Cartesian-closedness and Subcategories of (L,M)-fuzzy Q-convergence Spaces

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Research Article

Keywords: Fuzzy topology, Fuzzy convergence structure, Category, Function space

Posted Date: April 5th, 2021

DOI: https://doi.org/10.21203/rs.3.rs-317340/v1

Version of Record: A version of this preprint was published at Soft Computing on July 29th, 2021. See the published version at https://doi.org/10.1007/s00500-021-06057-w.

Cartesian-closedness and subcategories of (L, M)-fuzzy Q-convergence spaces[☆]

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Abstract

In this paper, we first construct the function space of (L, M)-fuzzy Q-convergence spaces to show the Cartesian-closedness of the category (L, M)-**QC** of (L, M)-fuzzy Q-convergence spaces. Secondly, we introduce several subcategories of (L, M)-**QC**, including the category (L, M)-**KQC** of (L, M)-fuzzy Kent Q-convergence spaces, the category (L, M)-**LQC** of (L, M)-fuzzy Q-limit spaces and the category (L, M)-**PQC** of (L, M)-fuzzy pretopological Q-convergence spaces, and investigate their relationships.

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1. Introduction

In general topology, function spaces of topological spaces cannot be constructed in a satisfactory way. This means the category of topological spaces with continuous mappings as morphisms is not Cartesian closed. In order to overcome this deficiency, the concept of filter convergence spaces (convergence spaces in short) was proposed and discussed [3, 7, 13, 14]. In [24], Preuss gave a systematical collection of convergence structures, including function spaces and subcategories of convergence spaces as well as their connections with topological spaces.

With the development of the theory of fuzzy topology [2, 15, 25, 29], many types of fuzzy convergence structures have been proposed, such as stratified L-generalized convergence structure [9, 11, 17, 18], L-fuzzifying convergence structure [26, 27], L-convergence tower structure [8, 10, 22], L-ordered convergence structure [4, 5], (Enriched) (L, M)-fuzzy (Q-)convergence structure [20, 21, 23], $\top$-convergence structure [6, 12] and so forth. Fuzzy convergence structures are usually discussed from two aspects. On one hand, the categorical relationship between fuzzy convergence structures and fuzzy topologies are discussed and it is shown that the category of fuzzy topological spaces can be embedded in the category of fuzzy convergence spaces as a reflective subcategory. On the other hand, the categorical properties of fuzzy convergence spaces are investigated. It is proved that the category of fuzzy convergence spaces is Cartesian closed and subcategories of fuzzy convergence spaces have compatible relationships.

In the theory of fuzzy convergence spaces, many researchers usually show the Cartesian-closedness of fuzzy convergence spaces by constructing the corresponding function space, i.e., the power object in the category of fuzzy convergence spaces. Actually, there are different approaches to show the Cartesian-closedness of a category. For example, a topological category **A** is Cartesian closed if and only if the

[☆]This work is supported by the Natural Science Foundation of China (12071033, 11701122).

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functor $A \times - : \mathbf{A} \longrightarrow \mathbf{A} : B \longmapsto A \times B$ preserves final epi-sinks for each object A in $\mathbf{A}$. In this approach, Pang and Li showed the Cartesian-closedness of the categories of (L, M)-fuzzy convergence spaces [21] and L-fuzzy Q-convergence spaces [16], respectively. Later, Pang and Zhao [23] introduced the concept of stratified (L, M)-fuzzy Q-convergence spaces and proved that the resulting category is Cartesian closed. However, they failed to construct the corresponding function spaces. By this motivation, we will focus on (L, M)-fuzzy Q-convergence spaces (called stratified (L, M)-fuzzy Q-convergence spaces in [23]) and present the concrete form of the corresponding function spaces. Moreover, we will introduce several types of (L, M)-fuzzy Q-convergence spaces and study their mutual relationships from a categorical aspect.

This paper is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we construct the function space of (L, M)-fuzzy Q-convergence structures to show the Cartesian-closedness of the resulting category. In Sections 4–6, we propose the concepts of (L, M)-fuzzy Kent Q-convergence spaces, (L, M)-fuzzy Q-limit spaces, and (L, M)-fuzzy pretopological Q-convergence spaces and investigate their categorical relationships.

2. Preliminaries

Throughout this paper, both L and M denote completely distributive lattices and $'$ is an order-reversing involution on L. The smallest element and the largest element in L (M) are denoted by $\bot_L$ $(\bot_M)$ and $\top_L$ $(\top_M)$, respectively. For $a, b \in L$, we say that a is wedge below b, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geq b$ implies $d \geq a$ for some $d \in D$. An element a in L is called coprime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. The set of nonzero coprime elements in L is denoted by $J(L)$. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in J(L) \mid a \prec b\}$ for each $b \in L$. An element a in L is called prime if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. L^X is also a complete lattice when it inherits the structure of the lattice L in a natural way, by defining $\vee$, $\wedge$ and $\leq$ pointwisely. The smallest element and the largest element in L^X are denoted by $\bot_L^X$ and $\top_L^X$, respectively. For each $x \in X$ and $a \in L$, the L-subset x_a, defined by $x_a(y) = a$ if $y = x$, and $x_a(y) = \bot_L$ if $y \neq x$, is called a fuzzy point. The set of nonzero coprime elements in L^X is denoted by $J(L^X)$. It is easy to see that $J(L^X) = \{x_\lambda \mid x \in X, \lambda \in J(L)\}$. We say that a fuzzy point x_λ quasi-coincides with A, denoted by $x_\lambda \hat{q} A$, if $\lambda \nleq A'(x)$. For each $a \in L$, $\underline{a}$ denotes the constant mapping $X \longrightarrow L$, $x \longmapsto a$. Let $f : X \longrightarrow Y$ be a mapping. Define $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ by $f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$, respectively.

Definition 2.1 ([28]). A mapping $\mathcal{F} : L^X \longrightarrow M$ is called an (L, M)-fuzzy filter on X if it satisfies

(LMF1) $\mathcal{F}(\bot_L^X) = \bot_M, \mathcal{F}(\top_L^X) = \top_M$;

(LMF2) $\mathcal{F}(A \wedge B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$.

The family of all (L, M)-fuzzy filters on X is denoted by $\mathcal{F}_{LM}(X)$.

Example 2.2 ([20]). For each $x_\lambda \in J(L^X)$, we define $\hat{q}(x_\lambda) : L^X \longrightarrow M$ as follows:

$$\forall A \in L^X, \quad \hat{q}(x_\lambda)(A) = \begin{cases} \top_M, & x_\lambda \hat{q} A, \\ \bot_M, & otherwise. \end{cases}$$

Then $\hat{q}(x_\lambda)$ is an (L, M)-fuzzy filter.

On the set $\mathcal{F}_{LM}(X)$ of all (L,M)-fuzzy filters on X, we define an order by $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F}(A) \leq \mathcal{G}(A)$ for all $A \in L^X$. Then for a family of (L,M)-fuzzy filters $\{\mathcal{F}_j \mid j \in J\}$, the infimum is given by $(\bigwedge_{j\in J} \mathcal{F}_j)(A) = \bigwedge_{j\in J} \mathcal{F}_j(A)$. For a mapping $f : X \longrightarrow Y$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$, we define $f^{\Rightarrow}(\mathcal{F}) \in \mathcal{F}_{LM}(Y)$ by $f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{\leftarrow}(B))$ for $B \in L^Y$, which is called the image of $\mathcal{F}$ under f (see [28]). For each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $\mathcal{G} \in \mathcal{F}_{LM}(Y)$, we define $\mathcal{F} \times \mathcal{G} : L^{X\times Y} \longrightarrow M$ by $(\mathcal{F} \times \mathcal{G})(A) = \bigvee_{B\times C\leq A}(\mathcal{F}(B) \wedge \mathcal{G}(C))$ for each $A \in L^{X\times Y}$. Suppose that $\perp_L$ is prime in L. Then $\mathcal{F} \times \mathcal{G} \in \mathcal{F}_{LM}(X \times Y)$, which is called the product of $\mathcal{F}$ and $\mathcal{G}$. Furthermore, for each $\mathcal{H} \in \mathcal{F}_{LM}(X \times Y)$, it follows that $p_X^{\Rightarrow}(\mathcal{H}) \times p_Y^{\Rightarrow}(\mathcal{H}) \leq \mathcal{H}$, where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ denote the projection mappings, repectively (see [20]).

