions.

For Firm 1, the two constraints are binding, while there are also two variables. Then we have nothing to check for the second order conditions.

Firm 2’s bordered Hessian, given that the constraint on \( \Psi_2 \) is the only active, is:

\[ BH_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -2/\Delta & 0 \\ -1 & 0 & -a \end{pmatrix} \]

We have to consider the sign of the last principal minor, as there are two variables and one binding constraint. The last principal minor (the third) is equal to \( 2/\Delta \), which has the same sign as \( (-1)^3 \). Thus the second order conditions are satisfied for Firm 2.
(ii) Case $\Psi_1 = 1$ and $\Psi_2 < 1$. Conducting the same reasoning as in case (i), we obtain $p_2 = Y/2$ and the necessary condition $y < 2$ to have an interior equilibrium.

As $\Psi_2 < 1$, then $\mu_2 = 0$ and Equation 24 implies $\Psi_2 = y^2/4\alpha$, which is smaller than 1 iff $y \leq 2\sqrt{\alpha}$.

Now, from Equation 21 and the condition $\lambda_1 \geq 0$, we obtain $y \leq 2(\alpha/3)^{1/3}$.

Using Equation 23 and the condition $\mu_1 \geq 0$, we obtain $y - (y^4/8\alpha) - \alpha \geq 0$.
Notice finally that the latter condition implies $y < 2$.

To sum up, only the conditions $y - y^4/8\alpha - \alpha \geq 0$, $y \leq 2\sqrt{\alpha}$ and $y \leq 2(\alpha/3)^{1/3}$ are necessary.

*The second order conditions.*

For Firm 1, as in case (i), there is nothing to check, since there are two variables and two binding constraints.

As for Firm 2, given that no constraint is binding, we have to consider the Hessian:

$$H_2 = \begin{pmatrix} -2/\Delta & 0 \\ 0 & -\alpha \end{pmatrix}.$$  

It is obviously definite negative so that the second order conditions are satisfied as well for Firm 2.

(iii) Case $\Psi_1 < 1$ and $\Psi_2 < 1$. We have $\mu_1 = \mu_2 = 0$. Using simultaneously Equations 23 and 24, we express $\Psi_1$ and $\Psi_2$, each as a function of the two prices and thus obtain again Equations 10 and 11. Then we substitute these expressions into Equation 22, which yields again Equation 13, i.e.:

$$a^2\Delta^2 + p_1^2(a\Delta + p_2^2) - p_1(a\Delta^2 + 2a\Delta p_2 + p_2^3) = 0.$$  

In this equation substitute $\Delta y$ for $p_1$; $a\Delta$ for $a$ and $v_2\Delta$ for $p_2$, then we obtain Condition 5.

We are going to show that 1) this equation has one and only one acceptable real positive solution under the conditions indicated in Proposition 1 Case CIE(iii); 2) otherwise (when these conditions are not satisfied), either no solution exists or the solution is not acceptable.

The derivative of $P_3$ w.r.t. $v_2$ is given by:

$$P'_3(v_2) = -3yv_2^3 + 2y^2v_2 - 2y\alpha.$$  

The discriminant of this second order polynomial is given by $y^2(y^2 - 6\alpha)$, which is of the same sign as $(y^2 - 6\alpha)$. The analysis depends on this sign.

1) **Suppose first** that $y^2 \leq 6\alpha$, the discriminant of $P'_3$ is always negative, thus $P_3$ is always decreasing.

---

13This can be seen by using Mathematica’s RegionPlot function.
14It is too long to be reproduced here.
The limit of $P_3$ as $v_2$ goes to infinity is $-\infty$. Condition 5 has one and only one real non-negative root if and only if $P_3(v_2 = 0) \geq 0$.

We have $P_3(v_2 = 0) = \alpha(\alpha + y(y - 1))$.

