Tense operators on frameable equality algebras

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Research Article

Keywords: Frameable equality algebras, Tense operator, Tense deductive system, Representation

Posted Date: May 3rd, 2021

DOI: https://doi.org/10.21203/rs.3.rs-321896/v1

Tense operators on frameable equality algebras

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Abstract: In this paper, we introduce the frameable equality algebras and use the concept of tense operators on them to define tense equality algebras. We investigate some algebraic properties of tense equality algebras and prove the representation theory for strict strong tense equality algebras. Then we introduce the notions of (prime) tense deductive systems and tense congruences and obtain some structural theorems.

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1 Introduction

Novák introduced the concept of EQ-algebras in [20] as candidates for a possible algebraic semantics of fuzzy-type theory.
Jenei introduced equality algebras in [12]. The motivation for introducing equality algebras came from EQ-algebras. As Jenei mentioned, if the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original (viewed as a two-place function) we still obtain an EQ-algebra. This fact might make it difficult to obtain certain algebraic results. For this reason, Jenei introduced a new structure, called equality algebra, to find something similar to EQ-algebras but without a product. The equality algebras have two binary operations meet and equivalence, and a constant 1. In [19], F. Zebardast and et al. studied and proved that there are relations among equality algebras and some of other logical algebras. Some types of filters of equality algebras are introduced in [2] and the corresponding logic to equality algebras is introduced and studied in [16].
Jenei and Kóródi defined pseudo equality algebras as a generalization of equality algebras in [14]. Moreover, in [11] Dvurečenskij and Zahiri proposed that every pseudo-equality algebra in the sense of Jenei and Kóródi is an equality algebra and they defined and investigated a new concept of

pseudo-equality algebras. Recently, monadic pseudo-equality algebras are studied in [15]. Since then many researchers have worked on this area.
Classical tense logic is the propositional logic with two tense operators G which reveals the future and H which expresses the past. Burges [5] studied tense operators on Boolean algebra. Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in the logic.
Later, many researchers have investigated and studied tense operators on the other algebras. Diaconescu and Georgescu [10] studied the tense operators for MV - algebra and Łukasiewicz-Moisil algebras and Chirita studied tense θ Moisil propositional logic in [8]. Chajda and et al. studied tense operators for basic algebras, effect algebras, De Morgan algebras (see [4], [6], [7]). Recently, Bakhshi [1] studied the algebraic properties of tense operators for non-commutative residuated lattices and Liu [17] proved the representation theory for (strict) strong tensor non-commutative residuated lattices.
This paper is organized as follows: In Sec. 2, some basic definitions and results are mentioned. In Sec. 3, we introduce and study frameable equality algebras. Then, we investigate tense operators on frameable equality algebras. We present a construction of natural tense operators when a frameable equality algebra and a frame are given. We give the representation theory of strict strong tense involutive equality algebra. In Sec. 4, the notions of (prime) tense deductive systems are studied and several characterizations of them are obtained. Tense deductive system generated by a nonempty subset is characterized. We study the relation between tense deductive systems and tense congruences. Also, it is shown that the set of all tense deductive systems is algebraic lattice. In particular, we prove that every tense prelinear equality algebra is a subdirect product of a systems of linearly ordered tense equality algebras.

2 Preliminaries

In this section, we recall the basic definitions and some known results about equality algebras that we need in the rest of the paper.

Definition 2.1.([12]) An equality algebra is an algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ of the type (2, 2, 0) such that satisfies the following axioms for all $x, y, z \in A$:
(E1) $(A, \wedge, 1)$ is a meet-semilattice with top element 1,
(E2) $x \sim y = y \sim x$,

(E3) $x \sim x = 1$,
(E4) $x \sim 1 = x$,
(E5) $x \leq y \leq z$ implies $x \sim z \leq y \sim z$ and $x \sim z \leq x \sim y$,
(E6) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$,
(E7) $x \sim y \leq (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called meet (infimum) and $\sim$ is called an equality operation. We write $x \leq y$ (and $y \geq x$) iff $x \wedge y = x$. We define the following two derived operations, the implication and the equivalence operation on the equality algebra $\mathcal{A}$ by
(I) $x \to y = x \sim (x \wedge y)$,
(II) $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.
An equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called bounded if there exists an element $0 \in A$ such that $0 \leq x$, for all $x \in A$ and $x \sim 0 = x \to 0$ is denoted by $\neg x$. If $\neg\neg x = x$ for all $x \in A$, then the bounded equality algebra $\mathcal{A}$ is called involutive.([3], [19])

Proposition 2.2.([12]) Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality algebra and consider
(E5a) $x \sim (x \wedge y \wedge z) \leq x \sim (x \wedge y)$,
(E5a') $x \to (y \wedge z) \leq x \to y$,
Then (E5) is equivalent to (E5a), which in turn is equivalent to (E5a').

Definition 2.3.([19]) Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality algebra. Then
(1) $\mathcal{A}$ is called prelinear, if 1 is the unique upper bound of the set $\{x \to y, y \to x\}$ for all $x, y \in A$.
(2) $\mathcal{A}$ is called equivalential, if $\sim$ coincides with the equivalence operation of a suitably chosen equality algebra.
(3) A lattice equality algebra is an equality algebra which is a lattice.

Proposition 2.4.([19])(1) An equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is an equivalential if and only if $x \sim y = x \leftrightarrow y$ for all $x, y \in A$.
(2) Every prelinear equality algebra is an equivalential.
(3) Any prelinear equality algebra is a distributive lattice where

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

(4) if $\mathcal{A}$ is a prelinear equality algebra, then $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.
(5) if $\mathcal{A}$ is a lattice equality algebra, then $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Proposition 2.5.([12, 19]) Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality algebra. Then the following hold for all $x, y, z \in A$:
(1) $x \to y = 1$ iff $x \leq y$,
(2) $1 \to x = x$, $x \to 1 = 1$and $x \to x = 1$,
(3) $x \leq y \to x$,
(4) $x \leq y \to z$ iff $y \leq x \to z$,
(5) $x \to (y \to z) = y \to (x \to z)$,
(6) $x \leq y$ implies that $z \to x \leq z \to y$ and $y \to z \leq x \to z$.

Definition 2.6.([12]) Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality algebra and F be a subset of A. Then F is called a deductive system of $\mathcal{A}$ if for all $x, y \in A$,
(i) $1 \in F$,
(ii) if $x \in F$ and $x \leq y$, then $y \in F$,
(iii) if $x \in F$ and $x \sim y \in F$, then $y \in F$.

Proposition 2.7.([12]) Let $\mathcal{A} = (A, \wedge, \sim, 1)$ be an equality algebra and F be a subset of A. Then F is a deductive system of $\mathcal{A}$ if and only if
(i) $1 \in F$,
(ii) if $x \in F$ and $x \to y \in F$, then $y \in F$.

A deductive system F of an equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called a proper deductive system if $F \neq A$. If $\mathcal{A}$ is a bounded equality algebra, then a deductive system is proper if and only if it does not contain 0 (see [2]). Also, every deductive system of an equality algebra $\mathcal{A}$ is a subalgebra of $\mathcal{A}$.

Definition 2.8.([12]) An equivalence relation θ on an equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ is called congruence, if $(x, z), (y, w) \in \theta$, then $(x \sim y, z \sim w), (x \wedge y, z \wedge w) \in \theta$. The set of all congruence on $\mathcal{A}$ is denoted by $Con(\mathcal{A})$.

Proposition 2.9.([2, 13]) Let F be a deductive system of an equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$. Define the relation $\theta_{\overrightarrow{F}}$ and θ_F as follows:
$(x, y) \in \theta_{\overrightarrow{F}}$ iff $\{x \to y, y \to x\} \subseteq F$ and $(x, y) \in \theta_F$ iff $x \sim y \in F$,
then $\theta_{\overrightarrow{F}}$ and θ_F are congruences and $\theta_{\overrightarrow{F}} = \theta_F$.

