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Exact Solution, Symmetry Group and Conservation laws for Some (2+1)-dimensional Nonlinear Physical Models

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Abstract

In this paper, the Bernoulli sub-equation function method is used to construct new exact travelling wave solutions of two important physical models, (2+1)-dimensional hyperbolic nonlinear Schrödinger (HNLS) equation and (2+1)-dimensional Heisenberg ferromagnetic spin chain (HFSC) equation. These solutions helpful to see the behavior of the physical phenomena of these models and they are expressible in exponential functions and tanh functions. The infinitesimal generators and the symmetry group have been investigated using the Lie symmetry method. In addition, by using multiplier approaches, the conservation laws are constructed for these models. Graphical simulation of some solutions in the form of two-dimensional and three-dimensional are plotted.

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1 Introduction

Many physical phenomena that arising in various fields of science can be described by nonlinear evolution equations (NLEEs) for instance optics, plasma, solid state physics, chemical physics, and mathematical physics. To understand these nonlinear phenomena, many physicist and mathematicians have made efforts to get various exact solutions of them. The investigation of the exact solutions is an important to provide better information about the mechanism and the applications of NLEEs. Abundant methods are extensively studied to obtain exact solutions, for
example the Bäcklund transform \([1, 2]\), the extended tanh-function method \([3]\), the F-expansion method \([4]\), sine-cosine method \([5]\), the extended F-expansion method \([6]\), Jacobi elliptic function method \([7]\), the Kudryashov method \([8]\), the extended Kudryashov method \([9, 10]\), the Bernoulli sub-equation function method \([11]\), the lie point symmetry method \([12]-[14]\) and other methods \([15]-[18]\).

In this study, we apply the Bernoulli sub-equation function method to construct exact solutions of two nonlinear models the first model is (2+1)-dim hyperbolic nonlinear Schrödinger equation \([19]-[22]\), which given as
\[
i \Theta_y - \frac{1}{2} (\Theta_{xx} - \Theta_{tt}) + |\Theta|^2 \Theta = 0, \tag{1.1}
\]
and the second model (2+1)-dim Heisenberg ferromagnetic spin chain equation \([23]-[25]\), in the form
\[
i \Theta_t + \alpha_1 \Theta_{xx} + \alpha_2 \Theta_{yy} + \alpha_3 \Theta_{xy} - \alpha_4 |\Theta|^2 \Theta = 0, \tag{1.2}
\]
where \(\Theta(x, y, t)\) is a complex envelope of the electric field, \(x\) is the dimensionless variable, \(y\) represents the propagation coordinates and \(t\) is the time. Eq. (1.1) expresses the elevation of water wave surface for slowly modulated wave trains in deep water in hydrodynamics, and also governs the propagation of electromagnetic fields in self-focusing and normally dispersive planar wave guides in optics. Eq. (1.2) can be used to depict the propagation of long waves, which has many applications in the percolation of water.

Different methods have been used to extract solutions for equation (1.1). Apeanti et. al. \([19]\) using generalized elliptic equation rational expansion method to construct several types of exact solutions. Travelling wave solutions of (1.1) was considered by Tebue et. al. \([20]\). Also, Seadawy et. al. \([21]\) derived new many solutions such as dark, bright, combined dark-bright, singular and combined singular solitons by using the extended sinh-Gordon equation expansion method. Moreover, Guo and Lin \([22]\) applied the classical Lie group symmetry method to obtain the Lie-point symmetries and also derive exact solutions. Several types of solutions of equation (1.2) are obtained by many researchers with different techniques. This includes the \(\exp(-\varphi(\xi))\)-expansion and the extended tanh-function methods in \([23]\), the complete discrimination system method \([24]\) and modified version of the Jacobi elliptic expansion method \([25]\). Although, many authors studied (2+1)-dimensional HNLS and (2+1)-dimensional HFSC equations, we studied these models by the Bernoulli sub-equation function method to obtain other solutions. Furthermore, conservation laws for these models are derived by using multiplier approach.

This paper is formed as follows: Firstly, we summarized the analytical method that we will use to construct novel exact solutions for Eq. (1.1) and Eq. (1.2) in section 2. In section 3, we get the solutions of the studied models. The geometrical shape of some solutions in the form of two-dimensional and three-dimensional have been plotted. Furthermore, the Lie point symmetry, group transformations, new solutions corresponding to the symmetry groups and
the conservation laws for (1.1) and (1.2) are obtained in section 4 and section 5, respectively. Finally, conclusions of the paper are presented.

2 Description of the Bernoulli sub-equation function method

In this section, we summarize the Bernoulli sub-equation function method [11] as follows:

Consider a NLPDE with independent variables \( x, y \) and \( t \)

\[ g(p, px, py, pt, pxx, pyy, pt, ...) = 0. \] (2.1)

Step 1: Suppose that \( p(x, y, t) = p(\alpha) \), \( \alpha = x + \rho y - \sigma t + \alpha_0 \) where \( \rho \) and \( \sigma \) are constants to be evaluated later and \( \alpha_0 \) is an arbitrary constant. Then (2.1) is transformed to an ordinary differential equation (ODE)

\[ h(p, p', p'', ...) = 0, \] (2.2)

Step 2: We propose the solutions of (2.2) as the following:

\[ p(x, y, t) = p(\alpha) = \sum_{i=0}^{K} d_i W^i(\alpha), \] (2.3)

where

\[ W'(\alpha) = a W(\alpha) + b W^S(\alpha), \ a \neq 0, \ b \neq 0, \ S \in \mathbb{R} - \{0, 1\}. \] (2.4)

By balancing principle, we can calculate \( K \) and \( S \). Also, the solution of Bernoulli differential equation (2.4) given as

\[ W(\alpha) = \left[ \frac{b}{a} + \frac{c}{e^{a(S-1)\alpha}} \right]^{1/S}, \ a \neq b, \]

\[ W(\alpha) = \left[ \frac{(c-1)+(c+1) \tanh(\frac{a(1-S)\alpha}{2})}{1-\tanh(\frac{a(1-S)\alpha}{2})} \right]^{1/S}, \ a = b, \ c \in \mathbb{R}. \] (2.5)

Step 3: Putting (2.3) with (2.4) in (2.2), we can obtain a polynomial in \( W^i(\alpha) \). Setting each coefficients of it to zero, we get a system of algebraic equations for \( d_i, (i = 0, 1, 2, ..., K) \) and \( a \). Solving the system by using Mathematica or Maple and substituting the coefficients into (2.3), then the general formula solutions of the NLPDE (2.1) can be constructed.

Step 4: Substituting the solutions of \( W(\alpha) \) that given in (2.5) into the general formal solutions that we obtained, we get exact solutions of Eq. (2.1).