Definition 2.3 ([23]). A mapping $q : \mathcal{F}_{LM}(X) \longrightarrow L^X$ is called an (L,M)-fuzzy Q-convergence structure on X provided that

(LMQC1) $x_\lambda \leq q(\hat{q}(x_\lambda))$;

(LMQC2) $\mathcal{F} \leqslant \mathcal{G}$ implies $q(\mathcal{F}) \leq q(\mathcal{G})$;

(LMQC3) $x_\lambda \leq q(\mathcal{F})$ and $\lambda \nleq a'$ imply $\mathcal{F}(\underline{a}) = \top_M$.

For an (L,M)-fuzzy Q-convergence structure q on X, the pair (X,q) is called an (L,M)-fuzzy Q-convergence space.

A continuous mapping between (L,M)-fuzzy Q-convergence spaces (X,q_X) and (Y,q_Y) is a mapping $f : X \longrightarrow Y$ such that $x_\lambda \leq q_X(\mathcal{F})$ implies $f(x)_\lambda \leq q_Y(f^{\Rightarrow}(\mathcal{F}))$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $x_\lambda \in J(L^X)$, or equivalently, $q_X(\mathcal{F})(x) \leq q_Y(f^{\Rightarrow}(\mathcal{F}))(f(x))$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $x \in X$.

It is easy to check that (L,M)-fuzzy Q-convergence spaces and their continuous mappings form a category, denoted by (L,M)-**QC**.

Notice that (L,M)-fuzzy Q-convergence structures in Definition 2.3 are exactly stratified (L,M)-fuzzy Q-convergence structures in [23]. In this paper, we will focus on this kind of fuzzy convergence structures and explore its function spaces as well as its subcategories.

Definition 2.4 ([23]). Let $\{(X_j,q_j)\}_{j\in J}$ be a family of (L,M)-fuzzy Q-convergence spaces and $\{p_k : \prod_{j\in J} X_j \longrightarrow (X_k,q_k)\}_{k\in J}$ be the source formed by the family of the projection mappings $\{p_k : \prod_{j\in J} X_j \longrightarrow X_k\}_{k\in J}$. Then the (L,M)-fuzzy Q-convergence structure on $\prod_{j\in J} X_j$ defined by

$$\forall \mathcal{F} \in \mathcal{F}_{LM}(X),\ q_*(\mathcal{F}) = \bigwedge_{j\in J} p_j^{\leftarrow}(q_j(p_j^{\Rightarrow}(\mathcal{F})))$$

is called the product (L,M)-fuzzy Q-convergence structure, which is denoted by $\prod_{j\in J} q_j$. The pair $(\prod_{j\in J} X_j, \prod_{j\in J} q_j)$ is called the product space. For the product of two (L,M)-fuzzy Q-convergence spaces (X,q_X) and (Y,q_Y), we usually write $(X \times Y, q_X \times q_Y)$ or $(X \times Y, q_{X\times Y})$.

Theorem 2.5 ([23]). *(L,M)-**QC** is a topological category.*

For other notions related to category theory, we refer to [1, 24].

3. Function spaces of (L,M)-fuzzy Q-convergence spaces

In this section, we will construct the function space of (L,M)-fuzzy Q-convergence spaces. By means of the constructed function spaces, we will show the Cartesian-closedness of (L,M)-**QC**.

In order to guarantee the existence of the product of (L, M)-fuzzy filters, we assume that $\perp_L$ is prime in this section.

Let (X, q_X) and (Y, q_Y) be (L, M)-fuzzy Q-convergence spaces, $[X, Y]$ be the set of all continuous mappings from (X, q_X) to (Y, q_Y) and $ev : [X, Y] \times X \longrightarrow Y$ be the evaluation mapping. For each $\mathcal{H} \in \mathcal{F}_{LM}([X, Y])$ and $f \in [X, Y]$, we denote two subsets of L as follows:

$$\mathcal{R}_{\mathcal{H}}(f) = \{\nu \in J(L) \mid \forall \mu \leq \nu, \forall a \in L, \mu \nleq a' \text{ implies } \mathcal{H}(\underline{a}) = \top_M\}$$

and

$$\begin{aligned}\mathcal{S}_{\mathcal{H}}(f) = \; & \{\nu \in J(L) \mid \forall \mu \leq \nu, \forall (\mathcal{F}, x) \in \mathcal{F}_{LM}(X) \times X, x_\mu \leq q_X(\mathcal{F}) \\ & \text{implies } f(x)_\mu \leq q_Y(ev^{\Rightarrow}(\mathcal{H} \times \mathcal{F}))\}.\end{aligned}$$

Then we define $q_{[X,Y]} : \mathcal{F}_{LM}([X, Y]) \longrightarrow L^{[X,Y]}$ as follows:

$$q_{[X,Y]}(\mathcal{H})(f) = \bigvee \mathcal{R}_{\mathcal{H}}(f) \wedge \bigvee \mathcal{S}_{\mathcal{H}}(f).$$

In order to show $q_{[X,Y]}$ is an (L, M)-fuzzy Q-convergence structure on $[X, Y]$, the following lemma is necessary.

Lemma 3.1 ([21])**.** *Let $f_\lambda \in J(L^{[X,Y]})$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$. Then $ev^{\Rightarrow}(\hat{q}(f_\lambda) \times \mathcal{F}) \geq f^{\Rightarrow}(\mathcal{F})$.*

Now let us show that $q_{[X,Y]}$ defined above is an (L, M)-fuzzy Q-convergence structure on $[X, Y]$.

Theorem 3.2. *$q_{[X,Y]}$ is an (L, M)-fuzzy Q-convergence structure on $[X, Y]$.*

Proof. It suffices to verify that $q_{[X,Y]}$ satisfies (LMQC1)–(LMQC3). Indeed,

(LMQC1) Take each $f_\lambda \in J(L^{[X,Y]})$. Then

(1) For each $\mu \in J(L)$ with $\mu \leq \lambda$ and $a \in L$ with $\mu \nleq a'$, it follows that $\lambda \nleq a'$, which means $f_\lambda \hat{q} \underline{a}$. This implies $\hat{q}(f_\lambda)(\underline{a}) = \top_M$.

(2) For each $\mu \in J(L)$ with $\mu \leq \lambda$ and $(\mathcal{F}, x) \in \mathcal{F}_{LM}(X) \times X$ with $x_\mu \leq q_X(\mathcal{F})$, it follows from Lemma 3.1 that

$$q_Y(ev^{\Rightarrow}(\hat{q}(f_\lambda) \times \mathcal{F}))(f(x)) \geq q_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \geq q_X(\mathcal{F})(x) \geq \mu.$$

That is , $f(x)_\mu \leq q_Y(ev^{\Rightarrow}(\hat{q}(f_\lambda) \times \mathcal{F}))$.

By (1) and (2), we have $\lambda \in \mathcal{R}_{\hat{q}(f_\lambda)}(f) \cap \mathcal{S}_{\hat{q}(f_\lambda)}(f)$. This implies

$$q_{[X,Y]}(\hat{q}(f_\lambda))(f) = \bigvee \mathcal{R}_{\hat{q}(f_\lambda)}(f) \wedge \bigvee \mathcal{S}_{\hat{q}(f_\lambda)}(f) \geq \lambda,$$

which means $f_\lambda \leq q_{[X,Y]}(\hat{q}(f_\lambda))$.

(LMQC2) Straightforward.