Proposition 2.10.([2, 13]) Let F be a deductive system of an equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$ and $A/\theta_F = \{[x] : x \in A\}$, where $[x] = \{y \in A : (x, y) \in \theta_F\}$. Then $\mathcal{A}/\theta_F = (A/\theta_F, \wedge, \sim, 1)$ is an equality algebra, where for every $x, y \in A$, $1 := [1]$, $[x] \wedge [y] := [x \wedge y]$ and $[x] \sim [y] := [x \sim y]$.

Definition 2.11.([18]) A proper deductive system F of an equality algebra

$\mathcal{A} = (A, \wedge, \sim, 1)$ is called prime if $x \to y \in F$ or $y \to x \in F$ for all $x, y \in A$.

Proposition 2.12.([18]) Let F be a proper deductive system of a prelinear equality algebra $\mathcal{A} = (A, \wedge, \sim, 1)$. Then the following statements are equivalent:
(i) F is prime,
(ii) for each $x, y \in A$, if $x \vee y \in F$, then $x \in F$ or $y \in F$,
(iii) $\mathcal{A}/\theta_F$ is a chain.

3 Frameable and tense equality algebras

In this section, we introduce the concept of frameable equality algebras and tense operators on frameable equality algebras. We obtain some basic properties of them.

Definition 3.1. A bounded equality algebra $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is called frameable, if it satisfies the following for all $x, y, z, w \in A$

$$(x \sim y) \wedge (z \sim w) \leq (x \wedge z) \sim (y \wedge w).$$

Example 3.2.(1) Let $(A, \wedge, 0, 1)$ be a bounded meet-semilattice with a unique coatom c. Then $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is a frameable equality algebra where

$$x \sim y = \begin{cases} 1 & x = y \\ y & x = 1 \\ x & y = 1 \\ c & otherwise \end{cases} \tag{1}$$

(2) Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a frameable equality algebra, X be a non-empty subset and A^X be the set of all mapping from X to A with the following operations for all $f, g \in A^X$ ad for all $x, y \in X$:
$(f \wedge g)(x) = f(x) \wedge f(y)$, $(f \sim g)(x) = f(x) \sim f(y)$, $0(x) = 0$ and $1(x) = 1$. Then $\mathcal{A}^X = (A^X, \wedge, \sim, 0, 1)$ is a frameable equality algebra.

Proposition 3.3. Every bounded prelinear equality algebra is a frameable equality algebra.

Proof: Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a bounded prelinear equality algebra and $x, y, z, w \in A$. By Proposition 2.4 part (2) and then part (1), Proposition

2.5 part (6) and Proposition 2.4 part (4),

$$\begin{aligned}(x \sim y) \wedge (z \sim w) &= (x \leftrightarrow y) \wedge (z \leftrightarrow w)\\ &= ((x \to y) \wedge (y \to x)) \wedge ((z \to w) \wedge (w \to z))\\ &\leq (x \wedge z \to y) \wedge (y \wedge w \to x) \wedge (x \wedge z \to w) \wedge (y \wedge w \to z)\\ &= (x \wedge z \to y \wedge w) \wedge (y \wedge w \to x \wedge z)\\ &= (x \wedge z) \leftrightarrow (y \wedge w) = (x \wedge z) \sim (y \wedge w). \ \blacksquare\end{aligned}$$

Corollary 3.4. Every bounded chain equality algebra is a frameable equality algebra.

Proof: It is clear that every bounded chain equality algebra is a bounded prelinear equality algebra. Thus the result follows from Proposition 3.3. ■

There exists a frameable equality algebra which is not prelinear. Consider the following example:

Example 3.5. Let $A = \{0, a, b, c, 1\}$ such that $0 < a, b < c < 1$ and a, b are incomparable. Consider the operations $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	1
0	1	c	a	a	0
a	c	1	a	a	a
b	a	a	1	c	b
c	a	a	c	1	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	c	1	1
b	a	a	1	1	1
c	a	a	c	1	1
1	0	a	b	c	1

By routine calculations, we can see that $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ is a frameable equality algebra. Since $(a \to b) \vee (b \to a) = c$, then $\mathcal{A}$ is not prelinear.

There exists a bounded equality algebras which is not frameable. See the following example:

Example 3.6. Let $A = \{0, a, b, c, d, 1\}$ such that $0 < a, b < c < d < 1$ and a, b are incomparable. Consider the operations $\sim$ and $\to$ given by the following tables:

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

Then $(A, \wedge, \sim, 0, 1)$ is a bounded equality algebra ([19]). Since $(a \sim c) \wedge (c \sim b) = d \nleq c = (a \wedge c) \sim (c \wedge b)$, then $\mathcal{A}$ is not frameable.

Proposition 3.7. Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a frameable equality algebra. Then the following hold for all $x, y, z, w \in A$:
(1) $x \wedge y \leq x \sim y$,
(2) $(x \to y) \wedge (x \to z) = x \to (y \wedge z)$,
(3) $(x \to y) \wedge (z \to w) \leq (x \wedge z) \to (y \wedge w)$.

Proof:(1) Since $\mathcal{A}$ is frameable, then $x \wedge y = (x \sim 1) \wedge (1 \sim y) \leq x \sim y$ by (E2) and (E4).
(2) Using definition $\to$, (E2) and assumption, we obtain $(x \to y) \wedge (x \to z) = ((x \wedge y) \sim x) \wedge ((x \wedge z) \sim x) \leq (x \wedge y \wedge z) \sim x = x \to (y \wedge z)$. On the other hand, we have $x \to (y \wedge z) \leq (x \to y) \wedge (x \to z)$ by applying Proposition 2.5 part (6).
(3) Applying assumption and definition $\to$, we have

$$\begin{aligned}(x \to y) \wedge (z \to w) &= ((x \wedge y) \sim x) \wedge ((w \wedge z) \sim z) \\ &\leq ((x \wedge y) \wedge (w \wedge z)) \to (x \wedge z) \\ &= (x \wedge z) \to (y \wedge w). \blacksquare\end{aligned}$$

Definition 3.8. Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a frameable equality algebra and $G, H : A \to A$ be unary operations on A satisfying the following for all $x, y \in A$
(TE1) $G(1) = H(1) = 1$,
(TE2) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,
(TE3) $G(x \sim y) \leq G(x) \sim G(y)$, $H(x \sim y) \leq H(x) \sim H(y)$,
(TE4) $x \leq G(P(x))$, $x \leq H(F(x))$,
where $P(x) = \neg H(\neg x)$ and $F(x) = \neg G(\neg x)$. Then $(\mathcal{A}, G, H)$ is called a tense equality algebra and G, H are called tense operators.

Example 3.9. (1) Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a frameable equality algebra. We can easily check that id_A is a tense operator on A. So every frameable

equality algebra can be seen as a tense equality algebra where $H = G = id_A$.
(2) Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a bounded chain equality algebra. Define $G(1) = H(1) = 1$ and $G(x) = H(x) = 0$ for $x < 1$. Then $(\mathcal{A}, G, H)$ is a tense equality algebra.
(3) Let $(\mathcal{A}, G, H)$ be a tense equality algebra and X be a nonempty set. Then $(\mathcal{A}^X, G^X, H^X)$ is a tense equality algebra, where $G^X(f) = G \circ f$ and $H^X(f) = H \circ f$, for all $f \in A^X$.
(4) Let $A = \{0, a, b, 1\}$ be a lattice such that $0 < a, b < 1$ and a, b are incomparable. Consider the operation $\sim$ on A given as follows:

$\sim$	0	a	b	1
0	1	b	a	0
a	b	1	0	a
b	a	0	1	b
1	0	a	b	1

Then $(A, \wedge, \sim, 0, 1)$ is a bounded prelinear equality algebra ([19]). Thus it is a frameable equality algebra by Proposition 3.3.
(i) Define the operation $G = H$ as follows:
$G(0) = 0$, $G(a) = b$, $G(b) = a$ and $G(1) = 1$.
It is easy to check that $G = H$ is a tense operation on A. Therefore $(\mathcal{A}, G, H)$ is a tense equality algebra.
(ii) Define the operations G_1 and H_1 as follows:
$G_1(0) = G_1(a) = 0$, $G_1(b) = G_1(1) = 1$,
$H_1(0) = H_1(a) = H_1(b) = a$ and $H_1(1) = 1$.
Then (A, G_1, H_1) is a tense equality algebra.