3 The exact solutions

In this section, we apply the Bernoulli sub-equation function method to obtain the exact solutions of the studied models (1.1) and (1.2).
3.1 The exact solution of (2+1)-dimensional HNLS equation

To get the solutions of (1.1), we look for the solution as

\[ \Theta(x, y, t) = p(\alpha) e^{i \beta}, \quad \alpha = x + \rho y - \sigma t + \alpha_0, \quad \beta = l x + r y + \omega t, \quad (3.1) \]

where \( \rho, \sigma, l, r \) and \( \omega \) are constants to be calculated. Putting (3.1) into (1.1) and splitting the real and the imaginary parts, we get

\[ (\rho + l + \sigma \omega) p' = 0 \quad \Rightarrow \rho = -(l + \sigma \omega), \quad (3.2) \]

\[ (\sigma^2 - 1) p'' + (l^2 - \omega^2 + 2 r) p - 2p^3 = 0. \quad (3.3) \]

By applying the Bernoulli sub-equation function method and after balancing procedure, we get

\[ S = K + 1. \]

Now, we may choose the following three values of the parameter \( K \) to construct new exact solitary wave solutions of Eq. (1.1):

Case 1: If we put \( K = 1 \), then \( S = 2 \). Thus (2.3) and (2.4) take the form

\[ p(\alpha) = d_0 + d_1 W(\alpha), \quad (3.4) \]

\[ W'(\alpha) = a W(\alpha) + b W^2(\alpha), \quad (3.5) \]

where \( d_0, d_1, d_2, a \) and \( b \) are undetermined constants. Putting (3.4) into (3.3) and using (3.5), we get a polynomial in \( W(\alpha) \). Setting all coefficients of it equal to zero and with the help of Maple, we have

\[ a = \pm \sqrt{\frac{2l^2 - 2\omega^2 + 4r}{\sigma^2 - 1}}, \quad d_0 = \pm \frac{1}{2} \sqrt{2l^2 - 2\omega^2 + 4r}, \quad d_1 = \pm \sqrt{\sigma^2 - 1} b. \quad (3.6) \]

Substituting (3.6) into (3.4) with (2.5) and (3.1) we obtain the solution of model (1.1) as,

\[ \Theta(x, y, t) = \begin{cases} 
\pm \frac{1}{2} \sqrt{2l^2 - 2\omega^2 + 4r} \pm \frac{\sqrt{\sigma^2 - 1} b}{\pm \sqrt{2l^2 - 2\omega^2 + 4r} + \frac{c}{e^{\pm \sqrt{2l^2 - 2\omega^2 + 4r}}}} & a \neq b \\
\pm \frac{1}{2} \sqrt{2l^2 - 2\omega^2 + 4r} \left[ \frac{1 - \tanh(\pm \frac{1}{2} \sqrt{2l^2 - 2\omega^2 + 4r})}{(c-1)+(c+1)\tanh(\pm \frac{1}{2} \sqrt{2l^2 - 2\omega^2 + 4r})} \right] e^{i \beta} & a = b 
\end{cases} \quad (3.7) \]

where \( \alpha = x + \rho y - \sigma t + \alpha_0 \) and \( \beta = l x + r y + \omega t \).

Case 2: If we take \( K = 2 \), then \( S = 3 \). Therefor (2.3) and (2.4) take the form

\[ p(\alpha) = d_0 + d_1 W(\alpha) + d_2 W^2(\alpha), \quad (3.8) \]

\[ W'(\alpha) = a W(\alpha) + b W^3(\alpha). \quad (3.9) \]
Figure 1: (a-f) 3D and 2D the solitary wave solution (3.7) when $a = b$ are plotted with the parameters $l = \sigma = 1.5$, $r = 0.5$, $\omega = 1$, $\rho = 0.7$, $c = 0.8$, $\alpha_0 = 1$ and $y = 1$ for 3D plots, $y = 1$ and $t = 1$ for 2D plots.
Case 3: If we let \( K = 3 \), then \( S = 4 \). So (2.3) and (2.4) take the form

\[
p(\alpha) = d_0 + d_1 W(\alpha) + d_2 W^2(\alpha) + d_3 W^3(\alpha),
\]

\[
W'(\alpha) = a W(\alpha) + b W^4(\alpha).
\]

Substituting (3.12) into (3.3) and using (3.13), we get a polynomial in \( W(\alpha) \). Setting all coefficients of it equal to zero and with the aid of Maple, we obtain

\[
a = \pm \frac{1}{2} \sqrt{\frac{2 l^2 - 2 \omega^2 + 4 r}{\sigma^2 - 1}}, \quad d_0 = \pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r}, \quad d_1 = 0, \quad d_2 = \pm 2 \sqrt{\sigma^2 - 1} b.
\]

Putting (3.10) into (3.8) with (2.5) and (3.1) we obtain the solution of model (1.1) as,

\[
\Theta(x, y, t) = \begin{cases} 
\left( \pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \pm \frac{2 \sqrt{\sigma^2 - 1} b}{2 l^2 - 2 \omega^2 + 4 r} \right) e^{i \beta}, & a \neq b \\
\sqrt{2 l^2 - 2 \omega^2 + 4 r} \left( \pm \frac{1}{2} \pm \frac{1 - \tanh(\pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \alpha)}{(c-1)+(c+1) \tanh(\pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \alpha)} \right) e^{i \beta}, & a = b.
\end{cases}
\]

Putting (3.14) into (3.12) with (2.5) and (3.1) we have the solution of model (1.1) as,

\[
\Theta(x, y, t) = \begin{cases} 
\left( \pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \pm \frac{3 \sqrt{\sigma^2 - 1} b}{2 l^2 - 2 \omega^2 + 4 r} \right) e^{i \beta}, & a \neq b \\
\sqrt{2 l^2 - 2 \omega^2 + 4 r} \left( \pm \frac{1}{2} \pm \frac{1 - \tanh(\pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \alpha)}{(c-1)+(c+1) \tanh(\pm \frac{1}{2} \sqrt{2 l^2 - 2 \omega^2 + 4 r} \alpha)} \right) e^{i \beta}, & a = b.
\end{cases}
\]

From the previous cases when \( a \neq b \), the solutions that we obtained are similar to each others, but with a simple difference in the second term of these solutions. In the other hand at \( a = b \) the solutions are the same.
Figure 2: (a-f) 3D and 2D the solution (3.11) when $a \neq b$ are plotted with the parameters $l = \sigma = 1.5$, $r = 0.5$, $\omega = 1$, $\rho = 0.7$, $b = 0.6$, $c = 0.8$, $\alpha_0 = 1$ and $y = 1$ for 3D plots, $y = 1$ and $t = 1$ for 2D plots.