(LMQC3) Take each $\mathcal{H} \in \mathcal{F}_{LM}([X, Y]), a \in L$ and $f_\lambda \in J(L^{[X,Y]})$ such that $f_\lambda \leq q_{[X,Y]}(\mathcal{H})$ and $\lambda \nleq a'$. Then $\lambda \leq q_{[X,Y]}(\mathcal{H})(f) \leq \bigvee \mathcal{R}_{\mathcal{H}}(f)$. Thus, we have

$$\bigvee \mathcal{R}_{\mathcal{H}}(f) = \bigvee \{\nu \in J(L) \mid \forall \mu \leq \nu, \forall b \in L, \mu \nleq b' \text{ implies } \mathcal{H}(\underline{b}) = \top_M\} \nleq a'.$$

Then there exists $\nu_a \in J(L)$ such that $\nu_a \nleq a'$ and for each $\mu \leq \nu_a$ and $b \in L$, $\mu \nleq b'$ implies $\mathcal{H}(\underline{b}) = \top_M$. This implies $\mathcal{H}(\underline{a}) = \top_M$, as desired. □

Theorem 3.3. *The evaluation mapping $ev : ([X, Y] \times X, q_{[X,Y] \times X}) \longrightarrow (Y, q_Y)$ is continuous.*

Proof. Take each $(f,x)_\lambda \in J(L^{[X,Y]\times X})$ and $\mathcal{H} \in \mathcal{F}_{LM}([X,Y]\times X)$ such that $(f,x)_\lambda \prec q_{[X,Y]\times X}(\mathcal{H})$. That is,

$$\lambda \prec q_{[X,Y]\times X}(\mathcal{H})(f,x) = q_{[X,Y]}(p^{\Rightarrow}_{[X,Y]}(\mathcal{H}))(f) \wedge q_X(p^{\Rightarrow}_X(\mathcal{H}))(x).$$

Then $\lambda \prec q_{[X,Y]}(p_{[X,Y]}{}^{\Rightarrow}(\mathcal{H}))(f) \leq \bigvee \mathcal{S}_{p^{\Rightarrow}_{[X,Y]}(\mathcal{H})}(f)$ and $\lambda \leq q_X(p^{\Rightarrow}_X(\mathcal{H}))(x)$. By the definition of $\mathcal{S}_{p^{\Rightarrow}_{[X,Y]}(\mathcal{H})}(f)$, there exists $\nu \in J(L)$ such that $\lambda \leq \nu$ and for each $\mu \leq \nu$ and each $(\mathcal{F},x) \in \mathcal{F}_{LM}(X)\times X$, $x_\mu \leq q_X(\mathcal{F})$ implies $f(x)_\mu \leq q_Y(ev^{\Rightarrow}(p^{\Rightarrow}_{[X,Y]}(\mathcal{H})\times\mathcal{F}))$. Since $\lambda \leq \nu$ and $x_\lambda \leq q_X(p^{\Rightarrow}_X(\mathcal{H}))$, we have

$$ev(f,x)_\lambda = f(x)_\lambda \leq q_Y(ev^{\Rightarrow}(p^{\Rightarrow}_{[X,Y]}(\mathcal{H})\times p^{\Rightarrow}_X(\mathcal{H}))) \leq q_Y(ev^{\Rightarrow}(\mathcal{H})).$$

That is, $(f,x)_\lambda \leq ev^{\Leftarrow}(q_Y(ev^{\Rightarrow}(\mathcal{H})))$. By the arbitrariness of λ, we obtain $q_{[X,Y]\times X}(\mathcal{H})(f,x) \leq q_Y(ev^{\Rightarrow}(\mathcal{H}))(ev(f,x))$ for each $(f,x) \in [X,Y]\times X$. This shows the continuity of ev. □

Let $f : X\times Y \longrightarrow Z$ be a mapping. For each $x \in X$, define a mapping $f_x : Y \longrightarrow Z$, $y \longmapsto f(x,y)$ and a mapping $f^* : X \longmapsto Z^Y$, $x \longmapsto f_x$. Then the mapping $\varphi : Z^{X\times Y} \longrightarrow (Z^Y)^X$, $f \longmapsto f^*$ is called the exponential mapping.

Lemma 3.4. *If $f : (X\times Y, q_{X\times Y}) \longrightarrow (Z,q_Z)$ is continuous, then for each $x \in X$, $f_x : (Y,q_Y) \longrightarrow (Z,q_Z)$ is continuous.*

Proof. For each $x \in X$, define a mapping $\hat{x} : Y \longrightarrow X\times Y, y \longmapsto (x,y)$. Take each $y_\mu \in J(L^Y)$ and $\mathcal{G} \in \mathcal{F}_{LM}(Y)$ such that $y_\mu \leq q_Y(\mathcal{G})$. Then

$$\begin{aligned} q_{X\times Y}(\hat{x}^{\Rightarrow}(\mathcal{G}))(\hat{x}(y)) &= q_{X\times Y}(\hat{x}^{\Rightarrow}(\mathcal{G}))(x,y) \\ &= q_X((p_X\circ\hat{x})^{\Rightarrow}(\mathcal{G}))(x) \wedge q_Y((p_Y\circ\hat{x})^{\Rightarrow}(\mathcal{G}))(y) \\ &= q_X((p_X\circ\hat{x})^{\Rightarrow}(\mathcal{G}))(x) \wedge q_Y(\mathcal{G})(y), \end{aligned}$$

where the third equality holds since $p_Y\circ\hat{x} = id_Y$. Now for each $A \in L^X$ with $x_\mu\hat{q}A$, i.e., $\mu \nleq A'(x)$, it follows from $y_\mu \leq q_Y(\mathcal{G})$ and (LMQC3) that $\mathcal{G}(\underline{A(x)}) = \top_M$. Then

$$(p_X\circ\hat{x})^{\Rightarrow}(\mathcal{G})(A) = \mathcal{G}((p_X\circ\hat{x})^{\Leftarrow}(A)) = \mathcal{G}(\underline{A(x)}) = \top_M,$$

where the second quality holds since

$$(p_X\circ\hat{x})^{\Leftarrow}(A)(y) = A(p_X\circ\hat{x}(y)) = A(p_X(x,y)) = A(x).$$

This shows $(p_X\circ\hat{x})^{\Rightarrow}(\mathcal{G}) \geq \hat{q}(x_\mu)$. Then we have

$$\begin{aligned} q_{X\times Y}(\hat{x}^{\Rightarrow}(\mathcal{G}))(\hat{x}(y)) &= q_X((p_X\circ\hat{x})^{\Rightarrow}(\mathcal{G}))(x) \wedge q_Y(\mathcal{G})(y) \\ &\geq q_X(\hat{q}(x_\mu))(x) \wedge q_Y(\mathcal{G})(y) \\ &\geq \mu\wedge\mu \\ &= \mu, \end{aligned}$$

which means $\hat{x}(y)_\mu \leq q_{X\times Y}(\hat{x}^{\Rightarrow}(\mathcal{G}))$. This proves that $\hat{x} : (Y,q_Y) \longrightarrow (X\times Y, q_{X\times Y})$ is continuous. Considering the continuity of $f : (X\times Y, q_{X\times Y}) \longrightarrow (Z,q_Z)$, we obtain $f_x = f\circ\hat{x}$ (as the composition of two continuous mappings $\hat{x}$ and f) is continuous, as desired. □

Lemma 3.5 ([21]). *Let $\mathcal{F} \in \mathcal{F}_{LM}(X), \mathcal{G} \in \mathcal{F}_{LM}(Y)$ and $f : X\times Y \longrightarrow Z$ be a mapping. Then $ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F})\times\mathcal{G}) = f^{\Rightarrow}(\mathcal{F}\times\mathcal{G})$.*

Theorem 3.6. *If $f : (X \times Y, q_{X\times Y}) \longrightarrow (Z, q_Z)$ is continuous, then $\varphi(f) : (X, q_X) \longrightarrow ([Y,Z], q_{[Y,Z]})$ is continuous.*

Proof. By Lemma 3.4, we know the mapping $\varphi(f)$ is well defined. Take each $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \le q_X(\mathcal{F})$. In order to show $\varphi(f)(x)_\lambda \le q_{[Y,Z]}(\varphi(f)^{\Rightarrow}(\mathcal{F}))$, i.e., $\lambda \le q_{[Y,Z]}(\varphi(f)^{\Rightarrow}(\mathcal{F}))(\varphi(f)(x))$, it suffices to show (1) $\lambda \in \mathcal{R}_{\varphi(f)^{\Rightarrow}(\mathcal{F})}(\varphi(f)(x))$ and (2) $\lambda \in \mathcal{S}_{\varphi(f)^{\Rightarrow}(\mathcal{F})}(\varphi(f)(x))$.