Proposition 3.10. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. Then the following hold for all $x, y \in A$:
(1) $F(0) = P(0) = 0$,
(2) if $x \leq y$, then $G(x) \leq G(y)$, $H(x) \leq H(y)$, $P(x) \leq P(y)$ and $F(x) \leq F(y)$,
(3) $G(x \sim y) \leq F(x) \sim F(y)$, $H(x \sim y) \leq P(x) \sim P(y)$,
(4) $G(x \to y) \leq G(x) \to G(y)$, $H(x \to y) \leq H(x) \to H(y)$
(5) $P(G(x)) \leq \neg\neg x$, $H(F(x)) \leq \neg\neg x$,
(6) $P \leq P(G(P(x)))$ and $F \leq F(H(F(x)))$,
If $\mathcal{A}$ is involutive, then
(7) $P(G(x)) \leq x$, $F(H(x)) \leq x$,
(8) $P = P(G(P(x)))$ and $F = F(H(F(x)))$.

Proof. (1) We have $F(0) = \neg G(\neg 0) = \neg G(1) = \neg 1 = 0$ by (TE1). Similarly, we have $P(0) = 0$,
(2) Since $x \leq y$, then $x = x \wedge y$. By (TE2), we can show $G(x) \leq G(y)$, $H(x) \leq H(y)$. Also, we have have $\neg y \leq \neg x$ by Proposition 2.5 part (6). Thus $H(\neg y) \leq H(\neg x)$ and then $\neg H(\neg x) \leq \neg H(\neg y)$, that is $P(x) \leq P(y)$. Similarly, we can prove $F(x) \leq F(y)$.
(3) By (E7), we have $x \sim y \leq (x \sim 0) \sim (y \sim 0) = \neg x \sim \neg y$. Using part (2), (TE3) and then (E7), we obtain

$$\begin{aligned} G(x \sim y) &\leq G(\neg x \sim \neg y) \leq G(\neg x) \sim G(\neg y) \leq (G(\neg x) \sim 0) \sim (G(\neg y) \sim 0) \\ &= \neg G(\neg x) \sim \neg G(\neg y) = F(x) \sim F(y). \end{aligned}$$

(4) It follows from (TE2) and (TE3).
(5) Since $x \leq \neg\neg x$, then $G(x) \leq G(\neg\neg x)$ by part(2). So $\neg G(\neg\neg x) \leq \neg G(x)$ and then $H(\neg G(\neg\neg x)) \leq H(\neg G(x))$ by part (2). Thus

$$P(G(x)) = \neg H(\neg G(x)) \leq \neg H(\neg G(\neg\neg x)) = \neg H(F(\neg x)) \leq \neg\neg x.$$

(6) By (TE4), we have $x \leq G(P(x))$. Using part (2), we get the result.
(7) It follows from part (5) and $\neg\neg x = x$.
(8) By part (7), we have $P(G(P(x))) \leq P(x)$, $F(H(F(x))) \leq F(x)$. Thus by part (6), we obtain result.■

Recall that a frame is a couple (X, R) where X is a non-void set and R is a binary relation on X (see e.g. [5]). In the following, we will use a frame on frameable equality algebras to define tense equality algebras.

Proposition 3.11. Let $\mathcal{A} = (A, \wedge, \sim, 0, 1)$ be a frameable equality algebra and (X, R) be a frame. Define the unary operations G, H on A^X as follows: $\widehat{G}(f(x)) = \bigwedge\{f(y) : xRy\}$ and $\widehat{H}(f(x)) = \bigwedge\{f(y) : yRx\}$, for all $f \in A^X$. Then $(\mathcal{A}^X, \widehat{G}, \widehat{H})$ is a tense equality algebra.

Proof: We have $\widehat{G}(1(x)) = \bigwedge\{1(y) : xRy\} = 1$ and $\widehat{H}(1(x)) = \bigwedge\{1(y) : yRx\} = 1$. The proof of (TE2) is straightforward. Let $f, g \in A^X$ be arbitrary. Since $\mathcal{A}$ is a frameable equality algebra, then

$$\begin{aligned} \widehat{G}((f \sim g)(x)) &= \bigwedge\{(f \sim g)(y) : xRy\} \\ &= \bigwedge\{f(y) \sim g(y) : xRy\} \\ &\leq \bigwedge\{f(y) : xRy\} \sim \bigwedge\{g(y) : xRy\} \\ &= \widehat{G}(f(x)) \sim \widehat{G}(g(x)). \end{aligned}$$

Now, we will prove (TE4). Let xRy. We have

$$f(z) \leq \neg\neg f(z) \leq \neg(\bigwedge\{\neg f(z) : zRy\}) = \neg\widehat{H}(\neg f(y))$$

for all zRy. Thus $f(x) \leq \neg(\widehat{H}(\neg f)(y))$. So

$$\widehat{G}(\widehat{P}(f))(x) = \widehat{G}(\neg\widehat{H}(\neg f(x))) = \bigwedge\{\neg\widehat{H}(\neg f(y)) : xRy\} \geq f(x).$$

Similarly, we can prove $f(x) \leq \widehat{H}(\widehat{F}(f))(x)$.■

Suppose that a framable equality algebra and a frame are given. Applying Proposition 3.11, we can construct a tense equality algebra.

Conversely, suppose that a tense equality algebra $(\mathcal{A}, G, H)$ and a frameable equality algebra $\mathcal{B}$ are given, we would like to know that whether there exists a frame (X, R) such that the tense operators G, H can be derived by this construction in $\mathcal{B}$ where $\mathcal{A}$ is embedded into the power algebra $\mathcal{B}^X$. If such a representation exists, then $(\mathcal{A}, G, H)$ is said to be representable in $\mathcal{B}$ with respect to the frame (X, R).
In the following, we will prove this problem for a strict strong tense involutive equality algebra having a full set of strict semi-morphism into a complete framable equality algebra. Some proofs are similar to those in [6, 7, 17].

Definition 3.12. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. Then
(1) $(\mathcal{A}, G, H)$ is called a strong tense equality algebra, if $G(0) = H(0) = 0$.
(2) $(\mathcal{A}, G, H)$ is called strict, if $G(\neg x) = \neg G(x)$ and $H(\neg x) = \neg H(x)$ for all $x \in A$.

Proposition 3.13. Let $\mathcal{A}$ be an involutive prelinear equality algebra and (X, R) be a frame. Then $(\mathcal{A}^X, \widehat{G}, \widehat{H})$ is a strict strong tense involutive prelinear equality algebra.

Proof. It is easy to see that $\mathcal{A}^X = (A^X, \wedge, \sim, 0, 1)$ is an involutive prelinear equality algebra. By Proposition 3.11, $(\mathcal{A}^X, \widehat{G}, \widehat{H})$ is a tense equality algebra. We have $\widehat{G}(0(x)) = \bigwedge\{0(y) : xRy\} = 0$ and $\widehat{H}(0(x)) = \bigwedge\{0(y) : yRx\} = 1$. Then $(\mathcal{A}^X, \widehat{G}, \widehat{H})$ is strong. Using Proposition 3.7 part (4),

$$\begin{aligned}\widehat{G}(\neg f(x)) &= \bigwedge\{\neg f(y) : xRy\} = \bigwedge\{f(y) \to 0 : xRy\}\\ &\leq \bigwedge\{f(y) : xRy\} \to 0 = \neg(\bigwedge\{f(y) : xRy\}) = \neg\widehat{G}(f(x))\end{aligned}$$

Since $\mathcal{A}$ prelinear, then it is a distributive lattice. Applying Proposition 3.17 and Proposition 3.18 in [3]

$$\begin{aligned}\neg\widehat{G}(f(x)) &= \neg(\bigwedge\{f(y) : xRy\} = \bigwedge\{f(y) : xRy\} \to 0 = \bigvee\{f(y) \to 0 : xRy\}\\ &= \bigvee\{\neg f(y) : xRy\} \geq \bigwedge\{\neg f(y) : xRy\} = \widehat{G}(\neg f(x))\end{aligned}$$

Hencen $(\mathcal{A}^X, \widehat{G}, \widehat{H})$ is strict.■

Proposition 3.14. Let $(\mathcal{A}, G, H)$ be a strong tense involutive equality algebra.Then the following hold for all $x, y \in A$:

(1) $G(\neg x) \leq \neg G(x)$, $H(\neg x) \leq \neg H(x)$,
(2) $G(x) \leq F(x)$, $H(x) \leq P(x)$,
(3) $G(x) = \neg F(\neg x)$, $H(x) = \neg P(\neg x)$,
(4) $G(H(x)) \leq x$, $H(G(x)) \leq x$,
(5) $G(\neg x) = \neg G(x)$ iff $F(x) = G(x)$,
(6) $H(\neg x) = \neg H(x)$ iff $P(x) = H(x)$,
(7) $G = F$ iff $H \circ G = id_A$,
(8) $H = P$ iff $G \circ H = id_A$,
If $(\mathcal{A}, G, H)$ is strict, then
(9) $G(x) = F(x)$, $H(x) = P(x)$,
(10) G and H are inverse of each other,
(11) $F(\neg x) = \neg F(x)$, $P(\neg x) = \neg P(x)$.