(a) For $(\alpha, y)$ satisfying simultaneously $y(y - 1) + \alpha \geq 0$ and $y^2 \leq 6\alpha$, $v_2^*$, the unique real non-negative root of $P_3$ corresponds to an interior equilibrium only if $y > v_2^* > y - 1$ (so that $0 < \hat{\theta} < 1$).

Since $P_3$ is decreasing in this case and $P_3(v_2^*) = 0$, then Inequality $y > v_2^*$ is equivalent to $P_3(v_2 = y) < 0$, which writes as $-y^2 - y + \alpha < 0$, thus is equivalent to $y > \frac{1}{2}(-1 + \sqrt{1 + 4\alpha})$.

Using in the same way the decreasingness of $P_3$, $v_2^* > y - 1$ if and only if $P_3(v_2 = y - 1) > 0$, which is equivalent to $y^3 + \alpha^2 + y(1 + \alpha) - y^2(2 + \alpha) > 0$.

We prove graphically that the two conditions $y > \frac{1}{2}(-1 + \sqrt{1 + 4\alpha})$ and $y - y^4/8\alpha - \alpha \leq 0$ imply the condition $y^3 + \alpha^2 + y(1 + \alpha) - y^2(2 + \alpha) > 0$.

The interior equilibrium identified in case IE(iii) of Proposition 1 corresponds to the solution of the present system composed of Equations 21, 22, 23 and 24, with $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$.

The three equations 22, 23 and 24 are satisfied here in the same way as in case IE(iii). For $p_1^* = Y$ to correspond to the choice of Firm 1 at equilibrium, necessarily $y \leq v_1^{IE}(\alpha)$.

Equation 22 can we easily rewritten (replacing $p_1$ by $y$) as

$$1 - \Psi_1 + \Psi_1(y - 2v_2) = 0$$

Therefore $\Psi_1 < 1$ is equivalent to $v_2^* > y/2$.

But we have supposed $y^2 < 6\alpha$, which implies $P_3'(v_2) < 0$ for all $v_2 \geq 0$. Thus $P_3(y/2) > P_3(v_2^*) = 0$. We have

$$P_3(y/2) = \alpha^2 - \alpha y + y^4/8.$$  

This way, we have shown $\Psi_1^* < 1$ to be equivalent to $y - y^4/8\alpha - \alpha \leq 0$.

Finally, consider the value of $\Psi_2$ as given by Equation 11 and substitute $Y$ for $p_1$ and $p_2^*$ for $p_2$. We obtain that an inequality $\Psi_2 = g(\alpha, y) \leq 1$ must hold at equilibrium.\(^{15}\)

Graphical analysis\(^{16}\) shows that $y^3 + \alpha^2 + y(1 + \alpha) - y^2(2 + \alpha) \geq 0$ and $g(\alpha, y) \leq 1$ hold when $y - y^4/8\alpha - \alpha \leq 0$, $y \geq \frac{1}{2}(-1 + \sqrt{1 + 4\alpha})$ and $y \leq v_1^{Int}$.

Second order conditions.

Firm 1’s bordered Hessian is, given that only the constraint on $p_1$ is active:

\(^{15}\)g(y, \alpha) is too long to be reproduced here.

\(^{16}\)Using the RegionPlot function of Mathematica.
decreasing with $p$ of the proposition with a positive $y$ and on the other hand inequation

$$P = \Psi v$$

admits potentially 3 roots, depending on the sign of $P$ and $a$. We must satisfy $P(0) < 0$, which implies $P(v_2) < 0$ for all $v_2 \geq 0$. This means that the profit of Firm 2 is decreasing with $p_2$, thus the equilibrium candidate in this case is $v_2 = 0$, $\Psi_2 = 0$, $\Psi_1 = \frac{\alpha}{3}$. The inequation $\Psi_1 < 1$ is equivalent to $y < \alpha$. But $y < \alpha$ implies on the one hand $y(y - 1) + \alpha > 0$, and on the other hand $y - (y^2/8\alpha) - \alpha < 0$, for which the equilibrium corresponds to case (iii) of the proposition with a positive $p_2$. Thus no new case appears with a null price $p_2$.