For (1), take each $\mu \in J(L)$ such that $\mu \le \lambda$ and $a \in L$ such that $\mu \nleq a'$. It follows that $x_\mu \le x_\lambda \le q_X(\mathcal{F})$. By (LMQC3), we have $\mathcal{F}(\underline{a}) = \top_M$. This implies

$$\varphi(f)^{\Rightarrow}(\mathcal{F})(\underline{a}) = \mathcal{F}(\varphi(f)^{\leftarrow}(\underline{a})) = \mathcal{F}(\underline{a}) = \top_M.$$

This proves $\lambda \in \mathcal{R}_{\varphi(f)^{\Rightarrow}(\mathcal{F})}(\varphi(f)(x))$.

For (2), take each $\mu \in J(L)$ such that $\mu \le \lambda$ and $(\mathcal{G}, y) \in \mathcal{F}_{LM}(Y) \times Y$. If $y_\mu \le q_Y(\mathcal{G})$, then

$$\begin{aligned}
& q_Z(ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F}) \times \mathcal{G}))(\varphi(f)(x)(y)) \\
= \; & q_Z(ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F}) \times \mathcal{G}))(f_x(y)) \\
= \; & q_Z(f^{\Rightarrow}(\mathcal{F} \times \mathcal{G}))(f(x,y)) \quad \text{(by Lemma 3.5)} \\
\ge \; & q_{X\times Y}(\mathcal{F} \times \mathcal{G})(x,y) \\
\ge \; & q_X(\mathcal{F})(x) \wedge q_Y(\mathcal{G})(y) \\
\ge \; & \lambda \wedge \mu \\
= \; & \mu,
\end{aligned}$$

i.e., $\varphi(f)(x)(y)_\mu \le q_Z(ev^{\Rightarrow}(\varphi(f)^{\Rightarrow}(\mathcal{F}) \times \mathcal{G}))$. Thus, $\lambda \in \mathcal{S}_{\varphi(f)^{\Rightarrow}(\mathcal{F})}(\varphi(f)(x))$. □

By Theorems 3.2, 3.3 and 3.6, we have

Theorem 3.7. *The category (L, M)-**QC** is Cartesian closed.*

Actually, Pang and Zhao [23] showed the Cartesian-closedness of the category of (L, M)-fuzzy Q-convergence spaces (which is called stratified (L, M)-fuzzy Q-convergence space in [23]). However, they failed to construct the corresponding function spaces. In this section, we provide the concrete form of the corresponding function spaces, which gives an answer to the question proposed in [23].

4. (L, M)-fuzzy Kent Q-convergence spaces

In this section, we will generalize the notion of Kent convergence spaces to the (L, M)-fuzzy case and study its relations with (L, M)-fuzzy Q-convergence spaces.

Definition 4.1. An (L, M)-fuzzy Q-convergence structure q on X is called an (L, M)-fuzzy Kent convergence structure if it satisfies

(LMKQC) $\forall \mathcal{F} \in \mathcal{F}_{LM}(X)$, $x_\lambda \in J(L^X)$, $x_\lambda \le q(\mathcal{F})$ implies $x_\lambda \le q(\mathcal{F} \wedge \hat{q}(x_\lambda))$.

For an (L, M)-fuzzy Kent Q-convergence structure q on X, the pair (X, q) is called an (L, M)-fuzzy Kent Q-convergence space.

The full subcategory of (L, M)-**QC**, consisting of (L, M)-fuzzy Kent Q-convergence spaces, is denoted by (L, M)-**KQC**.

Next let us establish the relationship between (L, M)-fuzzy Kent Q-convergence spaces and (L, M)-fuzzy Q-convergence spaces.

Lemma 4.2. *Let (X,q) be an (L,M)-fuzzy Q-convergence space and define $q^r : \mathcal{F}_{LM}(X) \longrightarrow L^X$ by for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$,*

$$q^r(\mathcal{F}) = \bigvee\{x_\lambda \in J(L^X) \mid \exists \mathcal{G} \in \mathcal{F}_{LM}(X) \ s.t.\ x_\lambda \leq q(\mathcal{G}) \text{ and } \mathcal{G} \wedge \hat{q}(x_\lambda) \leq \mathcal{F}\}.$$

Then q^r is an (L,M)-fuzzy Kent Q-convergence structure on X.

Proof. It is enough to show that q^r satisfies (LMQC1)–(LMQC3) and (LMKQC). Indeed, (LMQC1) and (LMQC2) are straightforward.

(LMQC3) Take each $x_\lambda \in J(L^X)$, $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $a \in L$ such that $x_\lambda \leq q^r(\mathcal{F})$ and $\lambda \nleq a'$. This implies

$$q^r(\mathcal{F})(x) = \bigvee\{\lambda \in J(L) \mid \exists\, \mathcal{G} \in \mathcal{F}_{LM}(X), s.t.\ x_\lambda \leq q(\mathcal{G}),\ \mathcal{G} \wedge \hat{q}(x_\lambda) \leq \mathcal{F}\} \nleq a'.$$

Then there exists $\lambda_a \in J(L)$ such that $\lambda_a \nleq a'$ and there exists $\mathcal{G} \in \mathcal{F}_{LM}(X)$ such that $x_{\lambda_a} \leq q(\mathcal{G})$ and $\mathcal{G} \wedge \hat{q}(x_{\lambda_a}) \leq \mathcal{F}$. Since q satisfies (LMQC3), it follows from $x_{\lambda_a} \leq q(\mathcal{G})$ and $\lambda_a \nleq a'$ that $\mathcal{G}(\underline{a}) = \top_M$, and further $\mathcal{F}(\underline{a}) \geq \mathcal{G}(\underline{a}) \wedge \hat{q}(x_{\lambda_a})(\underline{a}) = \top_M$.

(LMKQC) Take each $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \leq q^r(\mathcal{F})$, i.e. $\lambda \leq q^r(\mathcal{F})(x)$. Then for each $\mu \prec \lambda$, there exists $\lambda_1 \in J(L)$ such that $\mu \leq \lambda_1$ and there exists $\mathcal{G} \in \mathcal{F}_{LM}(X)$ such that $x_{\lambda_1} \leq q(\mathcal{G})$ and $\mathcal{G} \wedge \hat{q}(x_{\lambda_1}) \leq \mathcal{F}$. Thus it follows that $x_\mu \leq q(\mathcal{G})$ and $\mathcal{G} \wedge \hat{q}(x_\mu) \leq \mathcal{F} \wedge \hat{q}(x_\mu) \leq \mathcal{F} \wedge \hat{q}(x_\lambda)$. This implies

$$\begin{aligned}\mu &\leq \bigvee\{\nu \in J(L) \mid \exists\, \mathcal{G} \in \mathcal{F}_{LM}(X), s.t.\ x_\nu \leq q(\mathcal{G}),\ \mathcal{G} \wedge \hat{q}(x_\nu) \leq \mathcal{F} \wedge \hat{q}(x_\lambda)\}\\ &= q^r(\mathcal{F} \wedge \hat{q}(x_\lambda))(x).\end{aligned}$$

By the arbitrariness of μ, we obtain $\lambda \leq q^r(\mathcal{F} \wedge \hat{q}(x_\lambda))(x)$. That is to say, $x_\lambda \leq q^r(\mathcal{F} \wedge \hat{q}(x_\lambda))$, as desired. □

Theorem 4.3. *(L,M)-**KQC** is a bireflective subcategory of (L,M)-**QC**.*

Proof. Let (X,q) be an (L,M)-fuzzy Q-convergence space. By Lemma 4.2, we know (X,q^r) is an (L,M)-fuzzy Kent Q-convergence space. Next we claim that $id_X : (X,q) \longrightarrow (X,q^r)$ is the (L,M)-**KQC**-bireflector. To this end, we need to show:

(1) $id_X : (X,q) \longrightarrow (X,q^r)$ is continuous.

(2) For each (L,M)-fuzzy Kent Q-convergence space (Y,q_Y) and each mapping $f : X \longrightarrow Y$, the continuity of $f : (X,q) \longrightarrow (Y,q_Y)$ implies the continuity of $f : (X,q^r) \longrightarrow (Y,q_Y)$.

For (1), it is easy to verify that $q(\mathcal{F}) \leq q^r(\mathcal{F})$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$.