Proof. The Proof is straightforward.■

Definition 3.15. Let $f : \mathcal{A}_1 \to \mathcal{A}_2$ be a map where $\mathcal{A}_1$ and $\mathcal{A}_2$ be bounded equality algebras. Then
(i) f is called a semi-morphism, if for all $x, y \in A_1$,
(h1) $f(0) = 0$, $f(1) = 1$,
(h2) $f(x \wedge y) = f(x) \wedge f(y)$,
(sh3) $f(x \sim y) \leq f(x) \sim f(y)$.
(ii) A semi-morphism f is called strict, if for all $x \in A_1$, $f(\neg x) = \neg f(x)$.
(iii) Let S be a set of semi-morphisms from $\mathcal{A}_1$ into $\mathcal{A}_2$. A subset $T \subseteq S$ is called full, if for $x, y \in A_1$

$$x \leq y \text{ if and only if } f(x) \leq f(y).$$

Lemma 3.16. Let $(\mathcal{A}, G, H)$ be a strict strong tense involutive equality algebra and f, g be two strict semi-morphisms from $\mathcal{A}$ into an involutive equality algebra $\mathcal{B}$. Then
(1) $f(x) \leq g(P(x))$ iff $f(G(x)) \leq g(x)$ iff $g(H(x)) \leq f(x)$ iff $g(x) \leq f(F(x))$ for all $x \in A$,
(2) $f \circ G$ and $f \circ H$ are strict semi-morphisms from $\mathcal{A}$ into the equality algebra $\mathcal{B}$.

Proof.(1) Suppose that $f(x) \leq g(P(x))$ for all $x \in A$. So $f(G(y)) \leq g(P(G(y)))$ for any $y \in A$. Since g is a semi-morphism, then it is order preserving. Applying Proposition 3.10 part (7), then $g(P(G(y))) \leq g(y)$. Hence $f(G(y)) \leq g(y)$ for any $y \in A$.
Let $f(G(x)) \leq g(x)$. Since f, g are strict and order preserving, then by (TE4), assumption Proposition 3.14 part (10) and part (9), we get

$\neg f(x) = f(\neg x) \leq f(G(P(\neg x))) \leq g(P(\neg x)) = \neg g(P(x)) = \neg g(H(x))$.
Since $\mathcal{B}$ is involutive, then $g(H(x)) \leq f(x)$.
Suppose that $g(H(x)) \leq f(x)$ for all $x \in A$. Using Proposition 3.14 part (9) and part (4), we have $g(x) \leq g(H(F(x)) \leq f(F(x))$ for all $x \in A$.
Let $g(x) \leq f(F(x))$ for all $x \in A$. Then $g(H(\neg x)) \leq f(F(H(\neg x))) \leq f(\neg x)$.
Thus $f(x) = f(\neg\neg x) = \neg f(\neg x) \leq \neg g(H(\neg x)) = g(\neg H(\neg x)) = g(P(x))$.
(2) The proof is straightforward.■

Theorem 3.17. Let $(\mathcal{A}, G, H)$ be a strict strong tense involutive equality algebra with a full set S of semi-morphisms into a frameable involutive equality algebra $\mathcal{B}$. Then
(1) there exists a full set T of strict semi-morphisms into $\mathcal{B}$ containing S .
(2) there exists a frame (T, R) such that for any $f, g \in T$

$$(f, g) \in R \text{ iff } f(G(x)) \leq g(x) \text{ for all } x \in A.$$

Moreover, we have
$f(G(x)) = \bigwedge\{g(x) : g \in T, fRg\}$, $f(H(x)) = \bigwedge\{g(x) : g \in T, gRf\}$.
(3) the map $\alpha : (\mathcal{A}, G, H) \to (\mathcal{B}^T, G^T, H^T)$ which sends x to $\alpha(x)$ is a strict semi-morphism of tense equality algebra to the powerset tense , where $\alpha(x)(f) = f(x)$ for all $x \in A$ and $f \in T$.
Moreover $(\mathcal{A}, G, H)$ is representable in $\mathcal{B}$ with respect to the frame (T, R).

Proof. (1) Let T be the smallest set consisting of strict semi-morphisms from $\mathcal{A}$ into $\mathcal{B}$ such that $S \subseteq T$ and $f \circ G, f \circ H \in T$.
(2) Put $R = \{(f, g) \in T \times T : f(G(x)) \leq g(x)\}$. Using Lemma 3.16 part (1),

$$\begin{aligned} R = &\{(f, g) \in T \times T : f(x) \leq g(P(x))\} \\ &\{(f, g) \in T \times T : g(H(x)) \leq f(x)\} \\ &\{(f, g) \in T \times T : g(x) \leq f(F(x))\}. \end{aligned}$$

We have $f \circ G, f \circ H \in T$ by Lemma 3.16 part (2). Thus $(f, f \circ G), (f \circ H, f) \in R$. Therefor $f(G(x)) \leq \bigwedge\{g(x) : g \in T, fRg\} \leq f(G(x))$ and $f(H(x)) \leq \bigwedge\{g(x) \in T : gRf\} \leq f(H(x))$.
(3) Now consider the frame (T, R). Since $\mathcal{A}$ is frameable, then $(\mathcal{A}^T, \widehat{G}, \widehat{H})$ is a tense equality algebra by Proposition 3.11. We will prove that α is an order reflecting strict semi-morphism of tense equality algebras. For all $x, y \in A$ and $f \in T$, we have
$\alpha(x \wedge y)(f) = f(x \wedge y) = f(x) \wedge f(y) = \alpha(x)(f) \wedge \alpha(y)(f) = (\alpha(x) \wedge \alpha(y))(f)$,
$\alpha(x \sim y)(f) = f(x \sim y) \leq f(x) \sim f(y)$
$= \alpha(x)(f) \sim \alpha(y)(f) = (\alpha(x) \sim \alpha(y))(f)$,

$\alpha(\neg x)(f) = f(\neg x) = \neg f(x) = \neg\alpha(x)(f)$,
$\alpha(G(x))(f) = f(G(x)) = \bigwedge\{g(x) \in T : fRg\} = \widehat{G}(\alpha(x))(f)$,
$\alpha(H(x))(f) = f(H(x)) = \bigwedge\{g(x) \in T : gRf\} = \widehat{G}(\alpha(x))(f)$.
Hence α is a tense strict semi-morphism. Since T is a full set of semi-morphisms, then α is one to one. Hence $(\mathcal{A}, G, H)$ is representable in $\mathcal{B}$ with respect to the frame (T, R).■

Corollary 3.18. Let $(\mathcal{A}, G, H)$ be a strict strong tense involutive equality. Then $S = \{id_A\}$ is a full set of semi-morphisms into a frameable involutive equality algebra $\mathcal{A}$. Then there exists a full set T of strict semi-morphisms into B containing S such that $(\mathcal{A}, G, H)$ is representable in $\mathcal{A}$ with respect to T.

4 Tense deductive systems

In this section, we will introduce the concept of a tense deductive system of tense equality algebras and obtain some characterizations and related results.

