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gravitation – galaxies: general – galaxies: clusters: general – galaxies: clusters: intracluster medium – cosmology: dark matter
Addressing the missing mass problem in galaxy clusters that is found with MOND-type theories

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ABSTRACT
The MOND-type theories of AQUAdratic Lagrangian (AQUAL) and GRavitational Anti-Screening (GRAS) are applied to galaxy clusters. It will be shown that treating the baryonic mass distribution of the intracluster gas as being continuous leads to neglecting an important contribution to the clusters overall gravitational field. This is the localized mass density distribution that falls out from the theory which surrounds the protons within the gas. Including this contributor leads to an additional boost of approximately 60\% to the gravitational field of clusters, over what MOND-type theories have previously been taken to provide. This accounts for most, if not all, of the shortfall of gravity or mass that has been attributed to clusters when MOND-type theories are used.

Key words: gravitation – galaxies: general – galaxies: clusters: general – galaxies: clusters: intracluster medium – cosmology: dark matter

1. INTRODUCTION
Currently, the theory of dark matter provides the most accepted explanation for the greater gravitational fields required to explain the dynamics of galaxies and galaxy clusters. This theory postulates that non-baryonic matter exists within galaxies and clusters which does not interact by any of the three non-gravitational forces of the standard particle model but does interact gravitationally with itself and baryonic matter. A particular success of the theory has been that the distribution of dark matter expected matches well with what is required to explain the hydrostatic stability of hot X-ray emitting intracluster gas in clusters. Indeed, it is the observations of galaxy clusters that initially led to the hypothesis that some sort of dark matter was required (Zwicky 1933). Although successful with galaxy clusters, the theory has had difficulty dealing with observations on the scale of galaxies. This includes the speed of galactic bars in disc galaxies (Roshan et al. 2021), the distribution of satellite galaxies (Pawlowski 2021), and the prediction that some types of dwarf galaxies should not contain dark matter (Krupa 2021).
A summary of such issues can be found in Banik&Zhao (2022). The most conspicuous observation that the theory of dark matter has difficulty with is that the rotational curves of spiral galaxies are found to flatten out and approach a constant value in their outer regions, with no apparent cut-off to this observation (Famaey&McGaugh 2012). The modeling of the expected dark matter distribution leads to a rotational profile (Navarro, Frenk & White 1996,1997) that does not naturally lead to this result (Wu&Kroupa 2015; Desmond 2017a,b). Going beyond just the rotational curves flattening out, it has been found that there is a direct correlation between a galaxy’s baryonic mass and this constant outer rotational speed. This again does not fall out from the theory of dark matter. This relationship is known as the Baryonic Tully Fisher Relationship (BTFR) (McGaugh et al. 2000; McGaugh 2012) and is given by

$$M_b = A_{BTFR} v_f^4$$

where $v_f$ is the velocity in the outer flattened section of a given spiral galaxy’s rotation curve, $M_b$ is the enclosed baryonic mass of the galaxy, and $A_{BTFR}$ is a fitted parameter. The value of $A_{BTFR}$ as given by McGaugh (2012) is

$$A_{BTFR} = (47 \pm 6) M_\odot \text{ km}^{-4} \text{s}^4.$$  

Recently the relationship between a galaxy’s baryonic mass and its rotational speed has been expanded to consider the complete rotational curve. This is highlighted by the work of McGaugh et al (2016) and Lelli et al (2017) who found a relationship between the radial acceleration as determined from the observed rotational curves and the predicted radial acceleration due to Newtonian gravity determined from the baryonic mass distribution of the galaxies. The following Radial Acceleration Relationship (RAR) was found to provide a very good fit to the data:

$$g = \frac{g_N}{1- e^{-\sqrt{g_N/g_0}/g_0}}.$$  

where $g$ is the true acceleration as determined from the observed rotational curve, $g_N$ is the Newtonian gravitational field as determined from the baryonic mass distribution and $g_0$ is a fitted parameter. From McGaugh et al’s (2016) data set and their adopted stellar mass-to-light ratio, the fitted value of this observational parameter was found to be

$$g_0 = (1.20 \pm 0.02) \times 10^{-10} \text{ms}^{-2}.$$  

In the limit where $g_N \ll g_0$, (2) leads to

$$g = (g_N g_0)^{1/2}.$$  

The value for $g_N$ in (4) must take into account that the gravitational field within the disc of the
A galaxy is greater than the average field at the specified distance. Correcting for this leads to the field within the disc being given by
\[ g_N = \alpha \frac{GM_b}{r^2}, \]
with \( \alpha = 1.32 \) (McGaugh & Blok 1998). Equations (4) and (5) lead directly to the BTFR.

It is the inability of the dark matter theory to naturally explain the BTFR, and now the RAR, that has been the primary motivation behind alternative theories to dark matter. Both the BTFR and the RAR strongly suggest that the baryonic mass of a galaxy is responsible for the total gravitational force that it experiences. This leads to two general possibilities.

The first possibility is that Newtonian theory needs to be modified. This falls broadly under MOdified Newtonian Dynamics or MilgrOmiaN Dynamics (MOND; Milgrom 1983a,b,c; Berkenstein & Milgrom 1984; Famaey & McGaugh 2012; Sanders 2014; Banik & Zhao 2022). AQUAL (AQUADratic Lagrangian) is one such version of MOND (Bekenstein and Milgrom 1984). AQUAL leads to a field equation that reduces in the spherically symmetric case and for \( g << g_o \) to the BTFR.

The second possibility for the baryonic mass being so tightly correlated with the larger than expected gravitational fields is that the baryonic mass itself is directly or indirectly responsible for an additional contribution to the gravitational field. GRAS (GRavitational Anti-Screening; Penner 2016a, 2016b, 2018) is one example of theories that involve fields of mass dipoles contributing to the gravitational field (Blanchet 2007a, 2007b, Blanchet and Le Tiec 2008, Hajdukovic 2011a, 2011b, 2020). In the case of Hajdukovic’s dipole theory, it is found to disagree with observations taken within the solar system (Banik & Kroupa 2020). In the case of GRAS, which is in agreement with solar system measurements (Penner 2020), it is hypothesised that baryonic masses are surrounded by a sea of virtual mass dipoles analogous to quantum electrodynamic theory (QED), where charges are surrounded by a sea of virtual electric dipoles. In QED, the alignment of the virtual electric dipoles results in a screening effect that leads to the observed charge of a particle being less than its actual bare charge. In GRAS, the alignment of the virtual mass dipoles results in an anti-screening effect that leads to the observed gravitational mass of a galaxy being greater than its baryonic mass.

