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Direct evaluation of J-Integrals by Gauss-Legendre quadrature in the Dual Boundary Element Method

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Abstract
The Dual Boundary Element Method (DBEM) has successfully been applied to fracture mechanics problems for the last three decades. Restricting the approach to Linear Elastic Fracture Mechanics (LEFM), a fundamental task in a numerical analysis is the evaluation of stress intensity factors, which completely characterizes the mechanical behavior at the crack tip. The use of the J-Integral procedure for this purpose is particularly interesting in the DBEM context, once the strain and stress fields at any internal point can be evaluated accurately by specific boundary integral equations. Such a methodology requires the solution of an integral around an arbitrary path enclosing the crack tip and historically it has been done by some discretization procedure (in constant or higher-order elements) of the referred path. In this paper, however, it is shown, by an extensive number of tests, that no discretization is actually necessary for an accurate evaluation of J-Integrals in the DBEM. A direct integration by Gauss-Legendre quadrature is sufficient, which represents less time consuming in a numerical analysis, mainly for problems involving multiple cracks propagation. Moreover, an optimal integration points quantity is proposed.

Keywords: Linear Elastic Fracture Mechanics, Dual Boundary Element Method, Stress Intensity Factors, J-Integral
1 Introduction

Crack propagation is one of the most promising applications of the Boundary Element Method (BEM), according to Aliabadi [1], due to its high accuracy in analysis that involves stress concentration. Another advantage, according to Portela et al. [2], is the ability of the mesh to follow the crack growth, without the need to re-mesh after each increment. In analysis of general cracks, a specific formulation of BEM can be applied, known as Dual Boundary Element Method (DBEM).

Restricting the analysis to Linear Elastic Fracture Mechanics (LEFM), it is necessary to evaluate the stress intensity factors (SIFs) at the crack tip. Several numerical methods can be used to evaluate SIFs, such as displacement correlation methods used by Zhu and Smith [3] and Fu et al. [4], crack closure integral applied by Carter et al. [5] and Maïtï et al. [6] and displacement fitting used by Cordeiro et al. [7]. However, in the DBEM context, J-Integral is the most effective procedure to determine stress intensity factors, since the elastic field can be accurately determined at any internal point. Furthermore, the performance of DBEM is excellent when J-Integral is used, as reported by Portela et al. [8]. The J-Integral method was proposed by Rice [9], as a path-independent integral. In order to calculate SIFs at the crack tip, the J-Integral is evaluated on an arbitrary contour, enclosing the crack tip. There are several methods for evaluating this integral, such as trapezoidal rule (Portela et al. [8], Santana and Portela [10]), Simpson’s rule (Sato et al. [11]) and polynomial interpolation over elements of different orders (Prasad et al. [12], Gonzalez et al. [13], Andrade and Leonel [14]). All these methods, however, consider a discretization of the integration path into segments or elements. Although the evaluation of a single integral based in such a discretization is not usually a computational processing time problem, it can become relevant for an analysis of multiple cracks propagation, especially in the DBEM, in which the evaluation of a mechanical field in an internal point requires the solution of integrals along all the problem’s boundary. Based on this, the application of direct integration of J-Integral by Gauss-Legendre quadrature is proposed. In order to prove the reliability of this approach, eight tests are performed, where J-Integral is evaluated by three different procedures: trapezoidal rule, quadratic polynomial interpolation over elements and the proposed direct integration. Moreover, based on these tests, an optimal quantity of integration points is proposed for this direct evaluation.

2 Dual Boundary Element Method

The Dual Boundary Element Method (DBEM), proposed by Hong and Chen [15] and systematized for application to LEFM problems by Portela et al. [2, 8, 16], is used in general crack analysis. DBEM consists of applying the displacement boundary equation to one of the crack faces, while the traction boundary equation is applied to the opposite face. The remaining boundary is discretized with the displacement equation. The modelling strategy is shown in Fig. 1.
The equations used in DBEM, considering smooth boundary points $\xi$, are:

\[
\frac{1}{2} u_i (\xi) + \int_\Gamma t_{ij}^* (\xi, x) u_j (x) d\Gamma = \int_\Gamma u_{ij}^* (\xi, x) t_j (x) d\Gamma, \quad (1)
\]