Definition 4.1. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. A subset D of A is called a tense deductive system, if it is a deductive system of equality algebra $\mathcal{A}$ and D is closed under the operations G and H.

Let $(\mathcal{A}, G, H)$ be a tense equality algebra. It is clear that $\{1\}$ is a tense deductive system. Also, if $G = H = id_A$, then every deductive system of an equality algebra $\mathcal{A}$ is a tense deductive system. But a deductive system may not be a tense deductive system in general. See the following example:

Example 4.2. Let $A = \{0, a, b, 1\}$ be a lattice such that $0 < a < b < 1$. Consider the operation $\sim$ on A as follows:

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	b
1	0	a	b	1

Then $(A, \wedge, \sim, 1)$ is a bounded chain equality algebra ([2]). Thus it is a frameable equality algebra by Corollary 3.4.

(i) Define the operations G and H as follows:
$G(0) = 0$, $G(a) = G(b) = b$ and $G(1) = 1$,
$H(0) = H(a) = 0$, $H(b) = b$ and $H(1) = 1$.
It is easy to check that G and H are tense operations on A. Therefore $(\mathcal{A}, G, H)$ is a tense equality algebra. We can see that $D_1 = \{b, 1\}$ is a tense deductive system. Now, consider the deductive system $D_2 = \{a, b, 1\}$. Since D_2 is not closed under the operation H, then it is not a tense deductive system.

Definition 4.3. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. Define the unary operation d on A by $d(x) = H(x) \wedge G(x) \wedge x$ for all $x \in A$. For $n \geq 0$, we define $d^0(x) = x$ and $d^{n+1} = d(d^n(x))$.

Proposition 4.4. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. Then the following hold for all $x, y \in A$ and $n \geq 0$:
(1) $d(x) \leq x$, $d^n(0) = 0$,
(2) $d^n(x) = 1$ if and only if $x = 1$,
(3) $d^n(x \wedge y) = d^n(x) \wedge d^n(y)$,
(4) if $x \leq y$, then $d^n(x) \leq d^n(y)$,
(5) $d^{n+1}(x) \leq d^n(x)$,
(6) if $x \leq d(y)$, then $d(x) \leq y$,
(7) $d(x \sim y) \leq d(x) \sim d(y)$,
(8) $d(x \to y) \leq d(x) \to d(y)$,

Proof. The proof is easy. So details have been deleted.■

Proposition 4.5. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. A deductive system F of $\mathcal{A}$ is a tense deductive system if and only if it is closed under the operation d.

Proof. Let F be a tense deductive system of $(\mathcal{A}, G, H)$ and $x \in A$. Since $H(x), G(x) \in F$, then $d(x) = H(x) \wedge G(x) \wedge x \in F$. Conversely, let F be closed under the operation d. Since $d(x) \leq H(x), G(x)$, then $H(x), G(x) \in F$ by Definition 2.6 part (ii).■

Remark: It is easy to verify that $Ker(d) = \{x \in A | d(x) = 1\}$ is a tense deductive system of a tense equality algebra $(\mathcal{A}, G, H)$. Also, by Proposition 4.4 part (2), we have $Ker(d) = \{1\}$.

Let $(\mathcal{A}, G, H)$ be a tense equality algebra and X be a nonempty subset of A. The tense deductive system generated by X is the smallest tense deductive system of $(\mathcal{A}, G, H)$ containing X and is denoted by $\langle X\rangle_T$. We write $\langle x\rangle_T$ instead of $\langle\{x\}\rangle_T$.

Proposition 4.6. Let X be a nonempty subset of a tense equality algebra $(\mathcal{A}, G, H)$. Then $\langle X\rangle_T = \{x \in A | d^{n_1}(x_1) \to (...(d^{n_k}(x_k) \to x)...) = 1$, $x_1, ..., x_k \in X$ and $n_1, ..., n_k \geq 0$ for some $k \geq 0\}$.

Proof. Suppose that $D = \{x \in A | d^{n_1}(x_1) \to (...(d^{n_k}(x_k) \to x)...) = 1$, for some $n \geq 0$ and $x_1, ..., x_k \in X\}$. We will prove that D is a tense deductive system. Since X is nonempty, then there exists $x \in X$. By Proposition 2.5 part (2), we have $d(x) \to (d^0(x) \to 1) = d(x) \to 1 = 1$. Hence $1 \in D$.
Let $a, a \to b \in D$ be arbitrary. Then there exist $x_1, ..., x_t, y_1, ..., y_s \in X$ and $n_1, ..., n_t, m_1, ..., m_s \geq 0$ such that $d^{n_1}(x_1) \to (...(d^{n_t}(x_t) \to a)...) = 1$ and $d^{m_1}(y_1) \to (...(d^{m_s}(y_s) \to (a \to b))...) = 1$. By Proposition 2.5 part (5), we get $a \to (d^{m_1}(y_1) \to (...(d^{m_s}(y_s) \to b)...)) = d^{m_1}(y_1) \to (...(d^{m_s}(y_s) \to (a \to b))...) = 1$, that is $a \leq d^{m_1}(y_1) \to (...(d^{m_s}(y_s) \to b)...)$. By Proposition 2.5 part (6), we obtain $d^{n_1}(x_1) \to (...(d^{n_t}(x_t) \to a)...) \leq d^{n_1}(x_1) \to (...(d^{n_t}(x_t) \to (d^{m_1}(y_1) \to (...(d^{m_s}(y_s) \to b))...)$. Thus $b \in D$. Hence D is a deductive system.
By Proposition 4.4 part (8), we have $1 = d(1) = d(d^{n_1}(x_1) \to (...(d^{n_t}(x_t) \to a)...)) \leq d^{n_1+1}(x_1) \to (...(d^{n_t+1}(x_t) \to d(a))...)$, that is $d(a) \in D$. Hence D is a tense deductive system.
Let $x \in X$. Since $d(x) \to (d^0(x) \to x) = d(x) \to 1 = 1$, then $d(x) \in D$. Thus $X \subseteq D$.
Let F be a tense deductive system such that $X \subseteq F$. If $a \in D$, then $d^{n_1}(x_1) \to (...(d^{n_t}(x_t) \to a)...) = 1$. Since $d^{n_1}(x_1), ..., d^{n_t}(x_t), 1 \in F$, then $a \in F$. Thus $D \subseteq F$. Therefore $\langle X\rangle_T = D$.■

Proposition 4.7. Let F be a deductive system of $(\mathcal{A}, G, H)$, $x \notin F$ and $a, b \in A$. Then
(1) $\langle F \cup \{x\}\rangle_T = \{a \in A | d^n(x) \to^m a \in F$ for some $n, m \geq 0\}$,
(2) $\langle a\rangle_T = \{x \in A | d^n(a) \to^m x = 1$ for some $n, m \geq 0\}$,
(3) if $a \leq b$, then $\langle b\rangle_T \subseteq \langle a\rangle_T$,
(4) $\langle d(a)\rangle_T = \langle a\rangle_T$,
(5) $\langle a\rangle_T \vee \langle b\rangle_T = \langle a \wedge b\rangle_T$,
(6) if $d(a) \vee d(b)$ exists, then $\langle d(a) \vee d(b)\rangle_T \subseteq \langle a\rangle_T \cap \langle b\rangle_T$.