Both AQUAL and GRAS lead to excellent agreement with the BTFR and the RAR. However, while these MOND-type theories are in good agreement with observations on the galactic scale, there are problems when applied to galaxy clusters (Sanders 2003, McGaugh...
Extrapolating these MOND-type theories to clusters of galaxies has led to gravitational fields that are too weak to explain the hydrostatic equilibrium of the X-ray emitting cluster gas. More recently, ultra diffuse galaxies (UDGs) have been discovered within clusters (Mihos et al. 2015, Koda et al 2015). Due to the relatively large gravitational fields of the clusters, MOND-type theories predict that the UDG’s should behave close to Newtonian and thereby suffer tidal disruptions (Bilek et al 2019, Hodson & Zhao 2017a). However, these UDG’s typically do not show any signs of tidal disruptions. These galaxies therefore also appear to need significantly stronger gravitational fields than that currently provided by MOND-type theories. A unique problem is found with the Bullet cluster, where the galaxies have been spatially separated from the majority of the intracluster gas. For this special cluster, the gravitational fields provided by MOND-type theories also appear to be far too weak, in this case to explain gravitational lensing results (Angus et al 2007). All three of these problems that MOND-type theories have, involve diffuse baryonic mass distributions which require significantly more gravity than what is apparently provided by the theory.

The focus of this manuscript will be on the first of these problems, namely the shortfall in the gravitational fields found in clusters. This is a major problem for MOND-type theories and several explanations have been made to explain the gravitational field discrepancy. One possibility that has been suggested over the years is that the gravitational shortfall is a missing mass problem. Is there an additional mass component within clusters that has not yet been detected? The most common proposal is that neutrinos are the missing mass required (Sanders 2003, Angus et al. 2008, Angus 2009, Angus&Diaferio 2011, Angus et al. 2013). Over the years the mass of neutrinos required in clusters to explain the observations has headed upwards. Simulations have shown that the neutrinos of mass 30-300 keV would be required to produce the distribution of cluster masses that are observed. Requiring an undetected dark matter-like mass component, in addition to a MOND-type gravity, goes against the spirit of MOND-type theories. However, this is not a reason to dismiss the idea.

Another possibility is that the shortfall in the gravitational field can be realized by modifying MOND. In terms of MOND-type theories, this involves having the single parameter in the theory, $a_o$ or $g_o$ depending on the version, being larger in clusters than in galaxies. Having a larger $a_o$ or $g_o$ will lead to a greater gravitational field for the cluster and thereby could solve the cluster problem. In EMOND (Zhao & Famaey 2012, Hodson & Zhao 2017b) $a_o$ is taken to be...
a function of the gravitational potential, which in general is greater in magnitude within clusters than in galaxies. EMOND has also been applied to UDGs (Hodson & Zhao 2017a). Given that for both clusters and UDGs, EMOND leads to better agreement with observations does suggest that these two issues are one in the same.

In modifications to MOND-type theories, it is required to preserve the theories’ behaviour on galaxy scales, where they work so well, while significantly increasing the gravitational fields within clusters and UDGs. In general, this requires going from just the single parameter, $a_o$ or $g_o$, in the theory to having 2 or more parameters. Although such modifications to MOND-type theories certainly lead to better results with regards to clusters and UDGs, this is to be expected with the increased number of parameters.

In this manuscript it will be shown that much, if not all, of the missing mass or missing gravity is not related to a dark matter-like mass component or to a modification of the MOND-type theories. The missing mass has been there all along in these theories but has been neglected through the manner in which the MOND-type theories have been applied.

2. THEORY

2.1 AQUAL and GRAS field equations

In Newtonian gravitational theory, the field equation is given by

$$\nabla \cdot \nabla \Phi_N = 4\pi G \rho_b.$$  

(6a)

where $\Phi_N$ is the Newtonian gravitational scalar potential and $\rho_b$ is the baryonic mass density. The Newtonian gravitational field, $g_N$, in turn is given by

$$g_N = -\nabla \Phi_N.$$  

(6b)

In AQUAL gravitational theory, Newtonian gravitational theory and GR are modified, but the gravitational source remains $\rho_b$. The nonrelativistic AQUAL field equation is given by

$$\nabla \cdot (\mu(g_A/g_o) \nabla \Phi_A) = 4\pi G \rho_b,$$  

(7a)

with the resultant gravitational field, $g_A$, given by

$$g_A = -\nabla \Phi_A.$$  

(7b)

The function $\mu(g_A/g_o)$ is referred to as the interpolating function with $g_o$ being a parameter fit by observations. In the case of spherical symmetry (7a) becomes

$$\mu(g_A/g_o) g_A = g_N.$$  

(8)
The interpolating function $\mu(g_A/g_o)$ therefore determines the deviation between the AQUAL and Newtonian gravitational fields.

To have agreement with both the Newtonian theory within the solar system (Hees et al 2014, Hees et al 2016, Aurelien et al 2016) and the BTFR, the interpolating function must be such that

$$\mu(g_A/g_o) \rightarrow 1 \quad \text{for } g_A >> g_o, \quad (9a)$$

and

$$\mu(g_A/g_o) \rightarrow \frac{g_A}{g_o} \quad \text{for } g_A << g_o. \quad (9b)$$

For $g_A << g_o$, by (8) and (9b) the relationship between the Newtonian gravitational field $g_N$ and the AQUAL gravitational field $g_A$, in the case of spherical symmetry, leads to (4), which in turn leads to the BTFR.

Although in AQUAL the only gravitational source is the baryonic mass, for computational purposes, especially when dealing with a non-spherically symmetric mass distribution, a phantom dark matter density $\rho_{PDM}$ can be defined as

$$\rho_{PDM} = \frac{1}{4\pi G} \nabla \cdot \left( (1 - \mu(g_A/g_o)) \nabla \Phi_A \right). \quad (10)$$

The field equation can then be expressed as

$$\nabla \cdot \nabla \Phi_A = 4\pi G (\rho_b + \rho_{PDM}). \quad (11)$$

Knowing $\rho_b$ and determining the distribution $\rho_{PDM}$, one can then use Newton’s gravitational theory to determine $g_A$.