\[
\frac{1}{2} u_i (\xi^*) + \frac{1}{2} u_i (\xi) + \int_\Gamma t_{ij}^* (\xi, x) u_j (x) d\Gamma = \int_\Gamma u_{ij}^* (\xi, x) t_j (x) d\Gamma, \quad (2)
\]

\[
n_j (\xi) \int_\Gamma u_{ijk}^* (\xi, x) t_k (x) d\Gamma - \frac{1}{2} t_i (\xi) + \frac{1}{2} t_i (\xi^*) = n_j (\xi) \int_\Gamma t_{ijk}^* (\xi, x) u_k (x) d\Gamma, \quad (3)
\]

where $\xi$ is the source point, $\xi^*$ is the source point coincident to $\xi$ located on the opposite face of the crack, $x$ is the field point, $u_i$ refers to displacement components, $t_i$ refers to traction components, $\Gamma$ is the boundary of the problem, $i$, $j$ and $k$ are cartesian components, which takes values equal to 1 or 2 in two-dimensional cases and $n_i$ are components of the normal vector. The kernels $u_{ij}^*$, $t_{ij}^*$, $u_{ijk}^*$ and $t_{ijk}^*$ are terms of Kelvin’s fundamental solution, which represent the solution associated to unitary forces applied at $\xi$ in a infinite domain. Explicit expressions for these terms can be found in many Boundary Element Method textbooks, e.g., Brebbia et al. [17], Brebbia [18], Aliabadi [19] and Katsikadelis [20]. Equation 1 is the displacement equation, applied to the general boundary of the problem, while eq. 2 is the specific displacement equation for points belonging to the crack and eq. 3 is the specific traction equation, applied to crack source points. The solution of these equations, after the discretization of the boundary into elements, gives the boundary results for displacement and traction. Based on these results, the variables at internal points, necessary for J-Integral evaluation, can be obtained through the following equations:

\[
u_i (\xi) = \int_\Gamma u_{ij}^* (\xi, x) t_j (x) d\Gamma - \int_\Gamma t_{ij}^* (\xi, x) u_j (x) d\Gamma, \quad (4)
\]

\[
\sigma_{ij} (\xi) = \int_\Gamma u_{ijk}^* (\xi, x) t_k (x) d\Gamma - \int_\Gamma t_{ijk}^* (\xi, x) u_k (x) d\Gamma, \quad (5)
\]

\[
u_{i,k} (\xi) = \int_\Gamma u_{ij,k}^* (\xi, x) t_j (x) d\Gamma - \int_\Gamma t_{ij,k}^* (\xi, x) u_j (x) d\Gamma, \quad (6)
\]
where \( u_{i,k} \) and \( \sigma_{ij} \) are the components of the displacement gradient and stress at internal points and \( \dot{u}_{ij,k}^* \) and \( t_{ij,k}^* \) refers to derivatives, relative to the source point, of the Kelvin’s displacement and traction, respectively, \( u_{ij}^* \) and \( t_{ij}^* \). The strain components are obtained from stress using Hooke’s Law.

3 Stress Intensity Factors Evaluation by J-Integral Approach

J-Integrals correspond to the energy release rate as the crack advances in the direction of each axis of the Cartesian system with origin defined at its tip. As J-Integrals are path independent, a circular contour centered at the crack tip can be chosen to evaluate the integral. The Cartesian system adopted and an example of circular contour used to evaluate J-Integrals are illustrated in Fig. 2.

![Cartesian system and example of contour](image)

Fig. 2 Cartesian system and example of contour

The J-Integrals are obtained by the formulation:

\[
J_k = \int_{\Gamma} (W n_k - t_i u_{i,k}) \, d\Gamma, \tag{7}
\]

where

\[
t_i = \sigma_{ij} n_j \tag{8}
\]

and

\[
W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}. \tag{9}
\]

Terms \( J_k \) are the J-Integrals, \( k \) are the Cartesian components that, in the two-dimensional case assume values equal to 1 or 2, \( W \) is the strain energy density (considering linear elastic media), \( n_k \) are the normal vector components, \( t_i \) refers to the traction vector components, \( u_{i,j} \) are the displacement gradient components and \( \varepsilon_{ij} \) refers to strain components.