Proof.(1) Let $D = \{a \in A | d^n(x) \to^m a \in F$ for some $n, m \geq 0\}$ and $a, a \to b \in D$ be arbitrary. Then there exist $n_1, n_2, m_1, m_2 \geq 0$ and $f_1, f_2 \in F$ such that $d^{n_1}(x) \to^{m_1} a = f_1$ and $d^{n_2}(x) \to^{m_2} (a \to b) = f_2$. By Proposition 2.5 part (5), we have $f_2 \leq a \to^{m_2} (d^{n_2}(x) \to b)$. Using Proposition 2.5 part (4), we obtain $a \leq f_2 \to^{m_2} (d^{n_2}(x) \to b)$. Applying Proposition 2.5 part (6) and part (5), we get

$$\begin{aligned} d^{n_1}(x) \to^{m_1} a &\leq d^{n_1}(x) \to^{m_1} (d^{n_2}(x) \to^{m_2} (f_2 \to b)) \\ &= f_2 \to (d^{n_1}(x) \to^{m_1} (d^{n_2}(x) \to^{m_2} b)). \end{aligned}$$

Thus $(d^{n_1}(x) \to^{m_1} (d^{n_2}(x) \to^{m_2} b) \in F$ by Definition 2.6 and Proposition 2.7. Put $n = max\{n_1, n_2\}$. Then

$$\begin{aligned} d^{n_1}(x) \to^{m_1} (d^{n_2}(x) \to^{m_2} b) &\leq d^{n}(x) \to^{m_1} (d^{n_2}(x) \to^{m_2} b) \\ &\leq d^{n}(x) \to^{m_1} (d^{n}(x) \to^{m_2} b). \end{aligned}$$

So $d^n(x) \to^{m_1+m_2} b \in F$. Hence $b \in F$. It is obvious that $1 \in D$. Therefor D is a deductive system.
Since F is a tense deductive system, we have $d(d^{n_1}(x) \to^{m_1} a) \in F$. By Proposition 4.4 part (8) and then Definition 2.6 part (ii), we get $d^{n_1+1}(x) \to^{m_1} d(a) \in F$. So $d(a) \in D$. Thus D is a tense deductive system
Let $f \in F$ be arbitrary. By Proposition 2.5 part (3), $f \leq d(x) \to f$. Thus $F \subseteq D$. Also, $d(x) \to x = 1 \in F$ by Proposition 2.5 part (1). Thus $F \cup \{x\} \subseteq D$.
Suppose that G is another tense deductive system such that $F \cup \{x\} \subseteq G$. Then $d^{n_1}(x) \to^{m_1} a \in G$ and $d^{n_1}(x) \in G$. Thus $a \in G$. Since $a \in D$ is arbitrary, then $D \subseteq G$. Hence $\langle F \cup \{x\}\rangle_T = D$.
(2) Suppose that $D = \{x \in A | d^n(a) \to^m x = 1 \text{ for some } n, m \geq 0\}$. Since $d^0(a) \to a = 1$ and $d^0(a) \to 1 = 1$, then $a, 1 \in D$. Let $x, x \to y \in D$. Then there exist $n_1, n_2, m_1, m_2 \geq 0$ such that $d^{n_1}(a) \to^{m_1} x = 1$ and $d^{n_2}(a) \to^{m_2} (x \to y) = 1$. By Proposition 2.5 part (5) and part (1), we get $x \leq d^{n_2}(a) \to^{m_2} y$. Using Proposition 2.5 part (6)

$$\begin{aligned} d^{n_1}(a) \to^{m_1} x &\leq d^{n_1}(a) \to^{m_1} (d^{n_2}(a) \to^{m_2} y) \\ &\leq d^{n_1}(a) \to^{m_1} (d^{n_1+n_2}(a) \to^{m_2} y) \\ &\leq d^{n_1+n_2}(a) \to^{m_1} (d^{n_1+n_2}(a) \to^{m_2} y) \\ &= d^{n_1+n_2}(a) \to^{m_1+m_2} y. \end{aligned}$$

Therefor $y \in D$. Also, we have $1 = d(1) = d(d^{n_1}(a) \to^{m_1} x) \leq d^{n_1+1}(a) \to^{m_1} d(x)$. So $d(x) \in D$ and then D is a tense deductive system containing a. Suppose that G is another tense deductive system containing a. Let $x \in D$ be arbitrary. Then $d^{n_1}(a) \to^{m_1} x = 1 \in G$. Since $d^{n_1}(a) \in G$, then $x \in G$. Therefor $D \subseteq G$. Hence $\langle a\rangle_T = D$.
(3) It follows from part (2), Proposition 4.4 part (3) and Proposition 2.5

part (6).
(4) The results follows from part (2) and part (3).
(5) By part (3), we have $\langle a\rangle_T, \langle b\rangle_T \subseteq \langle a \wedge b\rangle_T$. Thus $\langle a \wedge b\rangle_T$ is an upper bound of $\{\langle a\rangle_T, \langle b\rangle_T\}$. Suppose that F is an upper bound $\{\langle a\rangle_T, \langle b\rangle_T\}$ and $x \in \langle a \wedge b\rangle_T$. Then there exist $m, n \geq 0$ such that $d^n(a \wedge b) \to^m x = 1$. By Proposition 4.4 part (3), we obtain $(d^n(a) \wedge d^n(b)) \to^m x = 1 \in F$. Since F is a tense deductive system and $\langle a\rangle_T, \langle b\rangle_T \subseteq F$, then $d^n(a) \wedge d^n(b) \in F$. Thus $x \in F$, that is $\langle a\rangle_T \vee \langle b\rangle_T = \langle a \wedge b\rangle_T$.
(6) It follows from part (3).■

Corollary 4.8. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. The set of all tense deductive systems of $(\mathcal{A}, G, H)$ forms an algebraic lattice whose compact elements are the finitely generated tense deductive system.

Proof. It easy to see that that the mapping $X \mapsto \langle X\rangle_T$ is an algebraic closure operator on the power set of A. Hence the set of all tense deductive systems of $(\mathcal{A}, G, H)$ is an algebraic lattice.■

Definition 4.9. Let $(\mathcal{A}, G, H)$ be a tense equality algebra and $\theta \in Con(\mathcal{A})$. Then θ is called T-congruence on $(\mathcal{A}, G, H)$, if $(x, y) \in \theta$, then $(G(x), G(y))$, $(H(x), H(y)) \in \theta$.

Proposition 4.10. Let $(\mathcal{A}, G, H)$ be a tense equality algebra and $\theta \in Con(\mathcal{A})$. Then θ is a T-congruence iff $(x, y) \in \theta$ implies $(d(x), d(y)) \in \theta$.

Proof. Let $\theta \in Con(\mathcal{A})$ and $x \in [1]_\theta$ be arbitrary. Then $(x, 1) \in \theta$. By assumption $(d(x), 1) \in \theta$. So $d(x) \in [1]_\theta$. Since $[1]_\theta$ is a deductive system, then $[1]_\theta$ is a tense deductive system. Now, let $(x, y) \in \theta$ be arbitrary. Then $(x \sim y, 1) \in \theta$. Thus $x \sim y \in [1]_\theta$. By Definition 4.1, we obtain that $H(x \sim y), G(x \sim y) \in [1]_\theta$. Using (TE3) and then Definition 2.6, we get $H(x) \sim H(y), G(x) \sim G(y) \in [1]_\theta$. Thus $(H(x) \sim H(y), 1), (G(x) \sim G(y), 1) \in \theta$. Applying Lemma 5 in [13], we obtain $(H(x), H(y)), (G(x), G(y)) \in \theta$. Hence θ is a T-congruence. The converse is obvious.■

Corollary 4.11. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. There exists a one to one correspondence between the set of all tense deductive systems and the set of all T-congruences of $(\mathcal{A}, G, H)$.

Proof. The Proof is straightforward.■

The set of all tense deductive systems of a tense equality algebra $(\mathcal{A}, G, H)$ is denoted by $DS_T(\mathcal{A})$. Recall that a tense equality algebra $(\mathcal{A}, G, H)$ is a subdirectly irreducible if it has the least nontrivial tense congruence. By the above corollary, we can see that $(\mathcal{A}, G, H)$ is a subdirectly irreducible, if $\cap\{F : F \in DS_T(\mathcal{A}) - \{1\}\} \neq \{1\}$.

Theorem 4.12. $(\mathcal{A}, G, H)$ is a subdirectly irreducible tense equality algebra if and only if there exists $1 \neq a \in A$ such that $a \in \langle x\rangle_T$ for all $1 \neq x \in A$.