In GRAS’s gravitational theory, baryonic masses are taken to induce mass dipole moments in a surrounding sea of virtual mass dipoles. This leads to an anti-screening of the baryonic mass and hence a larger observed mass. The dependence that the resulting mass dipole moment density, $P$, has on the total gravitational field $g_G$ is given by the theory to be

$$P = \frac{1}{4\pi G} f(g_G/g_o) g_G, \quad (12)$$

with the function $f(g_G/g_o)$ incorporating any nonlinearity between $P$ and $g_G$. Analogous to a dielectric in electromagnetism, the equivalent mass density $\rho_{dipole}$ for this field of mass dipoles will be given by

$$\rho_{dipole} = -\nabla \cdot P \quad (13a)$$
In GRAS, Newtonian gravitational theory and GR hold but $\rho_{\text{dipole}}$ must now be included as a gravitational source. As such the nonrelativistic GRAS field equation is given by

$$\nabla \cdot \nabla \Phi_G = 4\pi G (\rho_b + \rho_{\text{dipole}}) \tag{14}$$

or by (13b)

$$\nabla \cdot \nabla \Phi_G = 4\pi G \rho_b + \nabla \cdot (f(g_G/g_o) \nabla \Phi_G) \tag{15}$$

In the case of spherical symmetry, (15) becomes

$$g_G = g_N + f(g_G/g_o) g_G \tag{16a}$$

or

$$(1 - f(g_G/g_o)) g_G = g_N \tag{16b}$$

In order have agreement with both Newtonian theory within the solar system and the BTFR, the interpolating function in GRAS must be such that

$$f(g_G/g_o) \to 0 \quad \text{for} \quad g_G >> g_o, \tag{17a}$$

and

$$f(g_G/g_o) \to 1 - \frac{g_G}{g_o} \quad \text{for} \quad g_G << g_o. \tag{17b}$$

From (7a) and (15), it can be seen that the field equations for AQUAL and GRAS are equivalent, with the interpolating functions for the two related by

$$\mu(g_A/g_o) = 1 - f(g_G/g_o). \tag{18}$$

The determined distributions for $\rho_{\text{dipole}}$ and $\rho_{\text{PDM}}$ and the generated gravitational and fields, $g_G$ and $g_A$, will be identical and as such will be referred to as $\rho_{GA}$ and $g_{GA}$ respectively throughout the remainder of the manuscript. The equivalence of both theories is very beneficial as previous results obtained specifically using AQUAL or specifically using GRAS will therefore apply to both theories. Although the two theories are equivalent, conceptually they are quite different. In the case of AQUAL, $\rho_{GA}$ is taken to be a phantom density, a tool used to assist in solving the field equation. In GRAS, $\rho_{GA}$ is a real gravitational source, equivalent to $\rho_b$. In GRAS and AQUAL the missing gravity or missing mass problem is a missing $\rho_{GA}$ problem.

In addition to having agreement with the BTFR and solar system observations, the interpolating function also needs to be in agreement with the more general RAR. The following interpolating function, found using GRAS (Penner 2023),
\[ f\left(\frac{g_{GA}}{g_o}\right) = \left(1 + \frac{g_{GA}}{2g_o}\right)^{-2}, \]  

satisfies (17) and leads to good agreement with the RAR. As such, (19) will be the interpolating function used in this manuscript. In the case of AQUAL, the equivalent interpolating function will be

\[ \mu\left(\frac{g_{GA}}{g_o}\right) = 1 - \left(1 + \frac{g_{GA}}{2g_o}\right)^{-2}. \]  

### 2.2 Solving the field equation

Given the nonlinear nature of the field equations for AQUAL and GRAS, determining the gravitational field surrounding an arbitrary non-spherically symmetric baryonic mass distribution \( \rho_b \) can be very computationally intensive. The general technique used by the author is as follows; given \( \rho_b \), the initial estimate of \( \Phi_{GA} \) is taken to be equal to \( \Phi_N \), i.e., that due solely to the baryonic masses. Then in the case of GRAS, (13), with an interpolating function \( f \) such as given by (19), is solved for an initial estimate of the distribution of \( \rho_{GA} \). For the next estimate, the \( \Phi_{GA} \) is then determined by integrating over the combined \( \rho_b \) and the estimate of \( \rho_{GA} \). This process is repeated, resulting in a new \( \rho_{GA} \) and a new \( \Phi_{GA} \), until the resulting values of the total gravitational field throughout the given volume obtained after a given iteration vary by less than a set amount from the previous iteration. These steps are listed in (21). In the case of AQUAL, the identical technique can be used, except now steps (21c) and (21d) are replaced by (10).

\[ \Phi_{GA}^{(i+1)} = -G \int \frac{\rho_{GA}^{(i)} + \rho_b}{r'} dV' \]  

\[ \rightarrow (21b) \]

For a non-spherically symmetric baryonic mass distribution it is necessary the range considered in the integral of (21e) must be large enough so that the distribution of \( \rho_{GA} \) tends towards spherical symmetry. This is one of the reasons why solving the field equation can be so computationally intensive. Just as with Newtonian theory, this is so that the gravitational field
contribution due to $\rho_{GA}$ beyond the considered volume goes to zero. An example of how this is handled is provided by the analysis of binary galaxies (Penner 2023).

Treating the baryonic mass distribution as being continuous and spherically symmetric makes things much easier. The baryonic mass distribution is taken to be a function of $r$, the distance from the centre of the distribution. From this $\rho_b(r)$, $g_N(r)$ can be calculated, and then by (8) or (16a), $g_{GA}(r)$ can be determined. An iterative method for determining $g_{GA}$ may or may not be required, depending on the interpolation function. Although treating the baryonic mass distribution as being continuous and spherically symmetric makes things computationally much easier, it may ignore local regions of $\rho_{GA}$, which as will be seen is a problem.

2.3 External field effects

In certain cases, there can be a significant complication when solving the field equation. For if there is a gravitational field present from other sources, outside of the volume of interest, one has what is known as the external field effect (EFE). In general, the volume over which $\rho_{GA}$ is calculated should include these external sources. Indeed, all baryonic sources that exhibit a significant gravitational field in the region of interest should be included. This is required as the total gravitational field is what is needed in the interpolating functions of AQUAL and GRAS. For example, in the case of determining the gravitational field of the Sun as far out as the Oort cloud, the gravitational field of the Galaxy needs to be taken into account. The volume considered for calculating $\rho_{GA}$ should therefore include the whole Galaxy! Fortunately, for cases like this, there is an alternative since the gravitational field due to the Galaxy in the region of the Sun can be taken to be constant. The volume of space around the Sun that needs to be considered then just needs to extend to where the gravitational field of the Galaxy dominates. Beyond this region $\rho_{GA}$ will fall off faster than $r^{-2}$, and its contribution to the gravitational field will tend to zero. The steps in calculating the gravitational field outlined by (21) remain the same except with the following change to (21b),

$$g_{GA}^{(i)} = -\nabla \phi_{GA}^{(i)} + g_{GA \text{\scriptsize \textsc{L}} \text{\scriptsize \textsc{A}} \text{\scriptsize \textsc{Y}}} \cdot$$