In order to evaluate stress intensity factors, the J-Integral related to a crack advance in \( x_1 \) direction, \( J_1 \), must be decomposed into a sum of its symmetrical and asymmetrical parts, i.e.:

\[
J_1 = J_1^I + J_1^{II}. \tag{10}
\]
The elastic stress and displacement fields are also decomposed as:

\[
\begin{align*}
\sigma_{11} &= \frac{1}{2} \left( \sigma_{11} + \sigma'_{11} \right), \\
\sigma_{22} &= \frac{1}{2} \left( \sigma_{22} + \sigma'_{22} \right), \\
\sigma_{12} &= \frac{1}{2} \left( \sigma_{12} + \sigma'_{12} \right), \\
\sigma_{11} &= \frac{1}{2} \left( \sigma_{11} - \sigma'_{11} \right), \\
\sigma_{22} &= \frac{1}{2} \left( \sigma_{22} - \sigma'_{22} \right), \\
\sigma_{12} &= \frac{1}{2} \left( \sigma_{12} - \sigma'_{12} \right),
\end{align*}
\]

(11a,b)

\[
\begin{align*}
u_1' &= \frac{1}{2} \left( u_1 + u'_1 \right), \\
u_2' &= \frac{1}{2} \left( u_2 + u'_2 \right),
\end{align*}
\]

(12a,b)

where \( \sigma' \) and \( u' \) are the symmetric parts of the elastic fields and \( \sigma'' \) and \( u'' \) are the asymmetric parts, \( \sigma_{ij} \) and \( u_i \) refer to stress and displacement components at the analyzed point and \( \sigma'_{ij} \) and \( u'_i \) are the components of stress and displacements at a point symmetric to the one analyzed, in relation to \( x_1 \) axis. Thus, each parcel in eq. 10 can be written as:

\[
J_{M} = \int_{\Gamma} (W_{M} n_{1} - t_{k} u_{M}) \, d\Gamma,
\]

(13)

where \( M = I, II \). From eqs. 12, it can be noted that the so called symmetric and asymmetric parts refer also to the distinct fracture modes I and II. Thus, solving these integrals, mode I and II stress intensity factors can be calculated by:

\[
J_{I} = \frac{K_{I}^{2}}{E'}, \quad J_{II} = \frac{K_{II}^{2}}{E'},
\]

(14a,b)

where \( K_{I} \) and \( K_{II} \) are the mode I and mode II stress intensity factors, respectively, \( E' = E \) for plane stress while \( E' = E/(1 - \nu^2) \) for plane strain, \( E \) is the elasticity modulus and \( \nu \) is the Poisson’s ratio.

Three different methods to evaluate eq. 13 are compared, in this paper. The first one is the trapezoidal rule, which consists of distributing points equally spaced around the crack tip, discretizing the circular contour into segments. Once the discretization is done, the integration proceeds as follows:

\[
J_{M} = \frac{1}{2} \sum_{j=1}^{n_p-1} \left[ (W_{M} n_{1} - t_{k} u_{M})_{j} + (W_{M} n_{1} - t_{k} u_{M})_{j+1} \right] r \theta,
\]

(15)

where index \( j \) indicates a point of the contour, \( n_p \) is the total number of points, \( r \) is the radius of the circular path centered in the crack tip and \( \theta \) is the angle between 2 adjacent points (\( \theta = 2\pi/n_p \)).

In the second method, the circular contour is divided into quadratic elements, which are individually integrated by Gauss-Legendre quadrature. The necessary variables are calculated based on the nodal values and the interpolation functions, so that:

\[
u_{k,1} = N_{\alpha}(\eta)(u_{k,1})\alpha, \quad t_{k} = N_{\alpha}(\eta)(t_{k})\alpha, \quad W_{M} = N_{\alpha}(\eta)(W_{M})\alpha,
\]

(16a,b,c)

where \( \alpha \) indicates the node position inside an element (1, 2 or 3), \( N_{\alpha} \) are the interpolation functions and \( \eta \) is a parametric coordinate that takes values between \(-1, \) at
node 1, and +1, at node 3, inside each element. Expressions for these interpolation functions are:

\[ N_1(\eta) = \frac{1}{2} \eta(\eta - 1), \quad N_2(\eta) = 1 - \eta^2, \quad N_3(\eta) = \frac{1}{2} \eta(\eta + 1). \]  

(17a,b,c)

In eq. 16, summation is implied by the repetition of index \( \alpha \). After defining the necessary variables, the integration is performed through two summations as follows:

\[ J_M^1 = \sum_{i=1}^{n_e} \left\{ \sum_{j=1}^{n_ip} \left[ W_M n_1^M \eta_j - t_k^M u_{k,1}^M \right] J(\eta_j) \omega_j \right\}. \]

(18)

In this equation, \( n_e \) is the number of elements, \( n_ip \) is the number of integration points used for Gauss-Legendre quadrature, \( J(\eta_j) \) is the Jacobian, at point \( \eta_j \), for coordinates transformation inside an element, and \( \omega_j \) is the weight associated with the integration point.