Proof. Let $(\mathcal{A}, G, H)$ be a subdirectly irreducible tense equality algebra. Then $\cap\{F : F \in DS_T(\mathcal{A}) - \{1\}\} \neq \{1\}$. We get $\cap\{\langle x\rangle_T : x \neq 1\} \neq \{1\}$. Then there exists $a \in \cap\{\langle x\rangle_T : x \neq 1\}$ such that $a \neq 1$. Hence $a \in \langle x\rangle_T$ for all $1 \neq x \in A$.
Conversely, let $F \in DS_T(\mathcal{A})$ such that $F \neq \{1\}$. Then there exists $x \in F$ such that $x \neq 1$. By assumption $a \in \langle x\rangle_T$. Thus $a \in \cap\{F : F \in DS_T(\mathcal{A}) - \{1\}\}$. Since $a \neq 1$, then $\cap\{F : F \in DS_T(\mathcal{A}) - \{1\}\} \neq \{1\}$. Hence $(\mathcal{A}, G, H)$ is a subdirectly irreducible. ■

Proposition 4.13. Let F be a tense deductive system of a tense equality algebra $(\mathcal{A}, G, H)$. Then $(\mathcal{A}/\theta_F, G_F, H_F)$ is a tense equality algebra, where $G_F([x]) = [G(x)]$ and $H_F([x]) = [H(x)]$ for all $x \in A$.

Proof The proof is straightforward.■

Definition 4.14. Let P be a tense deductive system of a tense equality algebra $(\mathcal{A}, G, H)$. Then P is called a prime tense deductive system of $(\mathcal{A}, G, H)$, if for every tense deductive systems F_1, F_2 of $(\mathcal{A}, G, H)$ whenever $F_1 \cap F_2 \subseteq P$, then $F_1 \subseteq P$ or $F_2 \subseteq P$.

Lemma 4.15. Let $(\mathcal{A}, G, H)$ be a tense equality algebra. Then $\langle a\rangle_T \cap \langle b\rangle_T = \langle U(a, b)\rangle_T$ where $U(a, b) = \{x \in A : d^n(a) \leq x, d^n(b) \leq x \text{ for some } n \geq 0\}$.

Proof. let $x \in \langle a\rangle_T \cap \langle b\rangle_T$. Then there exist $n_1, n_2, m_1, m_2 \geq 0$ such that $d^{n_1}(a) \to^{m_1} x = 1$ and $d^{n_2}(b) \to^{m_2} x = 1$. Put $n = max\{n_1, n_2\}$ and $m = max\{m_1, m_2\}$. Then $d^n(a) \to^m x = 1$ and $d^n(b) \to^m x = 1$. Proceeding inductively, we will show that there exist $z_1, z_2, ..., z_k \in U(a, b)$ and $r_1, r_2, ..., r_k \geq 0$ such that $d^{r_1}(z_1) \to (...(d^{r_k}(z_k) \to x)...) = 1$. Hence $x \in \langle U(a, b)\rangle_T$.
If $m = 1$, then we have $d^n(a) \to x = d^n(b) \to x = 1$. So $x \in U(a, b)$. We

take $z_1 = x$ and $r_1 = 0$. Now, Suppose that the statement holds for all positive integers $k < m$. Let $d^n(a) \rightarrow^{m+1} x = 1$ and $d^n(b) \rightarrow^{m+1} x = 1$. Since $d^n(a) \rightarrow^{m+1} x = 1$, then $d^n(a) \rightarrow (d^n(a) \rightarrow^m x) = 1$. We obtain

$$\begin{aligned} 1 = d^n(b) \rightarrow^m 1 &= d^n(b) \rightarrow^m (d^n(a) \rightarrow (d^n(a) \rightarrow^m x)) \\ &= d^n(a) \rightarrow (d^n(b) \rightarrow^m (d^n(a) \rightarrow^m x)). \end{aligned}$$

By Proposition 2.5 part (3), We have $x \leq d^n(a) \rightarrow^m x$. Applying Proposition 2.5 part (6) and then part (1), we get

$$1 = d^n(b) \rightarrow^{m+1} (d^n(a) \rightarrow^m x) = d^n(b) \rightarrow (d^n(b) \rightarrow^m (d^n(a) \rightarrow^m x)).$$

Then there exists $z'_1 \in U(a, b)$ and $r'_1 \geq 0$ such that

$$\begin{aligned} 1 &= d^{r'_1}(z'_1) \rightarrow (d^n(b) \rightarrow^m (d^n(a) \rightarrow^m x)) \\ &= d^n(b) \rightarrow^m (d^n(a) \rightarrow^m (d^{r'_1}(z'_1) \rightarrow x)) \\ &= d^n(b) \rightarrow (d^n(b) \rightarrow^{m-1} (d^n(a) \rightarrow^m (d^{r'_1}(z'_1) \rightarrow x)) \end{aligned}$$

by the first induction step.
By Proposition 2.5 part (3) part, We have $x \leq d^{r'_1}(z'_1) \rightarrow x$. Applying Proposition 2.5 part (6) and then part (1), we get

$$1 = d^n(a) \rightarrow^{m+1} (d^{r'_1}(z'_1) \rightarrow x) = d^n(a) \rightarrow (d^n(a) \rightarrow^m (d^{r'_1}(z'_1) \rightarrow x)).$$

Using Proposition 2.5 part (2) and part (5), we obtain

$$\begin{aligned} 1 = d^n(b) \rightarrow^{m-1} 1 &= d^n(b) \rightarrow^{m-1} (d^n(a) \rightarrow (d^n(a) \rightarrow^m (d^{r'_1}(z'_1) \rightarrow x)) \\ &= d^n(a) \rightarrow (d^n(b) \rightarrow^{m-1} (d^n(a) \rightarrow^m (d^{r'_1}(z'_1) \rightarrow x)). \end{aligned}$$

Then there exists $z'_2 \in U(a, b)$ and $r'_2 \geq 0$ such that

$$1 = d^{r'_2}(z'_2) \rightarrow (d^n(b) \rightarrow^{m-1} (d^n(a) \rightarrow^m d^{r'_1}(z'_1))).$$

by the first induction step. Repeating this procedure we obtain

$$d^n(a) \rightarrow^m (d^{r'_{m+1}}(z'_{m+1}) \rightarrow (...(d^{r'_1}(z'_1) \rightarrow x)...)) = 1$$

for some $z'_1, z'_2, ..., z'_{m+1} \in U(a, b)$ and $r'_1, r'_2, ..., r'_k \geq 0$.
Similarly, we can prove that

$$d^n(b) \rightarrow^m (d^{r''_1}(z''_{m+1}) \rightarrow (...(d^{r''_1}(z''_1) \rightarrow x)...)) = 1$$

for some $z''_1, z''_2, ..., z''_{m+1} \in U(a, b)$ and $r''_1, r''_2, ..., r''_k \geq 0$.
Put $z_i = z'_i \wedge z''_i$ and $r_i = max\{r'_i, r''_i\}$ for $i = 1, ..., m + 1$. Then

$$d^n(a) \rightarrow^m (d^{r_{m+1}}(z_{m+1}) \rightarrow (...(d^{r_1}(z_1) \rightarrow x)...)) = 1$$

$$d^n(b) \rightarrow^m (d^{r_{m+1}}(z_{m+1}) \rightarrow (...(d^{r_1}(z_1) \rightarrow x)...)) = 1.$$

Since $z_1, z_2, ..., z_{m+1} \in U(a, b)$, then there exist $z_{m+2}, z_{m+3}, ..., z_k \in U(a, b)$ and such that $d^{r_1}(z_1) \rightarrow (...(d^{r_k}(z_k) \rightarrow x)...) = 1$.
Conversely, let $x \in \langle U(a, b)\rangle_T$.Then $d^{n_1}(z_1) \rightarrow (...(d^{n_k}(z_k) \rightarrow x)...) = 1$, $z_1, ..., z_k \in U(a, b)$ and $n_1, ..., n_k \geq 0$ for some $k \geq 0$. Since $z_i \in U(a, b)$, then there exist $t_i \geq 0$ such that $d^{t_i}(a) \leq x, d^{t_i}(b) \leq x$ for$i = 1, ..., k$. Put

$n = max\{n_1+t_1, ..., n_k+t_k\}$. Using induction, we can prove that

$$d^{n_1}(z_1) \to (...(d^{n_k}(z_k) \to x)...) \leq d^n(a) \to^k x,$$
$$d^{n_1}(z_1) \to (...(d^{n_k}(z_k) \to x)...) \leq d^n(b) \to^k x.$$

Therefore $d^n(a) \to^k x = 1$ and $d^n(b) \to^k x = 1$. Hence $x \in \langle a \rangle_T \cap \langle b \rangle_T$.■

Proposition 4.16. Let P be a tense deductive system of a tense equality algebra $(\mathcal{A}, G, H)$. Then the following are equivalent:
(i) P is a prime tense deductive system of $(\mathcal{A}, G, H)$,
(ii) if $U(x, y) \subseteq P$, then $x \in P$ or $y \in P$.