For (2), take each $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \leq q^r(\mathcal{F})$. For each $\mu \in J(L)$ with $\mu \prec \lambda$, it follows that $\mu \prec q^r(\mathcal{F})(x)$. Then there exists $\lambda_1 \in J(L)$ such that $\mu \leq \lambda_1$ and there exists $\mathcal{G} \in \mathcal{F}_{LM}(X)$ such that $x_{\lambda_1} \leq q(\mathcal{G})$ and $\mathcal{G} \wedge \hat{q}(x_{\lambda_1}) \leq \mathcal{F}$. Since $f : (X,q) \longrightarrow (Y,q_Y)$ is continuous, it follows that $f(x)_{\lambda_1} \leq q_Y(f^{\Rightarrow}(\mathcal{G}))$. By (LMKQC), we have

$$f(x)_{\lambda_1} \leq q_Y(\hat{q}(f(x)_{\lambda_1}) \wedge f^{\Rightarrow}(\mathcal{G})) = q_Y(f^{\Rightarrow}(\hat{q}(x_{\lambda_1}) \wedge \mathcal{G})) \leq q_Y(f^{\Rightarrow}(\mathcal{F})).$$

This implies $f(x)_\mu \leq q_Y(f^{\Rightarrow}(\mathcal{F}))$. By the arbitrariness of μ, we obtain $f(x)_\lambda \leq q_Y(f^{\Rightarrow}(\mathcal{F}))$. This proves the continuity of $f : (X,q^r) \longrightarrow (Y,q_Y)$. □

Lemma 4.4. *Let (X,q) be an (L,M)-fuzzy Q-convergence space and define $q^c : \mathcal{F}_{LM}(X) \longrightarrow L^X$ by*

$$\forall \mathcal{F} \in \mathcal{F}_{LM}(X),\ q^c(\mathcal{F}) = \bigvee\{x_\lambda \in J(L^X) \mid \forall \mu \prec \lambda, x_\mu \leq q(\mathcal{F} \wedge \hat{q}(x_\mu))\}.$$

Then q^c is an (L,M)-fuzzy Kent Q-convergence structure on X.

Proof. (LMQC1) and (LMQC2) are easy to be verified and omitted.
(LMQC3) Take each $x_\lambda \in J(L^X), \mathcal{F} \in \mathcal{F}_{LM}(X)$ and $a \in L$ such that $x_\lambda \le q^c(\mathcal{F})$ and $\lambda \nleq a'$. It follows that

$$q^c(\mathcal{F})(x) = \bigvee\{\lambda \in J(L) \mid \forall \mu \prec \lambda,\ x_\mu \le q(\mathcal{F} \wedge \hat{q}(x_\mu))\} \nleq a'.$$

Then there exists $\lambda_a \in J(L)$ such that $\lambda_a \nleq a'$ and for each $\mu \prec \lambda_a, x_\mu \le q(\mathcal{F} \wedge \hat{q}(x_\mu))$. Since $\lambda_a \nleq a'$, there exists $\mu_a \prec \lambda_a$ such that $\mu_a \nleq a'$. This implies $x_{\mu_a} \le q(\mathcal{F} \wedge \hat{q}(x_{\mu_a}))$ and $\mu_a \nleq a'$. Since q satisfies (LMQC3), we have $(\mathcal{F} \wedge \hat{q}(x_{\mu_a}))(\underline{a}) = \top_M$. This implies $\mathcal{F}(\underline{a}) = \top_M$.

(LMKQC) Take each $x_\lambda \in J(L^X)$, $\mathcal{F} \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \le q^c(\mathcal{F})$, i.e., $\lambda \le q^c(\mathcal{F})(x)$. For each $\nu \in J(L)$ with $\nu \prec \lambda$, it follows that

$$\nu \prec q^c(\mathcal{F})(x) = \bigvee\{\lambda_1 \in J(L) \mid \forall \mu \prec \lambda_1,\ x_\mu \le q(\mathcal{F} \wedge \hat{q}(x_\mu))\}.$$

Then there exists $\lambda_1 \in J(L)$ such that $\nu \le \lambda_1$ and for each $\mu \prec \lambda_1$, $x_\mu \le q(\mathcal{F}\wedge\hat{q}(x_\mu))$. Thus, for each $\mu \in J(L)$ with $\mu \prec \nu$, it follows that

$$q(\mathcal{F} \wedge \hat{q}(x_\lambda) \wedge \hat{q}(x_\mu)) = q(\mathcal{F} \wedge \hat{q}(x_\mu)) \ge x_\mu.$$

This implies

$$q^c(\mathcal{F} \wedge \hat{q}(x_\lambda))(x) = \bigvee\{\gamma \in J(L) \mid \forall \mu \prec \gamma,\ x_\mu \le q(\mathcal{F} \wedge \hat{q}(x_\lambda) \wedge \hat{q}(x_\mu))\} \ge \nu.$$

By the arbitrariness of ν, we obtain $\lambda \le q^c(\mathcal{F} \wedge \hat{q}(x_\lambda))(x)$, that is, $x_\lambda \le q^c(\mathcal{F} \wedge \hat{q}(x_\lambda))$, as desired. □

Theorem 4.5. *(L, M)-**KQC** is a bicoreflective subcategory of (L, M)-**QC**.*

Proof. Let (X, q) be an (L, M)-fuzzy Q-convergence space. By Lemma 4.4, we obtain q^c is an (L, M)-fuzzy Kent Q-convergence structure on X. Next we claim that $id_X : (X, q^c) \longrightarrow (X, q)$ is the (L, M)-**KQC**-bicoreflector.

For this it suffices to show:

(1) $id_X : (X, q^c) \longrightarrow (X, q)$ is continuous.

(2) For each (L, M)-fuzzy Kent Q-convergence space (Y, q_Y) and each mapping $f : Y \longrightarrow X$, the continuity of $f : (Y, q_Y) \longrightarrow (X, q)$ implies the continuity of $f : (Y, q_Y) \longrightarrow (X, q^c)$.

For (1), it is easy to show $q^c(\mathcal{F}) \le q(\mathcal{F})$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$.

For (2), take each $\mathcal{G} \in \mathcal{F}_{LM}(Y)$ and $y_\lambda \in J(L^Y)$ such that $y_\lambda \le q_Y(\mathcal{G})$. Then for each $\mu \prec \lambda$, it follows that $y_\mu \le q_Y(\mathcal{G})$. Since (Y, q_Y) satisfies (LMKQC), we have $y_\mu \le q_Y(\mathcal{G} \wedge \hat{q}(y_\mu))$. By the continuity of $f : (Y, q_Y) \longrightarrow (X, q)$, we obtain $f(y)_\mu \le q(f^\Rightarrow(\mathcal{G}) \wedge \hat{q}(f(y)_\mu))$. From the definition of q^c, we get

$$q^c(f^\Rightarrow(\mathcal{G}))(f(y)) = \bigvee\{\nu \in J(L) \mid \forall \mu \prec \nu,\ f(y)_\mu \le q(f^\Rightarrow(\mathcal{G}) \wedge \hat{q}(f(y)_\mu))\} \ge \lambda.$$

This shows $f(y)_\lambda \le q^c(f^\Rightarrow(\mathcal{G}))$, as desired. □

Lemma 4.6 ([24]). *Suppose that **A** is a topological category. If **B** is a bicoreflective (full and isomorphic closed) subcategory of **A** which is closed under formation of finite products in **A**, then **B** is Cartesian closed whenever **A** is Cartesian closed.*

Theorem 4.7. *Suppose that $\bot_L$ is prime in L. Then (L, M)-**KQC** is a Cartesian closed.*

Proof. By Theorem 4.3, we know (L, M)-**KQC** is closed under formation of finite product in (L, M)-**QC**. Further, it is easy to see that (L, M)-**KQC** is a full and isomorphic closed subcategory of (L, M)-**QC**. Then it follows from Theorems 2.5, 3.7 and 4.5, and Lemma 4.6 that (L, M)-**KQC** is Cartesian closed. □

5. (L,M)-fuzzy Q-limit spaces

In this section, we will propose the concept of (L,M)-fuzzy Q-limit spaces, which is a generalization of limit spaces in general topology. Then we will study its relationship with (L,M)-fuzzy Kent Q-convergence spaces from a categorical aspect.

Definition 5.1. An (L,M)-fuzzy Q-convergence structure q on X is called an (L,M)-fuzzy Q-limit structure if it satisfies

(LMLQC) $\forall \mathcal{F},\mathcal{G} \in \mathcal{F}_{LM}(X)$, $q(\mathcal{F}) \wedge q(\mathcal{G}) \leq q(\mathcal{F} \wedge \mathcal{G})$.