The third method refers to the direct evaluation of J-Integral by Gauss-Legendre quadrature. The complete circular contour is directly divided into Gaussian integration points and the variables are calculated only at these points. After that, J-Integrals can be obtained by:

\[ J_M^1 = \sum_{j=1}^{n_ip} \left[ W_j^M (n_1^M)_{\eta_j} - (t_k^M)_{\eta_j} (n_k^M)_{\eta_j} \right] r \omega_j \]

(19)

After determining \( J_M^I \) and \( J_M^{II} \) through eqs. 15, 18 or 19, mode I and II stress intensity factors can be obtained by eq. 14.

4 Numerical Results

In order to compare the three different integration methodologies, eight tests have been performed, using different crack geometries, under different loading conditions, which cause mode I, mode II and mixed-mode fracture. Reference results were taken from various literature sources. For the material properties, an elasticity modulus, \( E = 100 \), and a Poisson’s ratio, \( \nu = 0.25 \), have been considered. The models are discretized into quadratic elements, with corner elements being semi-discontinuous, while crack elements are discontinuous. The circular path for J-Integral evaluation is defined beginning and finishing at the second node before the crack tip, i.e., at the middle node of the last element. The results are normalized, according to the equation:

\[ K_{\text{normalized}} = \frac{K}{\sigma \sqrt{\pi a}}. \]

(20)

In this equation, \( K \) is the stress intensity factor, \( \sigma \) is the remote applied stress and \( a \) is the characteristic crack length. The results are presented in comparative tables including trapezoidal rule, quadratic elements and direct integration, each method
with 6 and 20 elements (or integration points in the case of direct integration). When the path is discretized into quadratic elements, each element is integrated by Gauss-Legendre quadrature using 4 integration points. A second table is included for each test, containing the results obtained with direct integration only, using 4, 6, 8, 10, 12 and 20 points. Different crack sizes are analyzed in each example and the maximum error between them for each approach is also reported in the tables.

4.1 Test 1: Square Plate with an edge crack

The first test refers to a square plate with an edge crack, under tensile loading. The plate boundary is divided into 64 elements, being 8 elements in the width \( w \), 16 elements in the height \( h \) and 8 elements per crack face, which are graded towards the crack tip with ratios 0.2, 0.2, 0.15, 0.15, 0.1, 0.1, 0.05 and 0.05. The dimensions of the plate, as well as the boundary elements mesh and the boundary conditions are shown in Fig. 3.

![Fig. 3 Test 1 - Dimensions, mesh and boundary conditions.](image)

The applied load causes pure mode I at the crack tip. Mode I SIF is calculated for 5 different ratios \( a/w \) and the results are compared with reference values, obtained by Civelek and Erdogan [21]. Table 1 contains the results obtained with all the integration methodologies, using 6 and 20 elements or points.
Table 1 Test 1 - Normalized $K_I$ results for different methods of J-Integral evaluation.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>1.487</td>
<td>1.488</td>
<td>1.487</td>
</tr>
<tr>
<td>0.3</td>
<td>1.844</td>
<td>1.847</td>
<td>1.847</td>
</tr>
<tr>
<td>0.4</td>
<td>2.315</td>
<td>2.321</td>
<td>2.320</td>
</tr>
<tr>
<td>0.5</td>
<td>2.990</td>
<td>2.998</td>
<td>2.997</td>
</tr>
<tr>
<td>0.6</td>
<td>4.107</td>
<td>4.119</td>
<td>4.116</td>
</tr>
<tr>
<td></td>
<td>Max. Error %</td>
<td>1.07</td>
<td>0.81</td>
</tr>
</tbody>
</table>

The results obtained with direct integration considering different number of integration points are shown in table 2.