Proof. (i)⇒(ii) Suppose $U(x, y) \subseteq P$ for some $x, y \in A$. By Lemma 4.15, we get $\langle x \rangle_T \cap \langle y \rangle_T = \langle U(x, y) \rangle_T \subseteq P$. By Definition 4.14, we obtain $x \in \langle x \rangle_T \subseteq P$ or $y \in \langle x \rangle_T \subseteq P$.
(ii)⇒(i) let $F_1 \cap F_2 \in DS_T(\mathcal{A}) - \{P\}$ such that $F_1 \cap F_2 \subseteq P$. If $x \in P \backslash F_1$ and $y \in P \backslash F_2$, then $U(x, y) \subseteq P$. By assumption, we get $x \in P$ or $y \in P$ which is a contradiction. ■

Lemma 4.17. Let $(\mathcal{A}, G, H)$ be a tense prelinear equality algebra. Then $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in A$.

Proof. Let $x, y \in A$ be arbitrary. Using Proposition 2.5 part (6), Proposition 2.4 part (5), we obtain

$$\begin{aligned} d(x \vee y) &= d((x \to y) \to y) \wedge d((y \to x) \to x) \\ &\leq (d(x \to y) \to (d(x) \vee d(y))) \wedge (d(y \to x) \to (d(x) \vee d(y))) \\ &\leq (d(x \to y) \vee d(y \to x)) \to (d(x) \vee d(y)) \\ &= 1 \to (d(x) \vee d(y)) = d(x) \vee d(y). \end{aligned}$$

Applying Proposition 4.4 part (4), we obtain $d(x) \vee d(y) \leq d(x \vee y)$.■

Proposition 4.18. Let P be a tense deductive system of a tense prelinear equality algebra $(\mathcal{A}, G, H)$. Then the following are equivalent:
(1) P is a prime tense deductive system of $(\mathcal{A}, G, H)$,
(2) $x \vee y \in P$, then $x \in P$ or $y \in P$,
(3) $d(x) \to d(y) \in P$ or $d(y) \to d(x) \in P$,
(4) $d(x) \vee d(y) \in P$, then $d(x) \in P$ or $d(y) \in P$,
(5) $(\mathcal{A}/\theta_P, G_P, H_P)$ is a tense chain equality algebra.

Proof. (1)⇒(2) Let P be a prime tense deductive system and $x \vee y \in P$. Using Lemma 4.17, we get $U(x, y) \subseteq P$. By Proposition4.16, we have $x \in P$ or $y \in P$.
(2)⇒(3) Since $\mathcal{A}$ is a prelinear, then $(x \to y) \vee (y \to x) = 1$. By Lemma

4.17 and Proposition 4.4 part (2), we get $d(x \to y) \vee d(y \to x) = 1 \in P$. Using assumption, $d(x \to y) \in P$ or $d(y \to x) \in P$. Hence $d(x) \to d(y) \in P$ or $d(y) \to d(x) \in P$ by Proposition 4.4 part (8) and then Definition 2.6.
(3)⇒(4) Suppose that $d(x) \to d(y) \in P$ and $d(x) \vee d(y) \in P$. Applying Proposition 2.4 part (5), we have
$(d(x) \vee d(y)) \to d(y) = (d(x) \to d(y)) \wedge (d(y) \to d(y)) = d(x) \to d(y) \in P$. Hence $d(y) \in P$ by Proposition 2.7.
(4)⇒(5) Since $(d(x \to y) \vee d(y \to x) = d((x \to y) \vee (y \to x)) = 1 \in P$, then $d(x \to y) \in P$ or $d(y \to x) \in P$. Using Proposition 4.4 part (1) and Definition 2.6, we have $x \to y \in P$ or $y \to x \in P$. Hence $[x] \leq [y]$ or $[y] \leq [x]$.
(5)⇒(1) Let $[x] \leq [y]$ and $U(x, y) \subseteq P$. Then $x \vee y \in P$ and $x \to y \in P$. By Proposition 2.4 part (5), we have $(x \vee y) \to y = x \to y \in P$. Thus $y \in P$. Hence P is a prime tense deductive system of $(\mathcal{A}, G, H)$ by Proposition 4.15.■

Proposition 4.19. Let $(\mathcal{A}, G, H)$ be tense prelinear equality algebra and $1 \neq a \in A$. Then there exists a prime tense deductive system of $(\mathcal{A}, G, H)$ such that $a \notin P$.

Proof. Suppose that $\sum = \{P \in: P \text{ is prime and } a \notin P\}$.Since $\{1\} \in \sum$, then $\sum$ is a nonempty set. It is easy to prove that $\sum$ is a partially set under inclusion relation and every chain in $\sum$ has an upper bound in $\sum$. Hence there exists a maximal element P in $\sum$ by Zorn's Lemma. Since $P \in \sum$, then P is a tense deductive system not containing a. We will prove that P is prime. If P is not prime, then there exist $x, y \in A$ such that $d(x) \to d(y), d(y) \to d(x) \notin P$. Since P is strictly contained in $\langle P, d(x) \to d(y)\rangle_T$ and $\langle P, d(y) \to d(x)\rangle_T$, then $\langle P, d(x) \to d(y)\rangle_T \notin \sum$ and $\langle P, d(y) \to d(x)\rangle_T \notin \sum$ by the maximality of P. Thus $a \in \langle P, d(x) \to d(y)\rangle_T$ and $a \in \langle P, d(y) \to d(x)\rangle_T$. Then there exist $n_1, n_2, m_1, m_2 \geq 0$ such that $d^{n_1}(d(x) \to d(y)) \to^{m_1} a \in P$ and $dn^2(d(y) \to d(x)) \to^{m_2} a \in P$ by Proposition 4.7 part (i). let $n = max\{n_1, n_2\}$ and $m = max\{m_1, m_2\}$. Then $d^n(d(x) \to d(y)) \to^m a \in P$ and $d^n(d(y) \to d(x)) \to^m a \in P$. Applying induction n, we can show that there exists $l \geq 0$ such that $(d^n(d(x) \to d(y)) \vee d^n(d(y) \to d(x))) \to^l a \in P$. By Lemma 4.17, we obtain $(d^n((d(x) \to d(y)) \vee (d(y) \to d(x))) \to^l a \in P$. Sice $\mathcal{A}$ is prelinear, $a \in P$ which is a contradiction.■

Proposition 4.20. Every tense prelinear equality algebra is a subalgebra of the direct product of a system of linearly ordered tense equality algebras.

Proof. Let $Spec_T(A)$ be the class of all prime tense deductive system of a tense prelinear equality algebra $(\mathcal{A}, G, H)$. By Proposition 4.18part (5) $B = \prod_{P \in Spec_T(A)} \mathcal{A}/\theta_P$ is a direct product of linearly ordered tense equality algebra. We define $f : A \to B$ by $f(x) = \{[x]_{\theta_P} : P \in Spec_T(A)\}$. It is easy to prove that f preserves operations. We will prove that f is one to one. Suppose that $x, y \in A$ such that $x \neq y$. Then $x \not\leq$ or $y \not\leq x$. Suppose that $x \not\leq$. Then $x \to y \neq 1$. By Proposition 4.17, there exists a prime tense deductive system P such that $x \to y \notin P$. Thus $[x]_{\theta_P} \not\in [y]_{\theta_P}$ in $\mathcal{A}/\theta_P$. Hence $f(x) \neq f(y)$.■

5 Compliance with ethical standards

Conflict of interest The author declares that she has no conflict of interest.
Human and animal rights This article does not contain any studies with human participants or animals performed by any of the authors.

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