2) Suppose now that $y^2 > 6\alpha$, $P_3^\prime$ is a second degree polynomial that admits two roots:

$$v_2' = (1/3)(y - \sqrt{y^2 - 6\alpha})$$

and

$$v_2'' = (1/3)(y + \sqrt{y^2 - 6\alpha})$$

with $v_2' < v_2'' < y$. $P_3^\prime$ is positive between the two roots and negative outside, which means that $P_3$ is decreasing before $v_2'$, increasing between $v_2'$ and $v_2''$, then becomes decreasing.

We have that $P_3(v_2) = \alpha(-y - y^2 + \alpha)$, which is negative when $y^2 > 6\alpha$. Polynomial $P_3$ admits potentially 3 roots, depending on the sign of $P_3(0)$, $P_3(v_2')$ and $P_3(v_2'')$.

When they exist these roots ($R_i$) satisfy necessarily $R_1 < v_2'$, $v_2' < R_2 < v_2''$ and $v_2'' < R_3 < y$.

The root $R_1$, whenever it exists, is never relevant because $R_1 < v_2' < y/2$.

The root $R_2$, exists if and only if $P_3(v_2') < 0$, $P_3(v_2'') > 0$. For this root to be acceptable, it must satisfy $R_2 > y/2$ and the multiplier $\lambda$ calculated at $R_2$ must satisfy $\lambda_1 \geq 0$. Recall that

$$\Psi_1(v_2) = \frac{y(\alpha - yv_2 + v_2^2)}{\alpha^2 - y(y - v_2)v_2(1 - y + v_2)}$$

$$\Psi_2(v_2) = \frac{v_2(\alpha + y^2 - y(1 + v_2))}{\alpha^2 - y(y - v_2)v_2(1 - y + v_2)}$$
\[ \lambda_1(v_2) = \Psi_1(v_2) (1 + \Psi_2(v_2) (v_2 - 2y)) . \]

The representation of the set of \((\alpha, y)\) such that we have simultaneously: \(y^2 > 6\alpha, P_3(v_2') > 0, P_3(v_2') < 0, R_2 > y/2\) and \(\lambda_1 \geq 0\), leads to an empty set.

Finally, regarding \(R_3\), it exists if and only if \(P_3(v_2'') \geq 0\).

Again, for this root to be acceptable, \((\alpha, y)\) has to satisfy simultaneously \(y^2 > 6\alpha, P_3(v_2'') > 0, R_3 > y/2\) and \(\lambda_1 \geq 0\). And this set is proved graphically to be empty.

**Deviations.** Finally, we have to check that for each firm no profitable deviation exists among the prices such that its natural market is null\(^{17}\).

For Firm 1, such prices must satisfy \(p_1 \geq \Delta + p_2^*,\) or equivalently \(v_1 \geq v_2^* + 1\). Such prices do not exist as \(v_2^* + 1 > y\).

As for Firm 2, the prices such that it has no natural market must satisfy \(p_2 \geq p_1^* = Y\), thus \(p_2 = Y\), or \(v_2 = y\). But this price cannot be a profitable deviation for Firm 2 as \(v_2^*\) satisfies the first order conditions over the segment of prices \(y - 1 < v_2 < y\) and the profit is concave in \(v_2\) over this segment, thus it is decreasing in the neighborhood of \(y\).

**Proof of Proposition 1, Case 3 (Corner equilibrium).**

We deal successively with each possible case: first when Firm 2 has no natural market, second, when Firm 1 has no natural market. For each case, we identify the equilibrium candidate, then check whether profitable deviations exist.

1) If a corner equilibrium exists such that Firm 2 has no natural market, this means that \(\hat{\theta} \leq 0\) thus \(p_1 \leq p_2\).