(22)

An example of the EFE of the Galaxy on the gravitational field of the Sun is provided in Penner (2020). The determined gravitational field that falls out of GRAS and AQUAL in the case where the Sun is isolated from the Galaxy and all other baryonic masses is shown in Figure 1. Within ~2 kAU of the Sun, the gravitational field due to the baryonic mass of the Sun is
dominant and the relative contribution of $\rho_{GA}$ to the gravitational field is negligible and as such $g_{GA} \approx g_N$. At larger distances, $\rho_{GA}$ falls off as $r^{-2}$ and provides a greater and greater contribution to the field. At large enough distances, one ends up with the BTFR with the gravitational field due to $\rho_{GA}$ dominating with the gravitational field falling off as $1/r$. However, taking into account the EFE due to the Galaxy greatly changes things. At a distance of 5.45 kAU from the Sun, the solar gravitational field is equal to the Galaxy’s approximately constant field of $1.96 \times 10^{-10}$ m s$^{-2}$ or 1.63$g_o$ (taking the Sun’s orbital velocity to be 220 km s$^{-1}$ at a distance of 8.0 kpc from the Galactic centre). Beyond this distance $\rho_{GA}$ falls off faster then $r^{-2}$, and its contribution to the gravitational field rapidly tends to zero. Interestingly, at these distances the gravitational field therefore goes back to falling off as $1/r^2$, just as with Newtonian gravity. This can be seen in Figure 1, where $g_{GA}$ with the EFE added is included. The $\rho_{GA}$ distribution is like a finite halo surrounding the Sun. Although the gravitational field of the Sun behaves Newtonian, due to the $\rho_{GA}$ halo it behaves as if it has a gravitational mass approximately 1.26 times greater than its baryonic mass.

**Figure 1:** The magnitude of the gravitational field surrounding the Sun in the cases of Newton’s theory (black line), the AQUAL/GRAS theory with an isolated solar mass (blue line), and the AQUAL/GRAS theory including EFE from the Galaxy (red line).

The above behaviour of the gravitational field of a baryonic mass in the presence of a uniform external field is a general result. The gravitational field of any baryonic mass immersed
in a constant gravitational field behaves Newtonian at great distances, but with a larger effective gravitational mass. In the case of AQUAL, the asymptotic value of the gravitational field of a spherically symmetric baryonic mass distribution in a constant external field $g_{\text{ext}}$ has been determined to be (Chae & Milgrom 2022, Chae et al 2021)

$$g_{\text{GA,asym}} = g_N \left( \frac{1}{\mu(x)} (1 + 0.75\frac{d}{d\ln x} \ln \mu(x)) \right)^{-1/2}$$

(23a)

where $x = g_{\text{ext}}/g_o$ and

$$\mu(x) = \frac{d}{d\ln x} \ln \mu(x).$$

(23b)

The effective gravitational mass, $m_{b,\text{eff}}$, of a baryonic mass $m_b$ immersed in a constant gravitational field is therefore given by

$$m_{b,\text{eff}} = K(x)m_b$$

(24a)

with

$$K(x) = \frac{1}{\mu(x)} (1 + 0.75\frac{d}{d\ln x} \ln \mu(x))^{-1/2}.$$ (24b)

Using (20) as the interpolating function, Figure 2 shows how, in general, the effective gravitational mass of the baryonic source depends on the strength of the external field it is immersed in. As is seen, the effect certainly needs to be considered when the external field is $\lesssim 5g_o$.

It needs to be stressed that this result applies to any baryonic mass, be it a galaxy, a star, or a particle. If the external field that the baryonic mass is immersed in is $\lesssim 5g_o$, it will have an effective gravitational mass that can be significantly greater than its actual mass. For our purposes, what is of interest is the effective gravitational mass of a proton, the primary baryonic mass component of the cluster gas. The external field in this case is just the gravitational field of the cluster itself.

Figure 3 shows the distribution of $\rho_{\text{GA}}$ surrounding a proton in the case where it is isolated and where it is immersed in a cluster’s gravitational field, taken in this example to be equal to $g_o$. The distribution of $\rho_{\text{GA}}$ around a proton immersed in a uniform field behaves similarly to the distribution of $\rho_{\text{GA}}$ found around the Sun immersed in the Galaxy’s field. Within $\sim 1 \times 10^{-14}$ m of the proton, the gravitational field due to its baryonic mass is dominant and as such $g_{\text{GA}} \approx g_N$. At a distance of $3.05 \times 10^{-14}$ m from the proton, the proton’s gravitational field is equal to the external field $g_o$. Beyond this distance the EFE comes into play and $\rho_{\text{GA}}$ falls off faster than $r^{-2}$, with the result that its contribution to the gravitational field rapidly tends to zero. As
with the Sun, the $\rho_{GA}$ distribution is like a finite halo surrounding the proton. The effective gravitational mass for a proton immersed in a gravitational field of value $g_o$ is determined by (24) to be $1.52m_p$.

**Figure 2:** The ratio of the effective gravitational mass to the baryonic mass, as given by $K(x)$ in (24b), as a function of $x = g_{\text{ext}}/g_o$.

**Figure 3:** The $\rho_{GA}$ density distribution a) surrounding an isolated proton; b) surrounding a proton immersed in a constant gravitational field of magnitude $g_o$ directed along the $+y$-axis. The solid outer black line corresponds to a mass density of $1.0 \times 10^{12}$ kg m$^{-3}$. The two solid inner red lines correspond to a mass density of $1.0 \times 10^{13}$ kg m$^{-3}$. The dashed lines correspond to mass densities of $2.0 \times 10^{12}$ kg m$^{-3}$ and $5.0 \times 10^{12}$ kg m$^{-3}$ respectively as one moves inward. Units along both axes are $10^{15}$ m.
3. GALAXY CLUSTER SAMPLE AND MODEL

For this manuscript, 12 clusters have been selected from the Chandra galaxy cluster sample (Vikhlinin et al. 2006) that Hodson and Zhao (2017b) used in their analysis. These 12 clusters are all nearby (z<1) and are in the baryonic mass range $10^{13}$ to $10^{14} \, M_\odot$.