3.2 The exact solution of (2+1)-dimensional HFSC equation

To construct the solutions of (1.2). Putting (3.1) in (1.2) and splitting the real and the imaginary parts, we have

$$[l (2 \alpha_1 + \rho \alpha_3) + r (2 \alpha_2 + \alpha_3) - \sigma] u' = 0 \implies \sigma = l (2 \alpha_1 + \rho \alpha_3) + r (2 \alpha_2 + \alpha_3),$$

(3.16)
\[(\alpha_1 + \alpha_2 \rho^2 + \alpha_3 \rho) p'' - (\omega + \alpha_1 l^2 + \alpha_2 r^2 + \alpha_3 r) p - \alpha_4 p^3 = 0. \quad (3.17)\]

By using the Bernoulli sub-equation function method and after balancing procedure, we get \( S = K + 1 \). Now, we may choose the following three values of the parameter \( K \) to construct new exact solitary wave solutions of Eq. (1.2):

**Case 1**: If we put \( K = 1 \), then \( S = 2 \). Thus (2.3) and (2.4) are take the same form that given in (3.4) and (3.5), respectively. Putting (3.4) into (3.17) and using (3.5), we get a polynomial in \( W(\alpha) \). Setting all coefficients of it equal to zero and with the help of Maple, we have

\[
a = \pm \sqrt{-\frac{2}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1}} \left(-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}\right), \quad d_0 = \pm \sqrt{-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}},
\]

\[
d_1 = \pm \sqrt{2 \left(\frac{\rho \alpha_2 + \alpha_1 + \rho \alpha_3}{\alpha_4}\right)} b.
\]

Substituting (3.18) into (3.4) with (2.5) and (3.1) we obtain the solution of model (1.2) as,

\[
\Theta(x, y, t) = \begin{cases} 
\left[ \pm \sqrt{-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}} \left( \pm \frac{1}{2} \pm \frac{\sqrt{-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1}}}{e} + \frac{1}{2} \pm \frac{\sqrt{-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1}}}{e} \right) \right] e^{i \beta}, & a \neq b \\
\left[ \pm 1 \pm \frac{1}{2} \pm \frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1} \right] e^{i \beta}, & a = b,
\end{cases}
\]

where \( a = x + \rho y - \sigma t + \alpha_0 \) and \( \beta = l x + r y + \omega t \).

**Case 2**: If we take \( K = 2 \), then \( S = 3 \). Thus (2.3) and (2.4) take the same form that we obtained in (3.8) and (3.9), respectively. Setting (3.8) into (3.17) and using (3.9), we get a polynomial in \( W(\alpha) \). Putting all coefficients of it equal to zero and with the aid of Maple, we get

\[
a = \pm \sqrt{-\frac{2}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1}} \left(-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}\right), \quad d_0 = \pm \sqrt{-\frac{2 l^2 \alpha_1 + l r \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}}, \quad d_1 = 0,
\]

\[
d_2 = \pm 2 \sqrt{2 \left(\frac{\rho \alpha_2 + \alpha_1 + \rho \alpha_3}{\alpha_4}\right)} b.
\]

Putting (3.20) into (3.8) with (2.5) and (3.1) we obtain the solution of model (1.2) as,
corresponding to the symmetry group for two models that given in (1.1) and (1.2). In this section, we derive the Lie point symmetry, symmetry group and new solutions. A Lie symmetry of differential equations is a transformation that maps one solution to other solutions. If we let $K = 3$, then $S = 4$. So (2.3) and (2.4) become the same equations (3.12) and (3.13). Substituting (3.12) into (3.17) and using (3.13), we get a polynomial in $W(a)$. Setting all coefficients of it equal to zero and with the Maple software, we have

$$a = \pm \frac{1}{3} \sqrt{-\frac{2(\rho^2 \alpha_1 + l \rho_3 + r^2 \rho_2 + \omega)}{\rho^2 \alpha_2 + \rho \alpha_3 + \alpha_1}}, \quad d_0 = \pm \sqrt{-\frac{2 \alpha_1 + l \alpha_2 + r^2 \alpha_3 + \omega}{\alpha_4}}, \quad d_1 = d_2 = 0,$$

$$d_3 = \pm 3 \sqrt{-\frac{2(\rho^2 \alpha_3 + \alpha_1 + \rho \alpha_3)}{\alpha_4}} b. \quad (3.22)$$

Putting (3.22) into (3.12) with (2.5) and (3.1) we have the solution of model (1.2) as,

$$\Theta(x, y, t) = \begin{cases} \left[ \pm \sqrt{-\frac{2 \alpha_1 + l \alpha_3 + r^2 \alpha_2 + \omega}{\alpha_4}} \pm 2b \sqrt{\frac{2(\rho^2 \alpha_3 + \alpha_1)}{\alpha_4}} \right] e^{i \beta}, & a \neq b \\ \pm 3b \sqrt{-\frac{2(\rho^2 \alpha_3 + \alpha_1)}{\alpha_4}} e^{i \beta}, & a = b. \end{cases} \quad (3.23)$$

The solutions that we get in Case 1, Case 2 and Case 3 are similar to each others, but with a simple difference in the second term of these solutions at $a \neq b$. But when $a = b$ the solutions are the same in all cases.

4 Symmetry analysis and group transformation

A Lie symmetry of differential equations is a transformation, that maps one solution to other solution. In this section, we derive the Lie point symmetry, symmetry group and new solutions corresponding to the symmetry group for two models that given in (1.1) and (1.2).
Figure 3: (a-f) 3D and 2D the solitary wave solution (3.21) at $a \neq b$ are plotted with the parameters $l = \sigma = 1.5$, $r = 0.5$, $\omega = 1$, $\rho = 0.7$, $b = 0.6$, $c = 0.8$, $\alpha_0 = 1$ and $y = 1$ for 3D plots, $y = 1$ and $t = 1$ for 2D plots.
Figure 4: (a-f) 3D and 2D the solution (3.23) at $a = b$ are plotted with the parameters $l = \sigma = 1.5$, $r = 0.5$, $\omega = 1$, $\rho = 0.7$, $c = 0.8$, $\alpha_0 = 1$ and $y = 1$ for 3D plots, $y = 1$ and $t = 1$ for 2D plots.
4.1 Symmetry groups and new solutions of (2+1)-dimensional HNLS equation

First, we consider the transformation

$$\Theta(x, y, t) = n(x, y, t) + i m(x, y, t). \quad (4.1)$$

Substituting (4.1) into (1.1) and splitting the result into real and imaginary parts, we get the following system

$$m_y - \frac{1}{2} (n_{xx} - n_{tt}) - (n^3 + nm^2) = 0,$$
$$n_y + \frac{1}{2} (m_{xx} - m_{tt}) + (m^3 + mn^2) = 0. \quad (4.2)$$

The Lie point symmetries for (4.2) is generated by a vector field in the form

$$X = \tau^1(x, y, t, n, m) \partial_x + \tau^2(x, y, t, n, m) \partial_y + \tau^3(x, y, t, n, m) \partial_t + \nu^1(x, y, t, n, m) \partial_n + \nu^2(x, y, t, n, m) \partial_m. \quad (4.3)$$

Applying the prolongation Pr$^{(2)}$X to (4.2), we obtain a determined system of linear PDEs. Solving it by Maple, we get the infinitesimals as follows:

$$\tau^1 = \frac{1}{2} c_1 x + c_3 t + c_5, \quad \tau^2 = c_1 y + c_2, \quad \tau^3 = \frac{1}{2} c_1 t + c_3 x + c_4, \quad \nu^1 = -\frac{1}{2} c_1 n, \quad \nu^2 = -\frac{1}{2} c_1 m, \quad (4.4)$$

where $c_1, c_2, c_3, c_4$ and $c_5$ are constants. Substituting (4.4) into (4.3) we admit the algebra of Lie point symmetries generators of Eq. (4.2) as

$$X_1 = \frac{1}{2} (x \partial_x + 2y \partial_y + t \partial_t - n \partial_n - m \partial_m), \quad X_2 = \partial_y, \quad X_3 = t \partial_x + x \partial_t, \quad X_4 = \partial_t, \quad X_5 = \partial_x. \quad (4.5)$$