For an (L,M)-fuzzy Q-limit structure q on X, the pair (X,q) is called an (L,M)-fuzzy Q-limit space.

The full subcategory of (L,M)-**QC**, consisting of (L,M)-fuzzy Q-limit spaces, is denoted by (L,M)-**LQC**.

Obviously, (LMLQC) implies (LMKQC). That is to say, an (L,M)-fuzzy Q-limit space is an (L,M)-fuzzy Kent Q-convergence space. Thus, (L,M)-**LQC** is a full subcategory of (L,M)-**KQC**.

In order to show the further relationship between (L,M)-fuzzy Kent Q-convergence spaces and (L,M)-fuzzy Q-limit spaces, we first give the following lemma.

Lemma 5.2. *Let (X,q) be an (L,M)-fuzzy Kent Q-convergence space and define $q^l : \mathcal{F}_{LM}(X) \longrightarrow L^X$ by for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$,*

$$q^l(\mathcal{F}) = \bigvee\{x_\lambda \in J(L^X) \mid \exists\, \mathcal{F}_1,\cdots,\mathcal{F}_n \in \mathcal{F}_{LM}(X) \ s.t.\ x_\lambda \leq q(\mathcal{F}_i) \ and\ \mathcal{F} \geq \wedge_{i=1}^{i=n}\mathcal{F}_i\}.$$

Then q^l is an (L,M)-fuzzy Q-limit structure on X.

Proof. (LMQC1) and (LMQC2) are obvious. It suffices to show (LMQC3) and (LMLQC).

(LMQC3) Take each $\mathcal{F} \in \mathcal{F}_{LM}(X)$, $x_\lambda \in J(L^X)$ and $a \in L$ such that $x_\lambda \leq q^l(\mathcal{F})$ and $\lambda \nleq a'$. Then $q^l(\mathcal{F})(x) \nleq a'$. By the definition of $q^l(\mathcal{F})$, there exists $\lambda \in J(L)$ such that $\lambda \nleq a'$ and there exist $\mathcal{F}_1, \mathcal{F}_2,\cdots, \mathcal{F}_n \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \leq q(\mathcal{F}_i)$ and $\mathcal{F} \geq \wedge_{i=1}^{i=n}\mathcal{F}_i$. Since $x_\lambda \leq q(\mathcal{F}_i)$ and $\lambda \nleq a'$, it follows that $\mathcal{F}_i(\underline{a}) = \top_M$ for each $i = 1,\cdots,n$. This implies $\mathcal{F}(\underline{a}) \geq \wedge_{i=1}^{i=n}\mathcal{F}_i(\underline{a}) = \top_M$.

(LMLQC) Take $\mathcal{F}$, $\mathcal{G} \in \mathcal{F}_{LM}(X)$ and $x_\lambda \in J(L^X)$ such that $x_\lambda \leq q^l(\mathcal{F}) \wedge q^l(\mathcal{G})$. For each $\mu \in J(L)$ with $\mu \prec \lambda$, it follows that $\mu \prec q^l(\mathcal{F})(x)$ and $\mu \prec q^l(\mathcal{G})(x)$. Then there exist λ_1, $\lambda_2 \in J(L)$ and $\mathcal{F}_1, \mathcal{F}_2,\cdots,\mathcal{F}_n, \mathcal{G}_1, \mathcal{G}_2,\cdots,\mathcal{G}_m \in \mathcal{F}_{LM}(X)$ such that $\mu \leq \lambda_1$, $\mu \leq \lambda_2$, $x_{\lambda_1} \leq q(\mathcal{F}_i)$, $x_{\lambda_2} \leq q(\mathcal{G}_j)$, $\mathcal{F} \geq \wedge_{i=1}^{i=n}\mathcal{F}_i$ and $\mathcal{G} \geq \wedge_{j=1}^{j=m}\mathcal{G}_j$. Let $\{\mathcal{H}_k \mid k = 1,2,\cdots,m+n\} = \{\mathcal{F}_i \mid i = 1,2,\cdots,n\} \bigcup \{\mathcal{G}_j \mid j = 1,2,\cdots,m\}$. Then $x_\mu \leq q(\mathcal{H}_k)$ and $\mathcal{F} \wedge \mathcal{G} \geq \wedge_{k=1}^{k=m+n}\mathcal{H}_k$. This implies

$$\begin{aligned} & q^l(\mathcal{F} \wedge \mathcal{G})(x) \\ = \ & \bigvee\{\nu \in J(L) \mid \exists\, \mathcal{H}_1,\cdots,\mathcal{H}_p \in \mathcal{F}_{LM}(X) \ s.t.\ x_\nu \leq q(\mathcal{H}_k) \ and\ \mathcal{F} \wedge \mathcal{G} \geq \wedge_{k=1}^{k=p}\mathcal{H}_k\} \\ \geq \ & \mu. \end{aligned}$$

By the arbitrariness of μ, we obtain $\lambda \leq q^l(\mathcal{F} \wedge \mathcal{G})(x)$, that is, $x_\lambda \leq q^l(\mathcal{F} \wedge \mathcal{G})$, as desired. □

Theorem 5.3. (L,M)-**LQC** *is a bireflective subcategory of* (L,M)-**KQC**.

Proof. Let (X,q) be an (L,M)-fuzzy Kent Q-convergence space. By Lemma 5.2, we know q^l is an (L,M)-fuzzy Q-limit structure on X. Next we claim that $id_X : (X,q) \longrightarrow (X,q^l)$ is the (L,M)-**LQC**-bireflector. For this it suffices to verify

(1) $id_X : (X, q) \longrightarrow (X, q^l)$ is continuous.

(2) For each (L, M)-fuzzy Q-limit space (Y, q_Y) and each mapping $f : X \longrightarrow Y$, the continuity of $f : (X, q) \longrightarrow (Y, q_Y)$ implies the continuity of $f : (X, q^l) \longrightarrow (Y, q_Y)$.

For (1), it follows immediately from $q(\mathcal{F}) \leq q^l(\mathcal{F})$ for each $\mathcal{F} \in \mathcal{F}_{LM}(X)$.

For (2), take each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $x_\lambda \in J(L^X)$ such that $x_\lambda \leq q^l(\mathcal{F})$. Then for each $\mu \prec \lambda$, there exists $\lambda_\mu \in J(L)$ such that $\mu \leq \lambda_\mu$ and there exist $\mathcal{F}_1, \cdots, \mathcal{F}_n \in \mathcal{F}_{LM}(X)$ such that $x_{\lambda_\mu} \leq q(\mathcal{F}_i)$ and $\mathcal{F} \geq \wedge_{i=1}^{i=n} \mathcal{F}_i$. Since $f : (X, q) \longrightarrow (Y, q_Y)$ is continuous, it follows that $f(x)_{\lambda_\mu} \leq q_Y(f^{\Rightarrow}(\mathcal{F}_i))$ for each $i = 1, \cdots, n$. Then we have

$$\begin{aligned} f(x)_\mu \leq f(x)_{\lambda_\mu} &\leq \wedge_{i=1}^{i=n} q_Y(f^{\Rightarrow}(\mathcal{F}_i)) \\ &= q_Y(\wedge_{i=1}^{i=n} f^{\Rightarrow}(\mathcal{F}_i)) = q_Y(f^{\Rightarrow}(\wedge_{i=1}^{i=n} \mathcal{F}_i)) \leq q_Y(f^{\Rightarrow}(\mathcal{F})). \end{aligned}$$

By the arbitrariness of μ, we obtain $f(x)_\lambda \leq q_Y(f^{\Rightarrow}(\mathcal{F}))$. This proves $f : (X, q^l) \longrightarrow (Y, q_Y)$ is continuous. □

By Theorems 4.3 and 5.3, we have

Corollary 5.4. *(L, M)-**LQC** is a bireflective subcategory of (L, M)-**QC**.*

Next we discuss the Cartesian-closedness of (L, M)-**LQC**. To this end, the following two lemmas are necessary.