The dominant baryonic mass component of these clusters is the X-ray emitting gas, or more specifically the protons that make up the gas, with the contribution from the electrons being ignored. To model the density of this gas, Vikhlinin et al (2006) choose a three-term model leading to the following expression for $\rho_{gas}(r)$, the baryonic mass density profile of the cluster gas,

$$\rho_{gas}(r) = 1.252 m_p n_o \left( \frac{(r/r_c)^{-\alpha}}{1+(r/r_c)\alpha/2} \frac{1}{(1+(r/r_s)^\beta)\gamma/\gamma} + \frac{1}{(1+(r/r_c2)\beta/2)^{3\beta_2}} \right)^{1/2}$$

(25)

where $m_p$ is the mass of a proton and $n_o$ is the central number density. The value for $n_o$ and the values for the scale radii $r_c$, $r_s$, and $r_c2$, along with the dimensionless parameters $\alpha$, $\beta$, $\gamma$, $\varepsilon$, and $\beta_2$ are given in Vikhlinin et al (2006) as well as in Hodson and Zhao (2017).

The temperature profile of the gas is also needed. The profile, as given by Vikhlinin et al (2006), is

$$T(r) = T_0 \left( \frac{r/r_{cool}+T_{min}/T_0}{(r/r_{cool})^a+c_{cool}+1} \frac{(r/r_t)^{-a}}{(r/r_t)^{b+1}/c^{1/b}} \right).$$

(26)

The values for the gas parameters are as well given in Vikhlinin et al (2006).

Given the density and temperature profiles from (25) and (26), one can estimate $M_{dyn}(r)$, the total mass enclosed within the radius $r$, required to have the gas in hydrostatic equilibrium (Sarazin 1988). The following expression for $M_{dyn}(r)$ is, as given in Vikhlinin et al (2006),

$$M_{dyn}(r) = -3.68 \times 10^{13} M_\odot T(r) r \left( \frac{d \ln \rho_{gas}}{d \ln r} + \frac{d \ln T}{d \ln r} \right)$$

(27)

where $T$ is in units of keV and $r$ is in units of Mpc.

To apply the AQUAL and GRAS theories to clusters, one needs to include all the baryonic masses. After the gas, the only other significant baryonic mass component are the galaxies themselves. Following Hodson and Zhao (2017) it will be taken that only the brightest cluster galaxy (BCG), located in the centre of the cluster, will have a significant contribution to the cluster’s gravitational field. The other galaxies are taken to contribute a relatively small amount to the overall baryonic mass of the cluster and to the resulting gravitational field. For the density profile for the BGC, the following Hernquist model from Hodson and Zhao (2017b) will
be used
\[ \rho_{BCG}(r) = \frac{M_{BCG}h}{2\pi r(r+h)^3} \]  
where \( h \), the Hernquist scale length, is set equal to 30 kpc for all clusters, as per Hodson and Zhao. The mass, \( M_{BCG} \), of the BCG, following from the work of Schmidt and Allen (2007), is given by
\[ M_{BCG} = 5.3 \times 10^{11} \left( \frac{M_{500}}{3 \times 10^{14} M_\odot} \right)^{0.42} \]  
where \( M_{500} \) is the enclosed mass at \( r_{500} \), the radius at which the average density is 500 times the critical density of the universe. The values for \( M_{500} \) for 10 of the 12 clusters are provided by Vikhlinin et al (2006). For the smallest 2 clusters, A262 and RX J1159+5531, the values for \( M_{500} \) were not provided. For these 2 clusters a value of \( 7.7 \times 10^{13} M_\odot \) was used for \( M_{500} \), corresponding to the value for MKW 4, a similarly sized cluster. This value has little impact on the results.

It is taken that the baryonic mass has a continuous spherically symmetric density distribution with \( \rho_b(r) = \rho_{\text{gas}}(r) + \rho_{\text{BCG}}(r) \). Then, as outlined in Section 2.2, the gravitation field \( g_{GA} \) that falls out from GRAS and AQUAL can be determined. The equivalent enclosed mass profile for GRAS and AQUAL, \( M_{GA}(r) \), can then be determined by
\[ M_{GA}(r) = \frac{r^2 g_{GA}}{G} \]  
Figure 4 shows the enclosed mass profiles \( M_b(r) \), \( M_{\text{dyn}}(r) \), and \( M_{GA}(r) \) for the galaxy cluster A133. At \( r = 1 \) Mpc, the values for \( M_b(r) \), \( M_{\text{dyn}}(r) \), and \( M_{GA}(r) \) are \( 3.46 \times 10^{13} M_\odot \), \( 4.04 \times 10^{14} M_\odot \), and \( 1.86 \times 10^{14} M_\odot \) respectively. The baryonic mass at this distance therefore provides only 8.6% of the mass or gravitational field required to keep the gas in hydrostatic equilibrium. The theories of AQUAL and GRAS do better, in that they provide 46% of the gravitational field required. Obviously much better, but still only half of the gravitational field required. Hence the missing gravity or missing mass problem for MOND-type theories in the case of galaxy clusters that was discussed in the introduction.

The above method of determining the gravitational field due to a field of baryonic masses is an excellent approximation in the case where the baryonic masses are immersed in a gravitational field which is \( \gtrsim 5g_o \). However, if the baryonic masses are immersed in a field \( \lesssim 5g_o \), this method is ignoring the \( \rho_{GA} \) localized around the masses. In the case of the protons that make
up the cluster gas, this localized $\rho_{GA}$ surrounding them is significant and can’t be neglected when calculating the overall gravitational field. To illustrate this, Figure 5 shows the nature of the distribution of $\rho_{GA}$ expected within a region of the cluster gas. To determine the gravitational field due to such a region, one needs to consider the field directly due to the baryonic masses themselves, i.e. the protons, plus the global $\rho_{GA}$ due to the whole cluster, which is indicated as background + in the figure, plus the localized distribution of $\rho_{GA}$ around the individual protons. Treating the distribution of the baryonic mass as being continuous is ignoring this third contributor to the gravitational field, is ignoring this additional $\rho_{GA}$. This will not simply add to the overall $\rho_{GA}$, the localized $\rho_{GA}$ is coupled to the global $\rho_{GA}$. The localized $\rho_{GA}$ will affect the global $\rho_{GA}$, just as an additional $\rho_b$ would. There is no difference between the affects. Fundamentally the problem is that the baryonic mass distribution is not continuous, but discrete, i.e., the protons that make up the gas. This problem should properly be treated as a N-body problem, with N being the number of protons in the cluster! When integrating over $\rho_b$ and $\rho_{GA}$ in step (21e), during the iterative process outlined, the integral needs to include the space directly around the protons. Unfortunately, in practise this integral is done numerically, and the resolution would need to on the order of $10^{-15}$ m, to obtain the localized distribution of $\rho_{GA}$ around the protons, while the range would need to be that of the cluster itself, i.e., on the order of a Mpc!
This would be impracticable to solve via the general technique described in Section 2.2.