Table 2 Test 1 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>0.2</td>
<td>1.657</td>
</tr>
<tr>
<td>0.3</td>
<td>2.060</td>
</tr>
<tr>
<td>0.4</td>
<td>2.589</td>
</tr>
<tr>
<td>0.5</td>
<td>3.347</td>
</tr>
<tr>
<td>0.6</td>
<td>4.599</td>
</tr>
<tr>
<td></td>
<td>Max. Error %</td>
</tr>
</tbody>
</table>

4.2 Test 2: Rectangular Plate with an Internal Slant Crack

In the second test, a rectangular plate with an internal slant crack is analyzed. The crack is inclined at 45° and, because of that inclination, the tensile load applied creates mixed-mode fracture at the crack tip. The plate boundary is divided into 36 elements, being 4 elements in the width $2w$, 8 elements in the height $2h$ and 6 elements per crack face, which are graded from the middle of the crack towards the tip with ratios 0.25, 0.15 and 0.1. Figure 4 shows the dimensions, mesh and the boundary conditions of the plate.
As the applied load causes mixed-mode fracture, the results obtained for 5 different ratios $a/w$ are presented for both $K_I$ and $K_{II}$. The results are also divided per crack tip. The reference values are presented in Murakami [22]. Tables 3, 4, 5 and 6 contain the results obtained for $K_I$ and $K_{II}$ on crack tip A.

**Table 3** Test 2 - Normalized $K_I$ results for different methods of J-Integral evaluation on tip A.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem. 20 elem.</td>
<td>6 elem. 20 elem.</td>
<td>6 pts. 20 pts.</td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.519 0.520</td>
<td>0.520 0.520</td>
<td>0.523 0.520</td>
<td>0.518</td>
</tr>
<tr>
<td>0.3</td>
<td>0.542 0.543</td>
<td>0.543 0.543</td>
<td>0.546 0.543</td>
<td>0.541</td>
</tr>
<tr>
<td>0.4</td>
<td>0.574 0.575</td>
<td>0.574 0.575</td>
<td>0.578 0.575</td>
<td>0.572</td>
</tr>
<tr>
<td>0.5</td>
<td>0.616 0.616</td>
<td>0.615 0.616</td>
<td>0.619 0.616</td>
<td>0.612</td>
</tr>
<tr>
<td>0.6</td>
<td>0.666 0.666</td>
<td>0.666 0.666</td>
<td>0.670 0.666</td>
<td>0.661</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>0.82 0.79</td>
<td>0.69 0.76</td>
<td>1.33 0.76</td>
<td>-</td>
</tr>
</tbody>
</table>
Table 4 Test 2 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral on tip A.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.580 0.523 0.520 0.520 0.520 0.520 0.520 0.518</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.605 0.546 0.543 0.543 0.543 0.543 0.543 0.541</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.640 0.578 0.575 0.575 0.575 0.575 0.575 0.572</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.686 0.619 0.616 0.616 0.616 0.616 0.616 0.612</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.742 0.670 0.666 0.666 0.666 0.666 0.666 0.661</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Error %</td>
<td>12.21 1.33 0.77 0.76 0.76 0.76 -</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5 Test 2 - Normalized $K_{II}$ results for different methods of J-Integral evaluation on tip A.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.436 0.492 0.514 0.513 0.510 0.513 0.507</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.443 0.500 0.522 0.522 0.519 0.522 0.516</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.454 0.512 0.535 0.534 0.531 0.534 0.529</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.468 0.528 0.552 0.551 0.548 0.551 0.546</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.486 0.549 0.573 0.572 0.569 0.572 0.567</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Error %</td>
<td>14.32 3.21 1.32 1.19 0.61 1.19 -</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6 Test 2 - Normalized $K_{II}$ results for different number of integration points in direct evaluation of J-Integral on tip A.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.456 0.510 0.513 0.513 0.513 0.513 0.507</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.464 0.519 0.522 0.522 0.522 0.522 0.516</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.476 0.531 0.534 0.534 0.534 0.534 0.529</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.492 0.548 0.551 0.551 0.551 0.551 0.546</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.512 0.569 0.572 0.572 0.572 0.572 0.567</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max. Error %</td>
<td>10.08 0.61 1.18 1.19 1.19 1.19 -</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The results obtained on crack tip B are presented in tables 7, 8, 9 and 10.
### Table 7: Test 2 - Normalized $K_I$ results for different methods of J-Integral evaluation on tip B.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>0.521</td>
<td>0.522</td>
<td>0.522</td>
</tr>
<tr>
<td>0.3</td>
<td>0.545</td>
<td>0.545</td>
<td>0.545</td>
</tr>
<tr>
<td>0.4</td>
<td>0.577</td>
<td>0.578</td>
<td>0.577</td>
</tr>
<tr>
<td>0.5</td>
<td>0.619</td>
<td>0.619</td>
<td>0.618</td>
</tr>
<tr>
<td>0.6</td>
<td>0.669</td>
<td>0.669</td>
<td>0.668</td>
</tr>
</tbody>
</table>