**Group transformations:**

To get the group transformation $G_i : (x, y, t, n, m) \rightarrow (\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m})$ that is generated by the generator $X_i$ for $i = 1, 2, 3, 4, 5$. We want to solve the initial problems of ODEs that given as

$$\frac{d(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m})}{d\epsilon} = (\tau^1, \tau^2, \tau^3, \nu^1, \nu^2),$$

$$(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m})|_{\epsilon=0} = (x, y, t, n, m),$$

where $\tau^1 = \tau^1(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m}), \quad \tau^2 = \tau^1(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m}), \quad \tau^3 = \tau^1(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m}), \quad \nu^1 = \nu^1(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m}), \quad \nu^2 = \nu^2(\hat{x}, \hat{y}, \hat{t}, \hat{n}, \hat{m})$ and $\epsilon$ is a group parameter. Then the one-parameter symmetry groups $G_i$ corresponding to the generators $X_i$ that given in (4.5) can be obtained as

$$G_1 : (x, y, t, n, m) \rightarrow (xe^{2\epsilon}, ye^{\epsilon}, te^{\frac{1}{2}\epsilon}, ne^{-\frac{1}{2}\epsilon}, me^{-\frac{1}{2}\epsilon}),$$
$$G_2 : (x, y, t, n, m) \rightarrow (x, y + \epsilon, t, n, m),$$
$$G_3 : (x, y, t, n, m) \rightarrow (x + \epsilon t, y, t + \epsilon x, n, m),$$

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$G_4 : (x, y, t, n, m) \to (x, y, t + \epsilon, n, m)$.
$G_5 : (x, y, t, n, m) \to (x + \epsilon, y, t, n, m)$.

**New solutions via group transformations:**
Consider $n = \lambda(x, y, t)$ and $m = \mu(x, y, t)$ is a solution of the system in (4.2), by using the one-parameter symmetry groups $G_i (i = 1, 2, 3, 4, 5)$ that obtained above, we have a new solutions as the following:

\[
\begin{align*}
&n^{(1)} = e^{-\frac{1}{2}\epsilon} \lambda(x e^{-\frac{1}{2}\epsilon}, y e^{-\epsilon}, t e^{-\frac{1}{2}\epsilon}), & m^{(1)} = e^{-\frac{1}{2}\epsilon} \mu(x e^{-\frac{1}{2}\epsilon}, y e^{-\epsilon}, t e^{-\frac{1}{2}\epsilon}). \\
n^{(2)} = \lambda(x - \epsilon, y, t) , & m^{(2)} = \mu(x - \epsilon, y, t).
\end{align*}
\]

\[
\begin{align*}
&n^{(3)} = \lambda(x - \epsilon t, y, t - \epsilon x) , & m^{(3)} = \mu(x - \epsilon t, y, t - \epsilon x). \\
n^{(4)} = \lambda(x, y, t - \epsilon) , & m^{(4)} = \mu(x, y, t - \epsilon).
\end{align*}
\]

\[
\begin{align*}
&n^{(5)} = \lambda(x - \epsilon, y, t) , & m^{(5)} = \mu(x - \epsilon, y, t).
\end{align*}
\]

**4.2 Symmetry groups and new solutions of (2+1)-dimensional HFSC equation**

Substituting (4.1) into (1.2) and splitting the result into real and imaginary parts, we get the following system

\[
\begin{align*}
m_t - \alpha_1 n_{xx} - \alpha_2 n_{yy} - \alpha_3 n_{xy} + \alpha_4 (n^3 + n m^2) &= 0, \\
n_t + \alpha_1 m_{xx} + \alpha_2 m_{yy} + \alpha_3 m_{xy} - \alpha_4 (m^3 + m n^2) &= 0.
\end{align*}
\]

Applying the prolongation $\text{Pr}^{(2)} X$ to (4.6), we obtain a determined system of linear PDEs. Solving it by Maple, we get the infinitesimals as follows:

\[
\begin{align*}
t^1 = (c_1 - c_3) x + \frac{a_1}{a_3} (-c_1 + 2 c_3) y + c_5 , & \quad t^2 = c_3 y + \frac{a_2}{a_3} (c_1 - 2 c_3) x + c_4 , \quad t^3 = c_1 t + c_2 , \\
\nu^1 = -\frac{1}{2} c_1 m , & \quad \nu^2 = -\frac{1}{2} c_1 n.
\end{align*}
\]

Setting (4.7) into (4.3) Eq. (4.6) admits the algebra of Lie point symmetries generators as

\[
\begin{align*}
X_1 = (x - \frac{a_1}{a_3} y) \partial_x + \frac{a_2}{a_3} x \partial_y + t \partial_t - \frac{1}{2} m \partial_n - \frac{1}{2} n \partial_m , & \quad X_2 = \partial_t , \\
X_3 = (-x + \frac{2 a_1}{a_3} y) \partial_x + (y - \frac{2 a_2}{a_3} x) \partial_y , & \quad X_4 = \partial_y , \quad X_5 = \partial_x.
\end{align*}
\]

**Group transformations:**
The one-parameter symmetry groups $G_i$ corresponding to the generators $X_i$ that given in (4.8) can be obtained as follows:
\[ G_1 : (x, y, t, n, m) \rightarrow \left( \frac{\alpha_3^2 - \alpha_1^2 \epsilon}{\alpha_3^2} \right) \left[ x (1 + \epsilon) - \frac{\alpha_1 \epsilon y}{\alpha_3} \right] \left( \frac{\alpha_3^2 - \alpha_1^2 \epsilon}{\alpha_3^2} \right) \left( 1 + \epsilon \right) (x - \frac{\alpha_1 \epsilon}{\alpha_3} y) - y \right], t e^\epsilon, n - \frac{m \epsilon}{2}, m - \frac{n \epsilon}{2}, \]
\[ G_2 : (x, y, t, n, m) \rightarrow (x, y, t + \epsilon, n, m), \]
\[ G_3 : (x, y, t, n, m) \rightarrow (x (1 + \epsilon) + \frac{4 \alpha_1 \alpha_3 \epsilon}{\alpha_3^2 - 4 \alpha_1 \alpha_2} y, y (1 + \epsilon) + \frac{8 \alpha_1 \alpha_2 \epsilon}{\alpha_3^2 - 4 \alpha_1 \alpha_2} y, t, n, m), \]
\[ G_4 : (x, y, t, n, m) \rightarrow (x, y + \epsilon, t, n, m), \]
\[ G_5 : (x, y, t, n, m) \rightarrow (x + \epsilon, y, t, n, m). \]