Lemma 5.5 ([21]). *Suppose that $\perp_L$ is prime in L. Let $\mathcal{F}, \mathcal{K} \in \mathcal{F}_{LM}(X)$ and $\mathcal{G} \in \mathcal{F}_{LM}(Y)$. Then*

$$(\mathcal{F} \wedge \mathcal{K}) \times \mathcal{G} = (\mathcal{F} \times \mathcal{G}) \wedge (\mathcal{K} \times \mathcal{G}).$$

Lemma 5.6. *Suppose that $\perp_L$ is prime in L. (L, M)-**LQC** is closed under the formation of power objects in (L, M)-**QC**.*

Proof. For (L, M)-fuzzy Q-limit spaces (X, q_X) and (Y, q_Y). Let $q_{[X,Y]}$ be the corresponding (L, M)-fuzzy Q-convergence structure on $[X, Y]$ in (L, M)-**QC**. That is,

$$q_{[X,Y]}(\mathcal{H})(f) = \bigvee \mathcal{R}_{\mathcal{H}}(f) \wedge \bigvee \mathcal{S}_{\mathcal{H}}(f).$$

It suffices to verify that $q_{[X,Y]}$ satisfies (LMLQC). Take each $f_\lambda \in J(L^{[X,Y]})$, $\mathcal{H}$, $\mathcal{K} \in \mathcal{F}_{LM}([X, Y])$ such that $f_\lambda \leq q_{[X,Y]}(\mathcal{H}) \wedge q_{[X,Y]}(\mathcal{K})$, that is,

$$\lambda \leq q_{[X,Y]}(\mathcal{H})(f) \wedge q_{[X,Y]}(\mathcal{K})(f).$$

For each $\mu \in J(L)$ with $\mu \prec \lambda$, there exist ν_1, ν_2, γ_1, $\gamma_2 \in J(L)$ such that $\mu \leq \nu_1 \wedge \nu_2 \wedge \gamma_1 \wedge \gamma_2$ and

(1) for each $\nu \in J(L)$ with $\nu \leq \nu_i$ $(i = 1, 2)$ and for each $a \in L$ with $\nu \nleq a'$, $\mathcal{H}(\underline{a}) = \top_M$ and $\mathcal{K}(\underline{a}) = \top_M$, which implies $(\mathcal{H} \wedge \mathcal{K})(\underline{a}) = \top_M$.

(2) for each $\nu \in J(L)$ with $\nu \leq \gamma_i$ $(i = 1, 2)$ and for each $(\mathcal{F}, x) \in \mathcal{F}_{LM}(X) \times X$, $x_\nu \leq q_X(\mathcal{F})$ implies

$$\begin{aligned} f(x)_\nu &\leq q_Y(ev^{\Rightarrow}(\mathcal{H} \times \mathcal{F})) \wedge q_Y(ev^{\Rightarrow}(\mathcal{K} \times \mathcal{F})) \\ &= q_Y(ev^{\Rightarrow}((\mathcal{H} \times \mathcal{F}) \wedge (\mathcal{K} \times \mathcal{F}))) \\ &= q_Y(ev^{\Rightarrow}((\mathcal{H} \wedge \mathcal{K}) \times \mathcal{F})). \quad \text{(by Lemma 5.5)} \end{aligned}$$

Then for each $\gamma \in J(L)$ with $\gamma \leq \mu$, it follows that $\gamma \leq \nu_1 \wedge \nu_2 \wedge \gamma_1 \wedge \gamma_2$. Further, for each $a \in L$ with $\gamma \nleq a'$ and for each $(\mathcal{F}, x) \in \mathcal{F}_{LM}(X) \times X$, we have $(\mathcal{H} \wedge \mathcal{K})(\underline{a}) = \top_M$ and $x_\gamma \leq q_X(\mathcal{F})$ implies $f(x)_\gamma \leq q_Y(ev^{\Rightarrow}((\mathcal{H} \wedge \mathcal{K}) \times \mathcal{F}))$. This shows $\mu \in \mathcal{R}_{\mathcal{H}\wedge\mathcal{K}}(f) \cap \mathcal{S}_{\mathcal{H}\wedge\mathcal{K}}(f)$. This means

$$\mu \leq \bigvee \mathcal{R}_{\mathcal{H}\wedge\mathcal{K}}(f) \wedge \bigvee \mathcal{S}_{\mathcal{H}\wedge\mathcal{K}}(f) = q_{[X,Y]}(\mathcal{H} \wedge \mathcal{K})(f).$$

By the arbitrariness of μ, we have $\lambda \leq q_{[X,Y]}(\mathcal{H} \wedge \mathcal{K})(f)$, i.e., $f_\lambda \leq q_{[X,Y]}(\mathcal{H} \wedge \mathcal{K})$, as desired. □

Lemma 5.7 ([24])**.** *Suppose that* **A** *is a topological category. If* **B** *is a bireflective (full and isomorphic closed) subcategory of* **A** *which is closed under formation of power objects in* **A***, then* **B** *is Cartesian closed whenever* **A** *is Cartesian closed.*

Theorem 5.8. *Suppose that $\bot_L$ is prime in L. Then (L, M)-***LQC** *is Cartesian closed.*

Proof. It follows immediately from Theorems 2.5, 5.3 and 5.6, and Lemma 5.7. □

Remark 5.9. It is required that $\bot_L$ should be prime in several conclusions. This requirement seems to be strong. However, the real unit interval $I = [0, 1]$ at least fulfils this requirement. Moreover, I fulfills the assumption of being completely distributive lattice with an order reversing involution.

6. (L, M)-fuzzy pretopological and topological Q-convergence spaces

In this section, we will introduce the concept of (L, M)-fuzzy pretopological Q-convergence spaces and discuss its relations with (L, M)-fuzzy Q-limit spaces and (L, M)-fuzzy topological Q-convergence spaces [23]. For this, we first recall the following notation.

For an (L, M)-fuzzy Q-convergence space (X, q), define $\mathcal{F}^q_{x_\lambda} : L^X \longrightarrow M$ by

$$\mathcal{F}^q_{x_\lambda} = \bigwedge_{x_\lambda \leq q(\mathcal{F})} \mathcal{F}.$$

Then $\mathcal{F}^q_{x_\lambda}$ is an (L, M)-fuzzy filter on X satisfying $\mathcal{F}^q_{x_\lambda} \leq \hat{q}(x_\lambda)$.

Definition 6.1. An (L, M)-fuzzy Q-convergence structure q on X is called pretopological if it satisfies

(LMPQC) $x_\lambda \leq q(\mathcal{F}^q_{x_\lambda})$.

For an (L, M)-fuzzy pretopological Q-convergence structure q on X, the pair (X, q) is called an (L, M)-fuzzy pretopological Q-convergence space.

The full subcategory of (L, M)-**QC**, consisting of (L, M)-fuzzy pretopological Q-convergence spaces, is denoted by (L, M)-**PQC**.

Lemma 6.2. *If (X, q) is an (L, M)-fuzzy pretopological Q-convergence space, then (X, q) is an (L, M)-fuzzy Q-limit space.*

Proof. It suffices to show that (LMPQC) implies (LMLQC). Take each $\mathcal{F}, \mathcal{G} \in \mathcal{F}_{LM}(X)$, $x_\lambda \in J(L^X)$ such that $x_\lambda \leq q(\mathcal{F})$ and $x_\lambda \leq q(\mathcal{G})$. By the definition of $\mathcal{F}^q_{x_\lambda}$, it follows that $\mathcal{F}^q_{x_\lambda} \leq \mathcal{F}$ and $\mathcal{F}^q_{x_\lambda} \leq \mathcal{G}$. This implies $\mathcal{F}^q_{x_\lambda} \leq \mathcal{F} \wedge \mathcal{G}$. Thus, $x_\lambda \leq q(\mathcal{F}^q_{x_\lambda}) \leq q(\mathcal{F} \wedge \mathcal{G})$. By the arbitrariness of x_λ, we obtain $q(\mathcal{F}) \wedge q(\mathcal{G}) \leq q(\mathcal{F} \wedge \mathcal{G})$, as desired. □

Lemma 6.3. *Let (X, q) be an (L, M)-fuzzy Q-limit space and define $q^p : \mathcal{F}_{LM}(X) \longrightarrow L^X$ by*

$$\forall \mathcal{F} \in \mathcal{F}_{LM}(X),\ q^p(\mathcal{F}) = \bigvee \{x_\lambda \in J(L^X) \mid \mathcal{F}^q_{x_\lambda} \leq \mathcal{F}\}.$$

Then q^p is an (L, M)-fuzzy pretopological Q-convergence structure on X.