Max. Error % 1.26 1.22 1.11 1.19 1.75 1.18 -

### Table 8: Test 2 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral on tip B.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.2</td>
<td>0.582</td>
<td>0.525</td>
</tr>
<tr>
<td>0.3</td>
<td>0.608</td>
<td>0.548</td>
</tr>
<tr>
<td>0.4</td>
<td>0.643</td>
<td>0.581</td>
</tr>
<tr>
<td>0.5</td>
<td>0.689</td>
<td>0.622</td>
</tr>
<tr>
<td>0.6</td>
<td>0.745</td>
<td>0.673</td>
</tr>
</tbody>
</table>

Max. Error % 12.68 1.75 1.19 1.18 1.18 1.18 -

### Table 9: Test 2 - Normalized $K_{II}$ results for different methods of J-Integral evaluation on tip B.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>0.437</td>
<td>0.493</td>
<td>0.515</td>
</tr>
<tr>
<td>0.3</td>
<td>0.444</td>
<td>0.501</td>
<td>0.524</td>
</tr>
<tr>
<td>0.4</td>
<td>0.455</td>
<td>0.514</td>
<td>0.536</td>
</tr>
<tr>
<td>0.5</td>
<td>0.469</td>
<td>0.530</td>
<td>0.553</td>
</tr>
<tr>
<td>0.6</td>
<td>0.487</td>
<td>0.550</td>
<td>0.574</td>
</tr>
</tbody>
</table>

Max. Error % 14.19 3.04 1.49 1.36 0.79 1.36 -
Table 10: Test 2 - Normalized $K_{II}$ results for different number of integration points in direct evaluation of J-Integral on tip B.

<table>
<thead>
<tr>
<th>$a/w$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.457</td>
<td>0.511</td>
<td>0.514</td>
<td>0.514</td>
<td>0.514</td>
<td>0.514</td>
<td>0.507</td>
</tr>
<tr>
<td>0.3</td>
<td>0.465</td>
<td>0.520</td>
<td>0.523</td>
<td>0.523</td>
<td>0.523</td>
<td>0.523</td>
<td>0.516</td>
</tr>
<tr>
<td>0.4</td>
<td>0.477</td>
<td>0.533</td>
<td>0.536</td>
<td>0.536</td>
<td>0.536</td>
<td>0.536</td>
<td>0.529</td>
</tr>
<tr>
<td>0.5</td>
<td>0.493</td>
<td>0.549</td>
<td>0.552</td>
<td>0.552</td>
<td>0.552</td>
<td>0.552</td>
<td>0.546</td>
</tr>
<tr>
<td>0.6</td>
<td>0.513</td>
<td>0.570</td>
<td>0.573</td>
<td>0.573</td>
<td>0.573</td>
<td>0.573</td>
<td>0.567</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>9.92</td>
<td>0.79</td>
<td>1.35</td>
<td>1.36</td>
<td>1.36</td>
<td>1.36</td>
<td>-</td>
</tr>
</tbody>
</table>

When analyzing $K_{II}$, it is possible to notice that the trapezoidal rule cannot provide reliable results as the other two methodologies can. That is because the trapezoidal rule cannot accurately approximate the variation of the elastic field along the J-Integral contour in mode II deformation, as noted by Portela et al. [8]. The other two methodologies present the same maximum error when using 20 elements or points.