**New solutions via group transformations:**

Consider \( n = \lambda(x, y, t) \) and \( m = \mu(x, y, t) \) is a solution of the system in (4.6), by using the one-parameter symmetry groups \( G_i(i = 1, 2, 3, 4, 5) \), we have a new solutions as

\[ n^{(1)} = \lambda \left( \frac{\alpha_3^2 [\alpha_1 y + (\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2) x)]}{(\alpha_3^2 - \alpha_1 \alpha_2) [\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2)]} \right] \left[ \frac{\alpha_3^2 [\alpha_3 y - \alpha_2 x]}{(\alpha_3^2 - \alpha_1 \alpha_2) [\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2)]} \right], t e^{-\epsilon}, \]
\[ - \frac{\epsilon}{2} \mu \left( \frac{\alpha_3^2 [\alpha_1 y + (\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2) x)]}{(\alpha_3^2 - \alpha_1 \alpha_2) [\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2)]} \right] \left[ \frac{\alpha_3^2 [\alpha_3 y - \alpha_2 x]}{(\alpha_3^2 - \alpha_1 \alpha_2) [\alpha_1 \alpha_2 - (\alpha_3^2 - \alpha_1 \alpha_2)]} \right], t e^{-\epsilon}, \]
\[ n^{(2)} = \lambda(x, y, t - \epsilon), \quad m^{(2)} = \mu(x, y, t - \epsilon), \]
\[ n^{(3)} = \lambda \left( \frac{x}{1 + \epsilon} - \frac{4 \alpha_1 \alpha_3 \epsilon y}{\alpha_3^2 (1 + \epsilon)^2 - 4 \alpha_1 \alpha_2} \right] \left[ \frac{\alpha_3^2 - 4 \alpha_1 \alpha_2}{\alpha_3^2 (1 + \epsilon)^2 - 2 \alpha_1 \alpha_2 (2 - \epsilon)} \right] y, t, \]
\[ n^{(4)} = \lambda(x, y - \epsilon, t), \quad m^{(4)} = \mu(x, y - \epsilon, t). \]
\[ n^{(5)} = \lambda(x - \epsilon, y, t), \quad m^{(5)} = \mu(x - \epsilon, y, t). \]

**5 Conservation laws via multiplier approach**

First, we will mention some notations and definitions that used to obtain the conservation laws with multiplier approach. For simplicity, suppose that a Nth-order system of PDEs of \( \gamma \) dependent variables \( u = (u^1, u^2, \ldots, u^\gamma) \) and \( \nu \) independent variables \( x = (x^1, x^2, \ldots, x^\nu) \), define as

\[ \Gamma_\eta(x, u, u^{(1)}, \ldots, u^{(N)}) = 0, \quad \eta = 1, 2, \ldots, \gamma, \]

(5.1)

where, \( u^{(1)}, u^{(2)}, \ldots, u^{(N)} \) denote the collections of all first, second,...,Nth-order partial derivatives.

1- The Euler operator is given as

\[ \frac{\delta}{\delta u^\eta} = \frac{\partial}{\partial u^\eta} - D_i \frac{\partial}{\partial u^i} + D_i D_j \frac{\partial}{\partial u^i} - \ldots, \]

(5.2)
where
\[ D_i = \frac{\partial}{\partial x^i} + u^\eta_i \frac{\partial}{\partial u^\eta} + u^\eta_j \frac{\partial}{\partial u^\eta_j} + \ldots, \quad i = 1, 2, \ldots, \nu, \tag{5.3} \]
is the total derivative operator with respect to \( x^i \).

2- The \( \nu \)-tuple vector \( C = (C^1, C^2, \ldots, C^\nu) \) is a conserved vector of (5.1), such that
\[ D_i C^i = 0, \quad i = 1, 2, \ldots, \nu. \tag{5.4} \]

3- The multipliers \( M^\eta \) for the system (5.1) has the following property:
\[ D_i C^i = M^\eta \Gamma^\eta. \tag{5.5} \]

4- The determining equations for the multipliers are constructed by using variational derivative of (5.5) as,
\[ \frac{\delta}{\delta u^\eta}(M^\eta \Gamma^\eta) = 0. \tag{5.6} \]

5.1 Conservation laws of (2+1)-dimensional HNLS equation

To calculate the conservation laws for (1.1), we consider multipliers for (4.2) in the form
\[ M^1(t, x, y, n, m, t_n, m_n, n_x, m_x, n_y, m_y), \ M^2(t, x, y, n, m, t_n, m_n, n_x, m_x, n_y, m_y) \] and applying (5.6), we have
\[ \frac{\delta}{\delta n} [M^1 \Gamma_1 + M^2 \Gamma_2] = 0, \]
\[ \frac{\delta}{\delta m} [M^1 \Gamma_1 + M^2 \Gamma_2] = 0. \tag{5.7} \]
where, \( \frac{\delta}{\delta n} \) and \( \frac{\delta}{\delta m} \) are the standard Euler operators which we can obtain from (5.2) with (5.3).

Expanding (5.7) and separating the different combinations of derivatives of \( n \) and \( m \), we have system of equations for \( M^1 \) and \( M^2 \). Solving it by Maple software, we get

\[ M^1 = [(t n_t + x n_x + n) y + y^2 n_y + \frac{1}{2} (x^2 - t^2) m] c_1 + (t n_t + x n_x + 2 y n_y + n) c_2 + (y n_t - t m) c_3 - (x m + y n_x) c_4 - m c_5 + (x n_t + t n_x) c_6 + n_x c_7 + n_t c_8 + n_y c_9, \]
\[ M^2 = [(t m_t + x m_x + m) y + y^2 m_y + \frac{1}{2} (t^2 - x^2) n] c_1 + (t m_t + x m_x + 2 y m_y + m) c_2 + (y m_t + t n) c_3 + (x n - y m_x) c_4 + n c_5 + (x m_t + t m_x) c_6 + m_x c_7 + m_t c_8 + m_y c_9. \tag{5.8} \]

From (5.5) and (5.8), we get the following conserved vectors:
Case 1: When $M^1 = (t_n x + x_n + n) y + y^2 n_y + \frac{1}{2} (x^2 - t^2) m$ and $M^2 = (t_m x + m_x + m) y + y^2 n_y + \frac{1}{2} (t^2 - x^2) n$, then we get

\[
C^x = \left[\frac{1}{2} m^2 t x - 3 n^3 n_t x t + \frac{1}{2} n_x n t + \frac{1}{2} n_x n + \frac{1}{4} n_x^2 x + \frac{1}{2} n_t^2 x + \frac{1}{2} n_x m + \frac{1}{2} m_x^2 x \\
+ n_x m + \frac{1}{2} \frac{1}{2} m x m t | y + (\frac{1}{2} m y m n + \frac{1}{2} n_x n y + n y x n^3) y^2 - \frac{1}{2} n_x m x^2 - \frac{1}{2} n_y m x^3 \\
+ \frac{1}{2} n_y x m t^2 - \frac{1}{2} n x m t^2 + \frac{1}{2} m m x + \frac{1}{2} m_x n t^2 \right],
\]

\[
C^y = \left(-\frac{1}{4} m^2 + \frac{1}{4} m^4 + \frac{1}{4} m^2 - n^3 n_x x + \frac{1}{4} n_t^2 + \frac{1}{2} n^2 m^2 - \frac{1}{4} n_x^2 \right) y^2 + (-m n t - m x n x - m n) y \\
+ \frac{1}{4} t^2 m^2 - \frac{1}{4} x^2 m - \frac{1}{2} n n x t^2 x + \frac{1}{6} n x n x^3,
\]

\[
C^t = \left(-\frac{1}{2} n y m_t - \frac{1}{2} m y m_t \right) y^2 + [3 n^3 n_t x + n^4 t - \frac{1}{2} n t^2 + \frac{1}{4} m^4 t - \frac{1}{4} m^4 t - \frac{1}{4} n x^2 t - \frac{1}{2} m t m \\
- \frac{1}{2} n t^2 t + n y m t - \frac{1}{2} m x m t x + \frac{1}{2} m^2 n^2 t - \frac{1}{2} n x n x t + \frac{1}{2} m n t + \frac{1}{4} n m t^2 - \frac{1}{4} n t m x^2 + \frac{1}{4} m t n x^2 - \frac{1}{4} m t n t^2.
\]