Fortunately, there is an alternative. The gravitational field can be calculated, treating the baryonic mass distribution as being continuous and treating the problem as having spherical symmetry as is normally done, i.e., ignoring the \( \rho_{GA} \) halo around the protons. However, in order to include the gravitational affect of the \( \rho_{GA} \) halos, instead of using the mass of the proton \( m_p \) as the baryonic source of the gravitational field, the effective gravitational mass \( m_{p,\text{eff}} \) of the proton needs to be used. In terms of an effective baryonic density, this is given by

\[
\rho_{b,\text{eff}}(r) = K(g_{GAE}/g_0)\rho_b,
\]

(31)

where \( g_{GAE} \), the net gravitational field within the cluster, is the external field that the protons are immersed in. In the case of GRAS, instead of (16a), the equation to be solved is therefore

\[
g_{GAE} = g_{Ne} + f(g_{GAE}/g_0)g_{GAE}
\]

(32)

with

\[
g_{Ne} = \frac{G}{r^2} \int_0^r K(g_{GAE}/g_0)\rho_b \, dV.
\]

(33)

The iterative steps that are used by the author in solving (32) are

\[
g_{GAE}^{(i)} = g_N = \frac{G}{r^2} \int \rho_b \, dV'
\]

(34a)

\[
g_{Ne}^{(i)} = \frac{G}{r^2} \int K(g_{GAE}^{(i)}/g_0)\rho_b \, dV'
\]

(34b)
\[ g^{(1+1)}_{\text{GAe}} = f \left( \frac{g^{(i)}_{\text{GAe}}}{g_o} \right) g^{(i)}_{\text{GA}} + g^{(i)}_{\text{Ne}}. \]  
\[ \Rightarrow \]  
\[ (34b) \]

From the determined \( g_{\text{GAe}} \), the equivalent enclosed mass profile will then be

\[ M_{\text{GAe}}(r) = \frac{r^2 g_{\text{GAe}}}{G}. \]  
\[ (35) \]

4. RESULTS

The above model, using equations (32) and (33), was applied to all 12 clusters and the results are shown in Figures 6 and 7. Each figure shows the enclosed baryonic mass profile, \( M_{\text{bary}}(r) \), due to \( \rho_b \), the theoretical mass profile, \( M_{\text{dyn}}(r) \), based on the gas being in hydrostatic equilibrium, the equivalent enclosed mass profile, \( M_{\text{GA}}(r) \), due to the AQUAL/GRAS gravitational theory using (16a), and \( M_{\text{GAe}}(r) \), the enclosed mass profile due to the AQUAL/GRAS theory using (32) and (33), i.e. taking into account the effective gravitational mass of the protons. The radial range used for each cluster corresponds to the radial range used by Vikhlinin et al (2006), with \( r_{\text{min}} \) corresponding to the radial range used for their temperature profile fit, and \( r_{\text{max}} \) corresponding to the outer boundary of the Chandra field of view. These ranges are given in Table 1.

To quantify the results, a reference range \( r_{\text{peak}} \), corresponding to where the maximum of \( M_{\text{dyn}} \) occurs for each cluster, was used. For 8 of the 12 clusters this corresponds to \( r_{\text{max}} \). For the other 4 clusters, the maximum of \( M_{\text{dyn}} \) occurs before \( r_{\text{max}} \), which does indicate a breakdown of the hydrostatic model beyond this distance. The values for \( r_{\text{peak}} \) as well as the resulting values for the enclosed masses \( M_b, M_{\text{dyn}}, M_{\text{GA}}, \) and \( M_{\text{GAe}} \) at \( r_{\text{peak}} \) are included in Table 1. In Table 2 the ratios of these enclosed masses relative to \( M_{\text{dyn}} \) at \( r_{\text{peak}} \) are shown. Averaging these values over the 12 clusters results in \( M_b/M_{\text{dyn}} \text{peak} = 0.13 \pm 0.04, M_{\text{GA}}/M_{\text{dyn}} \text{peak} = 0.63 \pm 0.12, \) and \( M_{\text{GAe}}/M_{\text{dyn}} \text{peak} = 1.00 \pm 0.18. \) Of course, these values are dependent on the reference range used. Overall, including the halos of \( \rho_{\text{GA}} \) around the protons when determining \( M_{\text{GAe}} \) leads to excellent agreement with the total enclosed cluster mass within \( r_{\text{peak}} \) required to have the gas in hydrostatic equilibrium. The apparent missing mass in clusters for MOND-type theories has been there all along in the localized \( \rho_{\text{GA}} \) around the protons. There is no need for an additional mass component or a requirement to modify the theories of AQUAL and GRAS. The average boost given to the AQUAL/GRAS results, by including the \( \rho_{\text{GA}} \) halos around the protons, for the 12 clusters is \( M_{\text{GAe}}/M_{\text{GA}} \text{peak} = 1.60 \pm 0.12 \) or approximately 60%.
Figure 6: Enclosed masses in units of $M_\odot$ versus radial distance for 6 of the clusters. Enclosed baryonic mass $M_b$ (lower black line); Dynamic mass $M_{\text{dyne}}$ based on hydrostatic equilibrium (upper black line); enclosed mass $M_{GA}$ (blue line) as per AQUAL/GRAS theory; enclosed mass $M_{GAe}$ (red line) as per AQUAL/GRAS theory using the effective gravitational mass of the protons. The vertical scale is in units of $M_\odot$. 
Figure 7: Enclosed masses in units of $M_\odot$ versus radial distance for 6 of the clusters. Enclosed baryonic mass $M_b$ (lower black line); Dynamic mass $M_{\text{dyne}}$ based on hydrostatic equilibrium (upper black line); enclosed mass $M_{GA}$ (blue line) as per AQUAL/GRAS theory; enclosed mass $M_{GAe}$ (red line) as per AQUAL/GRAS theory using the effective gravitational mass of the protons. The vertical scale is in units of $M_\odot$. 
Table 1: Cluster sample; \( r_{\text{min}} \) corresponds to the inner boundary of the radial range used in Vikhlinin et al.’s (2006) temperature profile; \( r_{\text{max}} \) is the outer boundary of the Chandra field of view; \( r_{\text{peak}} \) is the radial distance where \( M_{\text{dyn}}(r) \) is a maximum.