Proof. (LMQC1) and (LMQC2) are straightforward.

(LMQC3) Take each $\mathcal{F} \in \mathcal{F}_{LM}(X)$, $x_\lambda \in J(L^X)$ and $a \in L$ such that $x_\lambda \leq q^p(\mathcal{F})$ and $\lambda \nleq a'$. It follows that

$$q^p(\mathcal{F})(x) = \bigvee\{\lambda \in J(L) \mid \mathcal{F}^q_{x_\lambda} \leq \mathcal{F}\} \nleq a'.$$

Then there exists $\lambda_a \in J(L)$ such that $\mathcal{F}^q_{x_{\lambda_a}} \leq \mathcal{F}$ and $\lambda_a \nleq a'$. This implies

$$\mathcal{F}(\underline{a}) \geq \mathcal{F}^q_{x_{\lambda_a}}(\underline{a}) = \bigwedge_{x_{\lambda_a} \leq q(\mathcal{F})} \mathcal{F}(\underline{a}) = \top_M.$$

(LMPQC) For each $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$ with $x_\lambda \leq q^p(\mathcal{F})$, take each $\mu \in J(L)$ such that $\mu \prec \lambda$. It follows that

$$\mu \prec \lambda \leq q^p(\mathcal{F})(x) = \bigvee\{\nu \in J(L) \mid \mathcal{F}^q_{x_\nu} \leq \mathcal{F}\}.$$

Then there exists $\nu \in J(L)$ such that $\mu \leq \nu$ and $\mathcal{F}^q_{x_\nu} \leq \mathcal{F}$. This implies $\mathcal{F}^q_{x_\mu} \leq \mathcal{F}^q_{x_\nu} \leq \mathcal{F}$. So we have $\mathcal{F}^q_{x_\mu} \leq \bigwedge_{x_\lambda \leq q^p(\mathcal{F})} \mathcal{F} = \mathcal{F}^{q^p}_{x_\lambda}$. Then it follows that

$$\mu \leq \bigvee\{\gamma \in J(L) \mid \mathcal{F}^q_{x_\gamma} \leq \mathcal{F}^{q^p}_{x_\lambda}\} = q^p(\mathcal{F}^{q^p}_{x_\lambda})(x).$$

By the arbitrariness of μ, we get $\lambda \leq q^p(\mathcal{F}^{q^p}_{x_\lambda})(x)$, i.e., $x_\lambda \leq q^p(\mathcal{F}^{q^p}_{x_\lambda})$, as desired. □

Theorem 6.4. *(L, M)-**PQC** is a bireflective subcategory of (L, M)-**LQC**.*

Proof. Let (X, q) be an (L, M)-fuzzy Q-limit convergence space. By Lemma 6.3, we know q^p is an (L, M)-fuzzy pretopological Q-convergence structure on X. Next we claim that $id_X : (X, q) \longrightarrow (X, q^p)$ is the (L, M)-**PQC**-bireflector. For this, it suffices to verify

(1) $id_X : (X, q) \longrightarrow (X, q^p)$ is continuous.

(2) For each (L, M)-fuzzy pretopological Q-convergence space (Y, q_Y) and each mapping $f : X \longrightarrow Y$, the continuity of $f : (X, q) \longrightarrow (Y, q_Y)$ implies the continuity of $f : (X, q^p) \longrightarrow (Y, q_Y)$.

For (1), take each $x_\lambda \in J(L^X)$ and $\mathcal{F} \in \mathcal{F}_{LM}(X)$ such that $x_\lambda \leq q(\mathcal{F})$. Then it follows that $\mathcal{F}^q_{x_\lambda} \leq \mathcal{F}$, which means $x_\lambda \leq q^p(\mathcal{F})$. This shows $q(\mathcal{F}) \leq q^p(\mathcal{F})$.

For (2), take each $\mathcal{F} \in \mathcal{F}_{LM}(X)$ and $x_\lambda \in J(L^X)$ such that $x_\lambda \leq q^p(\mathcal{F})$. Then for each $\mu \in J(L)$ with $\mu \prec \lambda$, it follows that

$$\mu \prec q^p(\mathcal{F})(x) = \bigvee\{\nu \in J(L) \mid \mathcal{F}^q_{x_\nu} \leq \mathcal{F}\}.$$

This means there exists $\nu \in J(L)$ such that $\mathcal{F}^q_{x_\nu} \leq \mathcal{F}$ and $\mu \leq \nu$. Then it follows that

$$\begin{aligned} \mathcal{F}^{q_Y}_{f(x)_\nu} = \bigwedge_{f(x)_\nu \leq q_Y(\mathcal{H})} \mathcal{H} &\leq \bigwedge_{f(x)_\nu \leq q_Y(f^\Rightarrow(\mathcal{G}))} f^\Rightarrow(\mathcal{G}) \\ &\leq f^\Rightarrow\left(\bigwedge_{x_\nu \leq q(\mathcal{G})} \mathcal{G}\right) = f^\Rightarrow(\mathcal{F}^q_{x_\nu}) \leq f^\Rightarrow(\mathcal{F}), \end{aligned}$$

which implies $f(x)_\mu \leq f(x)_\nu \leq q_Y(\mathcal{F}^{q_Y}_{f(x)_\nu}) \leq q_Y(f^\Rightarrow(\mathcal{F}))$. By the arbitrariness of μ, we obtain $f(x)_\lambda \leq q_Y(f^\Rightarrow(\mathcal{F}))$. This proves $f : (X, q^p) \longrightarrow (Y, q_Y)$ is continuous. □

Next let us recall the definition of (L, M)-fuzzy topological Q-convergence structures in [21].

Definition 6.5 ([21])**.** An (L, M)-fuzzy pretopological Q-convergence structure q on X is called topological if it satisfies

(LMTQC) $\mathcal{F}^q_{x_\lambda}(A) = \bigvee_{x_\lambda \hat{q} B \leqslant A} \bigwedge_{y_\mu \hat{q} B} \mathcal{F}^q_{y_\mu}(B)$.

For an (L,M)-fuzzy topological Q-convergence structure q on X, the pair (X,q) is called an (L,M)-fuzzy topological Q-convergence space.

The full subcategory of (L,M)-**PQC**, consisting of (L,M)-fuzzy topological Q-convergence spaces, is denoted by (L,M)-**TQC**.

Actually, combining Theorem 3.7 in [19] and Theorem 5.3 in [23], the authors had shown the relationship between (L,M)-fuzzy pretopological Q-convergence structures and (L,M)-fuzzy topological Q-convergence structures without the stratification condition (LMQC3). However, most of the proofs can be adopted. So we only present the final result and omit the proofs.

Theorem 6.6. *(L,M)-**TQC** is a bireflective subcategory of (L,M)-**PQC**.*

The following graph collects the main results of the previous sections:

$$
\begin{array}{ccccll}
 & & & (L,M)-\mathbf{QC} & & \text{(Cartesian closed)} \\
 & & & {\scriptstyle biref}\uparrow{\scriptstyle bicoref} & & \\
 & & & (L,M)-\mathbf{KQC} & & \text{(Cartesian closed)} \\
 & & & {\scriptstyle biref}\uparrow & & \\
(L,M)-\mathbf{TQC} & \xrightarrow{biref} & (L,M)-\mathbf{PQC} & \xrightarrow{biref} (L,M)-\mathbf{LQC} & & \text{(Cartesian closed)}
\end{array}
$$

7. Conclusions

In this paper, we mainly constructed the function spaces of (L,M)-fuzzy Q-convergence spaces, which ensured the Cartesian-closedness of the category (L,M)-**QC** of (L,M)-fuzzy Q-convergence spaces. This gave an answer to the problem proposed by Pang and Zhao in [23]. Furthermore, we made some investigations on subcategories of (L,M)-**QC**. In the future, we will consider further categorical properties of (L,M)-fuzzy Q-convergence spaces, such as Extensionality and Productivity of Quotient mappings.

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

Author contributions

Bin Pang contributed to the conception the work and final approval of the version. Lin Zhang contributed to drafting the work and revising it critically for important intellectual content.

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