Case 2: When $M^1 = (t_n x + x_n + 2 y n_y + n)$ and $M^2 = (t_m x + m_x + 2 y m_y + m)$, thus we have

\[
C^x = \frac{1}{4} \left(m^4 + m^2 - 12 n^3 n_t x + m^4 + n^2 g x + 2 m^2 n^2 + 8 n y m^3 + n_x^2 \right) x + (m y m x + n x n_y) y + \frac{1}{2} (n x n + n x n t + m x m t + m x m),
\]

\[
C^y = \left(-2 n^3 n_t y - m n x x + \frac{1}{4} (n_t^2 - m x^2 + m^2 + n^4 - n_x^2 + 2 m^2 n^2) y - t m n t - n m, \right.
\]

\[
C^t = \frac{1}{4} \left(m^4 t - m^2 t - 2 m t m + 2 m^2 n^2 t - 2 n t n - m_x^2 t - n_x^2 t - n_t^2 t \right) + n^4 t + n y m t \\
+ \frac{1}{2} (6 n^3 n_t x - n_t n_x - m x m t) x - (m y m t + n y n t) y.
\]

Case 3: When $M^1 = n t y - m t$ and $M^2 = m_y x + n t$, we obtain

\[
C^x = \frac{1}{2} \left[(2 n_y x n - n_x m + m_x n) t + (n x n_t + m_x m) \right] y, \quad C^y = -n n x t x - m n y t + \frac{1}{2} t m^2, \quad C^t = \frac{1}{2} \left((m m n - m t n) t + n m \right) + \frac{1}{4} \left(2 m^2 n^2 + 4 n m + m^4 - m_x^2 - n_t^2 \right), \quad (x, y).
\]

Case 4: If $M^1 = -m x - n x y$ and $M^2 = -m x y + n x$, then we have

\[
C^x = \frac{1}{4} \left(m_x^2 + n_t^2 + m_t^2 + 2 n^2 m^2 - 4 n^3 n_t x + n_x^2 + 4 n_y m + m^4 \right) y \\
+ \frac{1}{2} (n_y n x^2 - n m) + \frac{1}{2} (m x n - n m x) x,
\]

\[
C^y = \frac{1}{2} (x m^2 - n_x n x^2) + m n x y,
\]

\[
C^t = \frac{1}{2} \left[(n m t - m_t n) x + (n x n_t + m_x m_t - 2 n^3 n_t x) \right] y.
\]
Case 5: When $M^1 = -m$ and $M^2 = n$, thus we have

\[
\begin{align*}
C^x &= \frac{1}{2} (m_x n - n_x m + 2 n_y x n), \\
C^y &= \frac{1}{2} m_x^2 - n n_x x, \\
C^t &= \frac{1}{2} (-m_t n + n_t m). \\
\end{align*}
\] (5.13)

Case 6: If $M^1 = n_t x + n_x t$ and $M^2 = m_t x + m_x t$, then we have

\[
\begin{align*}
C^x &= -\frac{1}{2} n_t n^3 t^2 + \frac{1}{4} (4 n_y m + 2 m^2 n^2 + n_x^2 + n_t^2 + m_t^2 + m_x^2 + m^4) t + \frac{1}{2} (n_t n_x + m_x m_t) x, \\
C^y &= -m n_t x - m n_x t, \\
C^t &= \frac{1}{2} n_x n^3 t^2 - \frac{1}{2} (m_x m_t + n_t n_x) x + \frac{1}{4} (2 m^2 n^2 + n^4 - n_t^2 - m_t^2 - m_x^2 + m^4 - n_x^2 + 4 n_y m) x. \\
\end{align*}
\] (5.14)

Case 7: If $M^1 = n_x$ and $M^2 = m_x$, then we have

\[
\begin{align*}
C^x &= m n_y - n^3 n_t t + \frac{1}{4} (m^4 + m_x^2 + n_x^2 + n_t^2 + m_t^2 + 2 m^2 n^2), \\
C^y &= -n_x m, \\
C^t &= -\frac{1}{2} (m_t m_x + n_t n_x) + n^3 n_x t. \\
\end{align*}
\] (5.15)

Case 8: At $M^1 = n_t$ and $M^2 = m_t$, we get

\[
\begin{align*}
C^x &= \frac{1}{2} (m_t m_x + n_t n_x), \\
C^y &= -n_t m, \\
C^t &= m n_y + \frac{1}{4} (n^4 + m^4 - m_x^2 - n_x^2 - m_t^2 - n_t^2 + 2 m^2 n^2). \\
\end{align*}
\] (5.16)

Case 9: If $M^1 = n_y$ and $M^2 = m_y$, then we obtain

\[
\begin{align*}
C^x &= \frac{1}{2} (n_y n_x + m_y m_x) + n_y x n^3, \\
C^y &= \frac{1}{4} (m^4 - m_x^2 - n_x^2 + m_t^2 + n_t^2 - 4 n^3 n_x x + 2 m^2 n^2), \\
C^t &= -\frac{1}{2} (m_t m_y + n_t n_y). \\
\end{align*}
\] (5.17)
5.2 Conservation laws of (2+1)-dimensional HFSC equation

To calculate the conservation laws for (1.2), we consider multipliers for (4.6) in the same form for (4.2) and by the same steps, we get

\[
M^1 = \left[ (t n_t + x n_x + y n_y + n) t + \frac{(a_1 y^2 + a_2 x^2 - a_3 x y) m}{(4 a_1 a_2 - a_3^2)} \right] c_1 + (2 t n_t + x n_x + 2 y n_y + n) c_2
- (2 a_1 t n_x + a_3 t n_y + m x) c_3 - (2 a_2 t n_y + a_3 t n_x + m y) c_4 - m c_5 +
(2 a_1 y n_x - 2 a_2 x n_y - a_3 (x n_x - y n_y)) c_6 + n_x c_7 + n_t c_8 + n_y c_9,
\]

\[
M^2 = \left[ (t m_t + x m_x + y m_y + m) t - \frac{(a_1 y^2 + a_2 x^2 - a_3 x y) n}{(4 a_1 a_2 - a_3^2)} \right] c_1 + (2 t m_t + x m_x + y m_y + m) c_2
- (2 a_1 t m_x + a_3 t m_y - n x) c_3 - (2 a_2 t m_y + a_3 t m_x - n y) c_4 + n c_5 +
(2 a_1 y m_x - 2 a_2 x m_y - a_3 (x m_x - y m_y)) c_6 + m_x c_7 + m_t c_8 + m_y c_9.
\]