<table>
<thead>
<tr>
<th>Cluster</th>
<th>( r_{\text{min}} ) kpc</th>
<th>( r_{\text{max}} ) kpc</th>
<th>( r_{\text{peak}} ) kpc</th>
<th>( M_b(r_{\text{peak}}) \times 10^{14} \ M_\odot )</th>
<th>( M_{\text{dyn}}(r_{\text{peak}}) \times 10^{14} \ M_\odot )</th>
<th>( M_{\text{GA}}(r_{\text{peak}}) \times 10^{14} \ M_\odot )</th>
<th>( M_{\text{GAe}}(r_{\text{peak}}) \times 10^{14} \ M_\odot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A133</td>
<td>40</td>
<td>1100</td>
<td>1100</td>
<td>0.385</td>
<td>4.50</td>
<td>2.15</td>
<td>3.58</td>
</tr>
<tr>
<td>A262</td>
<td>10</td>
<td>450</td>
<td>450</td>
<td>0.698</td>
<td>0.705</td>
<td>0.376</td>
<td>0.641</td>
</tr>
<tr>
<td>A383</td>
<td>25</td>
<td>800</td>
<td>800</td>
<td>0.407</td>
<td>2.94</td>
<td>1.66</td>
<td>2.50</td>
</tr>
<tr>
<td>A478</td>
<td>30</td>
<td>2000</td>
<td>2000</td>
<td>1.48</td>
<td>8.70</td>
<td>6.32</td>
<td>9.29</td>
</tr>
<tr>
<td>A1413</td>
<td>20</td>
<td>1800</td>
<td>1528</td>
<td>1.25</td>
<td>11.1</td>
<td>5.51</td>
<td>8.40</td>
</tr>
<tr>
<td>A1795</td>
<td>40</td>
<td>1500</td>
<td>1500</td>
<td>0.955</td>
<td>6.36</td>
<td>4.68</td>
<td>7.27</td>
</tr>
<tr>
<td>A1991</td>
<td>10</td>
<td>1000</td>
<td>792</td>
<td>0.184</td>
<td>1.40</td>
<td>1.07</td>
<td>1.15</td>
</tr>
<tr>
<td>A2029</td>
<td>20</td>
<td>2250</td>
<td>1684</td>
<td>1.60</td>
<td>9.04</td>
<td>6.90</td>
<td>10.43</td>
</tr>
<tr>
<td>RXJ1159+5531</td>
<td>10</td>
<td>600</td>
<td>600</td>
<td>0.070</td>
<td>0.987</td>
<td>0.492</td>
<td>0.865</td>
</tr>
<tr>
<td>MKW4</td>
<td>5</td>
<td>550</td>
<td>550</td>
<td>0.0572</td>
<td>0.632</td>
<td>0.407</td>
<td>0.732</td>
</tr>
<tr>
<td>A907</td>
<td>40</td>
<td>1300</td>
<td>1300</td>
<td>0.894</td>
<td>5.23</td>
<td>3.96</td>
<td>6.12</td>
</tr>
<tr>
<td>A2390</td>
<td>80</td>
<td>2500</td>
<td>2120</td>
<td>3.04</td>
<td>21.7</td>
<td>12.1</td>
<td>17.9</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td></td>
<td>0.128</td>
<td>0.63</td>
<td>1.00</td>
<td>1.60</td>
</tr>
</tbody>
</table>

Table 2: The enclosed masses at \( r_{\text{peak}} \) relative to \( M_{\text{dyn}}(r) \). The last column indicates the boost given to the AQUAL/GRAS theory when the effective gravitational mass is used.

<table>
<thead>
<tr>
<th>Cluster</th>
<th>( M_b/M_{\text{dyn}} ) at ( r_{\text{peak}} )</th>
<th>( M_{\text{GA}}/M_{\text{dyne}} ) at ( r_{\text{peak}} )</th>
<th>( M_{\text{GAe}}/M_{\text{dyn}} ) at ( r_{\text{peak}} )</th>
<th>( M_{\text{GAe}}/M_{\text{GA}} ) at ( r_{\text{peak}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A133</td>
<td>0.0856</td>
<td>0.478</td>
<td>0.796</td>
<td>1.67</td>
</tr>
<tr>
<td>A262</td>
<td>0.0990</td>
<td>0.533</td>
<td>0.909</td>
<td>1.71</td>
</tr>
<tr>
<td>A383</td>
<td>0.138</td>
<td>0.565</td>
<td>0.850</td>
<td>1.51</td>
</tr>
<tr>
<td>A478</td>
<td>0.170</td>
<td>0.726</td>
<td>1.07</td>
<td>1.47</td>
</tr>
<tr>
<td>A1413</td>
<td>0.113</td>
<td>0.496</td>
<td>0.757</td>
<td>1.53</td>
</tr>
<tr>
<td>A1795</td>
<td>0.150</td>
<td>0.736</td>
<td>1.14</td>
<td>1.55</td>
</tr>
<tr>
<td>A1991</td>
<td>0.138</td>
<td>0.764</td>
<td>1.29</td>
<td>1.69</td>
</tr>
<tr>
<td>A2029</td>
<td>0.177</td>
<td>0.763</td>
<td>1.15</td>
<td>1.51</td>
</tr>
<tr>
<td>RXJ1159+5531</td>
<td>0.0709</td>
<td>0.498</td>
<td>0.876</td>
<td>1.76</td>
</tr>
<tr>
<td>MKW4</td>
<td>0.0905</td>
<td>0.644</td>
<td>1.16</td>
<td>1.80</td>
</tr>
<tr>
<td>A907</td>
<td>0.171</td>
<td>0.757</td>
<td>1.17</td>
<td>1.55</td>
</tr>
<tr>
<td>A2390</td>
<td>0.140</td>
<td>0.558</td>
<td>0.825</td>
<td>1.48</td>
</tr>
<tr>
<td>Average</td>
<td>0.128 ± 0.036</td>
<td>0.63 ± 0.12</td>
<td>1.00 ± 0.18</td>
<td>1.60 ± 0.12</td>
</tr>
</tbody>
</table>
Although, there is good agreement between observations and theory with regards to the overall enclosed mass within $r_{\text{peak}}$, a significant difference between the distributions of $M_{\text{dyn}}(r)$ and $M_{\text{GAe}}(r)$ can be seen in Figures 6 and 7. Figure 8 shows examples of the densities corresponding to $M_{\text{dyn}}(r)$ and $M_{\text{GAe}}(r)$, using sample clusters A133 and A907. It is seen from this figure that more mass or gravity in the inner regions of the cluster would be required for hydrostatic equilibrium than is found with AQUAL/GRAS, even taking into account the effective gravitational mass of the protons. It may be that more baryonic mass is required in the inner regions, it may also be that the assumption that the gas is in hydrostatic equilibrium is too great of a simplification.

Figure 8: The density profiles corresponding to $M_{\text{dyn}}(r)$ (black line) and $M_{\text{GAe}}(r)$ (red line) for clusters A133 and A907. The units for $\rho$ on the vertical axis are kg m$^{-3}$.