We first prove that necessarily \(p_1^* > 0\). Indeed Firm 1’s profit writes:

\[ \pi_1 = p_1 \Psi_1 - (a/2)\Psi_2^2. \]

If ever \(p_1 = 0\) then necessarily \(\Psi_1 = 0\), which leads to profit \(\pi_1 = 0\).

Firm 2’s profit which writes:

\[ \pi_2 = p_2 \Psi_2 (1 - \Psi_1) - (a/2)\Psi_2^2 = p_2 \Psi_2 - (a/2)\Psi_2^2 \]

would be maximal for \(p_2 = Y\). Then Firm 1 has interest to deviate to a positive price \(p_1 < p_2\) and a sufficiently small \(\Psi_1^*\) that ensure a positive \(\pi_1\). Thus necessarily \(p_1^* > 0\).

Firm 2’s profit writes:

\[ \pi_2 = p_2 \Psi_2 (1 - \Psi_1) - (a/2)\Psi_2^2. \]

First note that necessarily \(\Psi_1^* < 1\). Indeed, otherwise the best profit in this situation would be \(\pi_2 = 0\) obtained at \(\Psi_2 = 0\), whereas Firm 2 may obtain a positive profit if it deviates to a price \(p_2 < p_1\) (which is possible since \(p_1^* > 0\)) and a sufficiently small \(\Psi_2\).

\(^{17}\)No profitable deviation exists among the prices ensuring to the firm the whole market as a natural market.
Hence $\pi_2$ is increasing with $p_2$. Thus $p_2^* = Y$ and $\Psi_2 = \frac{Y(1-\Psi_1)}{a}$.

Firm 1’s profit
$$\pi_1 = p_1 \Psi_1 - \frac{a}{2} \Psi_1^2.$$  

is increasing in $p_1$ then reaches its maximum at $p_1^* = p_2^* = Y$.

The optimal value of $\Psi_1$ is given by $\Psi_1 = \frac{\Psi_1^*}{a}$, since $\Psi_1 < 1$ then necessarily $Y < a$, or equivalently $y < \alpha$, which is thus a necessary condition. Hence we have $\Psi_1^* = \frac{Y}{a}$, which implies $\Psi_2 = \frac{Y}{a}(1 - \frac{\Psi_1}{a})$.

**Deviations.** For this equilibrium candidate to be an equilibrium, we have to check whether the firms have interest to deviate.

For Firm 1, when $p_2 = Y$, its profit has only one expression given by
$$\pi_1 = p_1 \Psi_1 - \frac{a}{2} \Psi_1^2, \quad \forall \quad p_1 \leq Y.$$  

The best option for Firm 1 in absolute terms is the one provided by the equilibrium candidate. This implies that Firm 1 has no interest to deviate.

As for Firm 2, if $Y > \Delta$,
$$\pi_2 = \begin{cases} 
  p_2 \Psi_2 - \frac{a}{2} \Psi_2^2 & \text{if } p_2 \leq Y - \Delta, \\
  p_2 \Psi_2(\hat{\theta} + (1 - \hat{\theta})(1 - \Psi_1^*)) - \frac{a}{2} \Psi_2^2 & \text{if } Y - \Delta < p_2 \leq Y
\end{cases}$$

If $Y \leq \Delta$, only the second line of the above profit applies. We have to consider two types of deviations: $Y - \Delta < p_2 < Y$ and $p_2 \leq Y - \Delta$, when $Y > \Delta$ and only $Y - \Delta < p_2 < Y$ for $Y \leq \Delta$.

Let us begin with deviations $Y - \Delta < p_2 < Y$.

The expression of $\pi_2$ for $Y - \Delta \leq p_2 \leq Y$ is a continuous and concave function\(^{18}\) in $(p_2, \Psi_2)$ which reaches necessarily its maximum. When an interior solution to the first order conditions exists, it is the maximum of the function, otherwise the maximum is reached on the borders.