From (5.5) and (5.18), we construct the conserved vectors of Eq. (1.2) as

**Case 1:** When \( M^1 = (t n_t + x n_x + y n_y + n) t + \frac{(a_1 y^2 + a_2 x^2 - a_3 x y) m}{(4 a_1 a_2 - a_3^2)} \) and

\[
M^2 = (t m_t + x m_x + y m_y + m) t - \frac{(a_1 y^2 + a_2 x^2 - a_3 x y) n}{(4 a_1 a_2 - a_3^2)},
\]

then we get

\[
C^x = \frac{1}{4} \left[ (4 m_t m_x t^2 + 2 m_x^2 t x + 4 m_x m_y t y + 4 n_t n_x t^2 + 2 n_x^2 t x + 4 n_x n_y t y + 4 m m_x t + 4 n m_x t) a_1 +
(-2 m_y^2 t x - 2 n_y^2 t x) a_2 + (2 m_t m_y t^2 + 2 m_y^2 t y + 2 n_t n_y t^2 + 2 n_y^2 t y + 4 m m_y t + 4 n m_y t) a_3 +
(4 n^3 n_t t^2 x - 4 n^3 n_y t x y - m^4 t x - 2 m^2 n^2 t x) a_4 + 4 m n_t t x + \frac{1}{4 a_1 a_2 - a_3^2} \left( (4 m m_x y^2 - 4 m_x n y^2) a_1^2 +
((4 m n_x x^2 - 4 m x n x^2 + 8 m n x) a_2 + (4 m n_x x y + 4 m x n x y - 4 m y n y^2 - 4 m n y) a_3 \right) a_1
-4 a_2 a_3 m n x^2 + 4 a_3^2 m n x y) \right],
\]

\[
C^y = \frac{1}{4} \left[ (2 m_x^2 t y + 2 n_x^2 t y) a_1 + (-4 m_t m_y t^2 - 4 m_x m_y t x - 2 m_y^2 t y - 4 n_t n_y t^2 - 4 n_x n_y t x
-2 n_y^2 t y - 4 m m_y t - 4 n m_y t) a_2 + (-2 m_t m_x t^2 - 2 m_x^2 t x - 2 n_t n_x t^2 - 2 n_x^2 t x) a_3 + (-4 n^3 n_x t x y
+4 m^4 t y + 2 m^2 n^2 t y) a_4 - 4 m n_t t y + \frac{1}{4 a_1 a_2 - a_3^2} \left( a_1 a_2 (m n_y y^2 - m y n y^2 + 2 m n y) + a_1 a_3 m n x y^2 +
a_2^2 (x (m n_y - 4 m n x)) a_2 + a_2 a_3 (m n_x x^2 - m n x y + m y n x + m x) - a_3^2 m y (n x + n) \right),
\]

\[
C^t = \frac{1}{4} \left[ (2 m_x^2 t^2 + 2 n_x^2 t^2) a_1 + (2 m_y^2 t^2 + 2 n_y^2 t^2) a_2 + (2 m_x m_y t^2 + 2 n x n y t^2) a_3 +
(4 n^3 n_x t^2 x + 4 m^4 t^2 + 2 m^2 n^2 t^2 + 2 n^4 t^2) a_4 + 4 m n_x t x + 4 m n_y t y + 4 m n t - \frac{(m^2 + n^2) (x^2 a_2 - x y a_3 + y^2 a_1)}{2 (4 a_1 a_2 - a_3^2)}. \right]
\]
Case 2: If \( M^1 = (2 t n_t + x n_x + 2 y n_y + n) \) and \( M^2 = (2 t m_t + x m_x + y m_y + m) \), then we have

\[
C^x = (2 m_t m_x t + \frac{1}{2} n_x^2 x + n_x y g + m m_x + n n_x + \frac{1}{2} m_x^2 x + m_x m_y y + 2 n_t n_x t) \alpha_1 + \\
(\frac{1}{2} m_y^2 x - \frac{1}{2} n_y^2 x) \alpha_2 + (m_t m_y t + m m_y + n n_y + \frac{1}{2} m_y^2 y + n_t n_y t + \frac{1}{2} n_y^2 y) \alpha_3 + \\
(-n^3 n_y x y - \frac{1}{4} m^4 x - \frac{1}{2} m^2 n^2 x + 2 n^3 n_t t x) \alpha_4 + m n t x,
\]

\[
C^y = \frac{1}{2} y \left( m_x^2 + n_x^2 \right) \alpha_1 + (n n_y + 2 m_t m_g t + m_x m_y x + \frac{1}{2} m_y^2 y + 2 n_t n_y t + n_x n_y x + \frac{1}{2} n_y^2 y
\]

\[
+ m m_y) \alpha_2 + (m_t m_x t + n_t n_x t + \frac{1}{2} m_x^2 x + \frac{1}{2} m_y^2 x) \alpha_3 + (n^3 n_x x y - \frac{1}{4} m^4 y - \frac{1}{2} m^2 n^2 y) \alpha_4 + m n t y,
\]

\[
C^t = (-m_x^2 t - n_x^2 t) \alpha_1 + (-m_y^2 t - n_y^2 t) \alpha_2 + (-m_x m_y t - n_x n_y t) \alpha_3 \\
+ (-n^4 t - 2 n^2 n_x t x - \frac{1}{4} m^4 t - m^2 n^2 t) \alpha_4 - m n_n x - m n_y y - m n n.
\]

Case 3: When \( M^1 = -(2 \alpha_1 t n_x + \alpha_3 t n_y + m x) \) and \( M^2 = -(2 \alpha_1 t m_x + \alpha_3 t m_y - n x) \), we obtain

\[
C^x = (-m_x^2 t - n_x^2 t) \alpha_1 + [(m_y^2 t + n_y^2 t) \alpha_2 + (-m_x m_y t - n_x n_y t) \alpha_3 + (m^2 n^2 t - n^3 n_t t^2 + \frac{1}{2} m^4 t) \alpha_4 \\
- 2 m n_t t - m n x x + m x n x - m n n x \alpha_1] + (\frac{1}{2} n y^2 t - \frac{1}{2} n y^2 t) \alpha_3 + (m^4 n^2 t x + m y n x) \alpha_3,
\]

\[
C^y = \left( (-2 m_x m_y t - 2 n_x n_y t) \alpha_2 + (-\frac{1}{2} n_x^2 t - \frac{1}{2} m_x^2 t) \alpha_3 \right) \alpha_1 + ((\frac{1}{2} m^2 n^2 t - 3 n^3 n t x + 2 m^4 t) \alpha_4 - m n t t - m n x x - m n) \alpha_3,
\]

\[
C^t = (\alpha_4 n^3 n_x t^2 + 2 m n_t t) \alpha_1 + \frac{1}{2} x n^2 + \alpha_3 m n y t + \frac{1}{2} m^2 x.
\]