5. APPLICATIONS TO GALAXIES

It is important to consider the expected effect on galaxies. Just as with clusters, when applying MOND-type theories to galaxies the baryonic mass distribution has been taken to be continuous. In the case of galaxies, it is the stars that are the major baryonic mass source. Stars in the outer regions of galaxies will typically be immersed in fields $\lesssim 5g_0$, and therefore will have a halo of $\rho_{\text{GA}}$ surrounding them just as with the protons within the cluster gas. As previously discussed, the Sun’s effective gravitational mass is 1.26 times its baryonic mass. The same method of dealing with clusters can be applied to galaxies.
The goal of this section is just to provide an estimate of the effect of considering the effective gravitational mass of stars within a galaxy. To estimate the effect of the $\rho_{GA}$ halos around the stars on the total gravitational field, the following spherically symmetric model of the baryonic mass distribution within a galaxy was used,

$$\rho_{\text{bulge}}(r) = \rho_{bo}(r/r_b)^{-2} e^{-(r/r_b)^2}$$

(36a)

$$\rho_{\text{disc}}(r) = \rho_{do}e^{-(r/r_d)}$$

(36b)

where $\rho_{bo}$ and $\rho_{do}$ are the central mass densities and $r_b$ and $r_d$ are scale lengths of the bulge and disc respectively. In the model the effective mass of the stars depends on the strength of the gravitational field they are immersed in. As such, to account for the stars in a real galactic disc being in a stronger field than that stemming from the spherical distribution as given by (36b), the value of $g_{GAe}$ in (33) was multiplied by the correction factor $\alpha$ indicated in (5). Although this is a relatively crude way of handling galaxies, the goal is to determine the relative impact of the $\rho_{GA}$ halos around the stars. Figure 9 shows examples, using this baryonic mass model, of the rotation curves derived from (16a), neglecting the halos of $\rho_{GA}$ around the stars, and from (32) and (33), including the halos. For these figures, galaxy masses of $25\times10^9$ $M_\odot$, $50\times10^9$ $M_\odot$, and $100\times10^9$ $M_\odot$ were used, with $M_{\text{disc}}/M_{\text{bulge}}$ set equal to 3 and with the $r_d$ and $r_b$ values given in the figure caption. The boost to the rotational curve velocities beyond the inner region for the 3 galaxies, due to the inclusion of the $\rho_{GA}$ halos, is approximately 5%, with the boost to the gravitational field being approximately 10%. This compares to the boost to the gravitational field in the case of the clusters being approximately 60%.

In general, it is also found with the model that the more dispersed the stars within a galaxy are, the greater the effect. As an extreme example, the gravitational field of a point baryonic mass of mass $10\times10^9$ $M_\odot$ was compared to a spherically symmetric uniform distribution of stars. The baryonic density of the spherical star distribution was set to $2.53\times10^{-24}$ kg m$^{-3}$ so that $g_N$ at 40 kpc was the same for both distributions, i.e., at $8.7\times10^{-13}$ m s$^{-2}$. Ignoring the $\rho_{GA}$ halos surrounding the stars, both distributions lead to an AQUAL/GRAS gravitational field of $1.0\times10^{-11}$ ms$^{-2}$ at 40 kpc, which can also be determined directly by (4). Taking into account the effective gravitational mass using (32) and (33), the gravitational field at 40 kpc for the uniform star distribution is found to be $2.4\times10^{-11}$ m s$^{-2}$, 2.4 times greater! Just as MOND-type theories provide a larger boost to smaller galaxies, they also apply a larger boost to more diffuse systems.
Given that $g_{GA}$ depends not only on $g_N$ but also on the distribution of the baryonic matter, relationships such as the BTFR and RAR are only valid in cases where $m_{loc} \cong m_b$, i.e., with the localized $\rho_{GA}$ around the baryonic masses being neglected.

![Figure 9](image)

**Figure 9:** The rotational velocity curves for modeled galaxy distributions with $M_{\text{disc}}/M_{\text{bulge}} = 3$ and; a) $M_{\text{total}} = 25 \times 10^9 M_\odot$, $r_b = 0.75 \text{ kpc}$ and $r_d = 1.875 \text{ kpc}$; b) $M_{\text{total}} = 50 \times 10^9 M_\odot$, $r_b = 1.0 \text{ kpc}$ and $r_d = 2.5 \text{ kpc}$; c) $M_{\text{total}} = 100 \times 10^9 M_\odot$, $r_b = 1.5 \text{ kpc}$ and $r_d = 3.75 \text{ kpc}$. Units for $v_R$ are km s$^{-1}$.

6. CONCLUSIONS

It has been shown that when calculating the AQUAL/GRAS gravitational field for galaxy clusters, the baryonic mass distribution cannot be treated as a continuous distribution. The baryonic mass distribution is in fact discrete with individual protons contributing to the gravitational field. These particles will have finite halos of $\rho_{GA}$ around them, which are a significant contributor to the gravitational field. In order to treat the baryonic mass distribution as being continuous, one must then consider the effective mass of the protons, i.e., the protons plus their $\rho_{GA}$ halos. For the sample of 12 clusters considered, including the $\rho_{GA}$ halos increased the total gravitational fields derived by AQUAL/GRAS by 60%. This would account for the majority, if not all, of the apparent missing gravity or missing mass found in clusters. However, the difference between the calculated density distribution, $\rho_{GA}$, and the expected density distribution, based on the gas being in hydrostatic equilibrium, still needs an explanation.
Certainly, a more complete study of the impact that the $\rho_{\text{GA}}$ halos around the stars in a galaxy have on the galaxy’s rotation curve is needed, but the analysis provided does show that the effect is significant, although much smaller than that found with clusters. In general, the BTFR and RAR cannot be extended as they are to clusters. The gravitational field within a cluster or galaxy will depend not only on $g_N$ but also on the distribution of the baryonic mass. Any possible relationship would need to be between the gravitational field and $M_{b,\text{effective}}$.

Hopefully, applying the model to UDG’s should provide the additional gravitational field that they require. The Bullet cluster is a very interesting case as it is an extremely diffuse baryonic mass system, lacking the normal amount of cluster gas. It would be expected that there would be a very large boost to the gravitational field, due to the $\rho_{\text{GA}}$ halos surrounding the galaxies and surrounding the protons in whatever gas remains. This should alleviate at least some of the tension between MOND-type theories and the Bullet cluster.

It is expected that including the halos of $\rho_{\text{GA}}$ around the particles within a gas and around the stars within a galaxy will help solve most of the issues that the MOND-type theories of AQUAL and MOND currently have. Ideally, the scientific community can then move on from the theory of dark matter.

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