The first order conditions yield:
$$\begin{cases} 
  \Psi_2 = \frac{p_2(1-\Psi_1^*+\hat{\theta} \Psi_1^*)}{\frac{a}{2}} \\
  p_2 = \Delta \frac{a^2+\alpha-y}{2y}
\end{cases}$$

For a deviation to a price $Y - \Delta < p_2 < Y$ to be non-profitable, it is necessary and sufficient that $p_2 = \Delta \frac{a^2+\alpha-y}{2y} \geq Y$, which is equivalent to
$$y \leq \frac{1}{2} \left(-1 + \sqrt{1 + 4\alpha}\right). \tag{25}$$

As $\frac{1}{2} \left(-1 + \sqrt{1 + 4\alpha}\right) < \alpha$, this implies $y < \alpha$. Condition 25 is thus more constraining than $y < \alpha$ and the couple of conditions reduces to the only Condition 25.

The proof ends here when $Y \leq \Delta$.

\(^{18}\)The Hessian matrix is definite negative.
For \( Y > \Delta \) and supposing Condition 25, we have also to consider deviations to prices \( p_2 \leq Y - \Delta \), for which the profit writes:

\[
\pi_2 = p_2 \Psi_2 - (a/2)\Psi_2^2,
\]

which is increasing with \( p_2 \) and concave in \( \Psi_2 \). Thus the best deviation of this nature is

\[
\begin{align*}
p_2^d &= Y - \Delta, \\
\Psi_2^d &= \frac{Y - \Delta}{a}.
\end{align*}
\]

Note that Condition 25 implies \( Y < a \) (or equivalently \( y < \alpha \)), so that \( \Psi_2^d < 1 \). The resulting profit is then equal to

\[
\pi_2^d = \frac{(Y - \Delta)^2}{2a},
\]

which is smaller than the equilibrium candidate profit \( \pi_2^* = \frac{Y_2^2}{2a} \) if and only if \( y \leq \sqrt{\alpha} \). But \( \frac{1}{2} \left( -1 + \sqrt{1 + 4\alpha} \right) < \sqrt{\alpha} \), meaning that Condition 25 implies \( y \leq \sqrt{\alpha} \). This ends the proof for the corner equilibrium such that Firm 2 has no natural market.

2) We now deal with possible corner equilibria such that Firm 1 has no natural market. We first identify the equilibrium candidate then we consider deviations. We will prove that the identified equilibrium candidate does not resist to unilateral deviations, thus is not an equilibrium.

That Firm 1 has no natural market means that \( \hat{\theta} \geq 1 \), i.e. \( p_1 \geq p_2 + \Delta \). This implies necessarily \( p_2 + \Delta \leq Y \).

Firm 1’s profit is then given by:

\[
\pi_1 = p_1 \Psi_1 (1 - \Psi_2) - (a/2)\Psi_1^2.
\]

Then necessarily \( \Psi_2 < 1 \) at equilibrium. \( \pi_1 \) is then increasing in \( p_1 \), thus \( p_1^* = Y \) and \( \Psi_1^* = \min(1, \frac{y(1-\Psi_2^*)}{a}) \).

As for Firm 2, \( \pi_2 = p_2 \Psi_2 - (a/2)\Psi_2^2 \), with \( p_2 \leq Y - \Delta \). Hence \( p_2^* = Y - \Delta \) and \( \Psi_2^* = \frac{Y - \Delta}{a} \) which is necessarily \( < 1 \).

At this step \( 1 < y < \alpha + 1 \) is a necessary condition.

Deviations. Now let us deal with the deviations.

We focus on Firm 2. Firm 2’s profit for all possible prices is given by:

\[
\pi_2 = \begin{cases} 
  p_2 \Psi_2 - (a/2)\Psi_2^2 & \text{if } p_2 \leq Y - \Delta, \\
  p_2 \Psi_2 (\hat{\theta} + (1 - \hat{\theta})(1 - \Psi_1^*)) - (a/2)\Psi_2^2 & \text{if } Y - \Delta < p_2 \leq Y
\end{cases}
\]

The identified equilibrium candidate corresponds for Firm 2 to the best \( p_2 \) and \( \Psi_2 \) for \( p_2 \leq Y - \Delta \), thus there exist no profitable deviations by Firm 2 among these prices.