Case 4: If \( M^1 = -(2 \alpha_1 t n_y + \alpha_3 t n_x + m y) \) and \( M^2 = -(2 \alpha_2 t m_y + \alpha_3 t m_x - n y) \), then we have

\[
C^x = 2 \alpha_2 a_4 n^4 n_y t x t - t \left( \frac{1}{4} n^3 n_t t - \frac{1}{4} m^4 - \frac{1}{2} m^2 n^2 \right) \alpha_3 \alpha_4 - 2 t (m_x m_y + n_x n_y) \alpha_1 \alpha_2 \\
- \frac{1}{2} (m_x^2 + \frac{1}{2} n_x^2) \alpha_1 \alpha_3 - \frac{1}{2} t (m_y^2 + n_y^2) \alpha_2 \alpha_3 - y (m n_x - m_x n) \alpha_1 - (m n_t t - m_y n y) \alpha_3,
\]

\[
C^y = (m_x^2 t + n_x^2 t) \alpha_2 \alpha_1 + (-m_y^2 t - n_y^2 t) \alpha_2 \alpha_2 + \left( \frac{1}{2} m^2 t - \frac{1}{2} n_x^2 t \right) \alpha_3^2 - \alpha_3 m n^x y
\]

\[
+ [(-m_x m_y t - n_x n_y t) \alpha_3 + (m^2 n^2 t - 2 n^3 n_x t x + \frac{1}{4} m^4 t) \alpha_4 - 2 m n_t t - m n y y + m y n y - m n] \alpha_2,
\]

\[
C^t = 2 \alpha_2 m n_y t + \left( \frac{1}{2} a_4 n^3 n_x t^2 + m n_t t \right) \alpha_3 + \frac{1}{2} y m^2 + \frac{1}{2} n^2 y.
\]
Case 5: If $M^1 = -m$ and $M^2 = n$, then we have
\begin{align*}
C^x &= \alpha_1 (-mn_x + m_x n) + \alpha_3 m_y n, \\
C^y &= \alpha_2 (-mn_y + m_y n) - \alpha_3 mn_x, \\
C^t &= \frac{1}{2} (m^2 + n^2).
\end{align*}

Case 6: If $M^1 = (2 \alpha_1 y n_x - 2 \alpha_2 x n_y - \alpha_3 (x n_x - y n_y))$ and $M^2 = (2 \alpha_1 y m_x - 2 \alpha_2 x m_y - \alpha_3 (x m_x - y m_y))$, then we have
\begin{align*}
C^x &= (m_x^2 y + n_x^2 y) \alpha_1^2 + ((-\frac{1}{2} m_y^2 x - \frac{1}{2} n_y^2 x) \alpha_3 + \alpha_4 n^3 y x^2) \alpha_2 + (\frac{1}{2} n_y^2 y + \frac{1}{2} m_y^2 y) \alpha_3^2 \\
&\quad + (2 n^3 t y - \frac{1}{2} m^4 y - n^2 n^2 y) \alpha_4 + 2 mn_t y \alpha_1 + \left(\frac{1}{4} m^4 + \frac{1}{2} m^2 n^2 x - n^3 n_y x y\right) \alpha_4 - mn_t x \alpha_3, \\
C^y &= [(m_x^2 x + 2 m_x m_y y + n_x^2 x + 2 n_x n_y y) \alpha_2 + (\frac{1}{2} n_x^2 y + \frac{1}{2} m_x^2 y) \alpha_3 \alpha_1 + (2 n^3 n_y x y - \frac{1}{2} m^4 n^2 y) \alpha_2 \alpha_3 + (\frac{1}{4} m^4 + \frac{1}{2} m^2 n^2 x - n^3 n_y x y) \alpha_4 + mn_t y \alpha_3, \\
C^t &= (-2 n^3 n_x ty \alpha_4 - 2 mn_x y) \alpha_1 + 2 \alpha_2 mn_y x + (mn_x x - mn_y y) \alpha_3.
\end{align*}

Case 7: If $M^1 = n_x$ and $M^2 = m_x$, then we have
\begin{align*}
C^x &= \frac{1}{2} (m_x^2 + n_x^2) \alpha_1 - \frac{1}{2} (m_y^2 + n_y^2) \alpha_2 + m_t n^3 t - \frac{1}{4} m^4 - \frac{1}{2} m^2 n^2 \alpha_4 + mn_t, \\
C^y &= (m_x m_y + n_x n_y) \alpha_2 + \frac{1}{2} (m_x^2 + n_x^2) \alpha_3, \\
C^t &= -n^3 t n_x \alpha_4 - mn_x.
\end{align*}

Case 8: If $M^1 = n_t$ and $M^2 = m_t$, then we have
\begin{align*}
C^x &= (m_t m_x + n_t n_x) \alpha_1 + \frac{1}{2} (m_t m_y + n_t n_y) \alpha_3, \\
C^y &= (m_t m_y + n_t n_y) \alpha_2 + \frac{1}{2} (m_t m_x + n_t n_x) \alpha_3, \\
C^t &= -\frac{1}{4} \left[2 (m_x^2 + n_x^2) \alpha_1 + 2 (m_y^2 + n_y^2) \alpha_2 + 2 (m_x m_y + n_x n_y) \alpha_3 + (m^4 + 2 m^2 n^2 + n^4) \alpha_4 \right].
\end{align*}
Case 9: If $M^1 = n_y$ and $M^2 = m_y$, then we have

$$C^x = (m_x m_y + n_x n_y) \alpha_1 + \frac{1}{2} (m_y^2 + n_y^2) \alpha_3 - \alpha_4 n^3 n_y x,$$

$$C^y = -\frac{1}{4} [2 (n_x^2 + m_x^2) \alpha_1 - 2 (m_y^2 + n_y^2) \alpha_2 + (m^4 + 2 m^2 n^2 - 4 x n^3 n_x) \alpha_4 + mn],$$

$$C^t = -m n y.$$ (5.27)

6 Conclusion

In this paper, we study two physical models are (2+1)-dimensional HNLS equation and (2+1)-dimensional HFSC equation via the Bernoulli sub-equation function method to investigate the exact traveling wave solutions. By this method, we discuss the obtained solutions at different values of $K$ and $S$ in equations (2.3) and (2.4), respectively by using the relation $S = K + 1$ that we get from the balancing procedure. We observe that when $a \neq b$, the solutions that we obtained are similar to the other and at $a = b$ the solutions are the same. The computer systems like as Maple is used to solve the complicated algebraic equations to get these solutions. To the best of our knowledge, some of the obtained solutions for these models are new. The geometrical shape for some of the obtained results are plotted in the form of two-dimensional and three-dimensional for various choices of the parameters that appear in the results which may help researchers to known some physical meaning of these models. Furthermore, by applying the Lie symmetry method, we construct the infinitesimal generators and using them to construct the symmetry group. Also, we investigate novel solutions of the studied models by the group transformations. In addition, the conservation laws for the models are obtained by multiplier approach. We hope that the obtained solutions are useful in the study of optics and other important equations of mathematical physics.

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References