We now consider the deviations to prices \( p_2 \) such that \( Y - \Delta < p_2 < Y \). \( \pi_2 \) is a concave function in \( p_2 \). The first order condition w.r.t. \( p_2 \) is independent of \( \Psi_2 \) and gives:

\[
p_2^d = \frac{Y - \Delta}{2} + \frac{\Delta}{2\Psi_1^*}.
\]
For deviations to such prices to be non-profitable, it is necessary and sufficient to have \( p_2^D \leq Y - \Delta \), which is equivalent to
\[
\frac{\Delta}{\Psi_1^*} \leq Y - \Delta
\] (26)

But \( \Psi_1^* \) may have two expressions.
When \( \frac{Y(a-Y+\Delta)}{\alpha^2} \geq 1 \), then \( \Psi_1^* = 1 \) and Equation 26 is equivalent to \( Y \geq 2\Delta \) or equivalently \( y \geq 2 \).

When \( \frac{Y(a-Y+\Delta)}{\alpha^2} < 1 \), then \( \Psi_1^* = \frac{Y(a-Y+\Delta)}{\alpha^2} \) and Equation 26 is equivalent to \( y(y-1)(1+\alpha-y) \geq \alpha^2 \).

We consider now the deviations of Firm 1. Firm 1’s profit writes:
\[
\pi_1 = \begin{cases} 
  p_1 \Psi_1 (1 - \hat{\theta} \Psi_2^*) - (a/2) \Psi_1^2 & \text{if } Y - \Delta < p_1 \leq Y, \\
  p_1 \Psi_1 - (a/2) \Psi_1^2 & \text{if } p_1 < Y - \Delta 
\end{cases}
\]

We consider first the deviations of Firm 1 to prices \( p_1 \) such that \( Y - \Delta < p_1 < Y \). For such prices, \( \pi_1 = p_1 \Psi_1 (1 - \hat{\theta} \Psi_2^*) - (a/2) \Psi_1^2 \), which is a concave function in \( p_1 \). The first order condition w.r.t. \( p_1 \), which is independent of \( \Psi_1 \) yields:
\[
p_1^D = \frac{Y - \Delta}{2} + \frac{\Delta a}{2(y - \Delta)}
\]

For such deviations to be non-profitable it is necessary and sufficient to have \( p_1^D \geq Y \), which is equivalent to \( y \leq \sqrt{\alpha + 1} \).

Considered for \( p_1 < Y - \Delta \), the profit \( \pi_1 \) is increasing in \( p_1 \). When \( y \leq \sqrt{\alpha + 1} \), \( \pi_1 \) is increasing for all prices, reaching its maximal value at \( p_1 = Y \). Hence no deviation is profitable for Firm 1.

To summarize, the equilibrium candidate is an equilibrium if and only if \((\alpha, y)\) satisfies either Conditions 1 \( (2 \leq y \leq \sqrt{\alpha + 1} \text{ and } \frac{y(\alpha+1-y)}{\alpha^2} \geq 1) \) or Conditions 2 \( (\frac{y(\alpha+1-y)}{\alpha^2} < 1 \text{ and } y(y-1)(1+\alpha-y) \geq \alpha^2 \text{ and } 1 \leq y \leq \sqrt{\alpha + 1}) \).

As depicted in Figure 17, when one of Conditions 1 is satisfied, another one \( (y \geq 2) \) is violated. Figure 18 also shows that no \((\alpha, y)\) satisfies all Conditions 2.
Figure 17: Representing Conditions 1 in the ($\alpha$, $y$)-space

Figure 18: Representing Conditions 2 in the ($\alpha$, $y$)-space