4.3 Test 3: Rectangular Plate with an Internal Kinked Crack

The third test refer to the analysis of a rectangular plate with an internal kinked crack, which has a horizontal segment and an inclined segment, at an angle of 45°. The plate boundary is divided into 48 quadratic elements, being 5 elements along the boundaries with width $2w$ and 10 elements along the boundaries with height $2h$. The horizontal segment of the crack is discretized by 5 elements on each face, graded towards the tip A with ratios 0.3, 0.25, 0.2, 0.15 and 0.1, while the inclined segment has 4 elements on each face, graded towards the tip B with ratios 0.4, 0.3, 0.2 and 0.1. Dimensions, mesh and boundary conditions are shown in Fig. 5.
For this test, the stress intensity factors are normalized by the length $c$, which is given by:

$$2c = a + \frac{b}{\sqrt{2}}.$$  \hfill (21)

The normalization equation, given by eq. 20, is then transformed into:

$$K_{\text{normalized}} = \frac{K}{\sigma \sqrt{\pi c}}.$$  \hfill (22)

The plate is under tensile loading, but, because of the geometry of the crack, it causes mixed-mode fracture. The results for normalized $K_I$ and $K_{II}$ are presented per crack tip, for three different ratios $b/a$. The reference values are obtained from Murakami [22]. Tables 11 and 12 present the results for mode I SIF for crack tip A.

**Table 11** Test 3 - Normalized $K_I$ results for different methods of J-Integral evaluation on tip A.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>1.029</td>
<td>1.010</td>
<td>1.003</td>
</tr>
<tr>
<td>0.4</td>
<td>1.020</td>
<td>1.004</td>
<td>0.997</td>
</tr>
<tr>
<td>0.6</td>
<td>1.014</td>
<td>1.000</td>
<td>0.994</td>
</tr>
</tbody>
</table>

Max. Error % 3.38 1.52 0.83 0.88 1.45 0.88 -
Table 12  Test 3 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral on tip A.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.111</td>
<td>1.009</td>
<td>1.003</td>
<td>1.003</td>
<td>1.003</td>
<td>1.003</td>
<td>0.995</td>
</tr>
<tr>
<td>0.4</td>
<td>1.106</td>
<td>1.003</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
<td>0.998</td>
<td>0.990</td>
</tr>
<tr>
<td>0.6</td>
<td>1.103</td>
<td>1.000</td>
<td>0.995</td>
<td>0.995</td>
<td>0.995</td>
<td>0.995</td>
<td>0.986</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>11.84</td>
<td>1.45</td>
<td>0.89</td>
<td>0.88</td>
<td>0.88</td>
<td>0.88</td>
<td>-</td>
</tr>
</tbody>
</table>

Still analyzing crack tip A, the results for $K_{II}$ are presented in tables 13 and 14.

Table 13  Test 3 - Normalized $K_{II}$ results for different methods of J-Integral evaluation on tip A.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.024</td>
<td>0.028</td>
<td>0.029</td>
<td>0.029</td>
</tr>
<tr>
<td>0.4</td>
<td>0.028</td>
<td>0.032</td>
<td>0.034</td>
<td>0.034</td>
</tr>
<tr>
<td>0.6</td>
<td>0.026</td>
<td>0.029</td>
<td>0.031</td>
<td>0.031</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>15.22</td>
<td>1.98</td>
<td>4.39</td>
<td>4.19</td>
</tr>
</tbody>
</table>

Table 14  Test 3 - Normalized $K_{II}$ results for different number of integration points in direct evaluation of J-Integral on tip A.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.4</td>
<td>0.025</td>
<td>0.029</td>
</tr>
<tr>
<td>0.6</td>
<td>0.030</td>
<td>0.034</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>10.92</td>
<td>3.58</td>
</tr>
</tbody>
</table>

The results for mode I SIF, obtained on crack tip B, are presented in tables 15 and 16.
Table 15 Test 3 - Normalized $K_I$ results for different methods of J-Integral evaluation on tip B.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
<td>20 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>0.606</td>
<td>0.607</td>
<td>0.607</td>
<td>0.608</td>
</tr>
<tr>
<td>0.4</td>
<td>0.578</td>
<td>0.579</td>
<td>0.579</td>
<td>0.579</td>
</tr>
<tr>
<td>0.6</td>
<td>0.572</td>
<td>0.573</td>
<td>0.573</td>
<td>0.573</td>
</tr>
</tbody>
</table>

Max. Error % 1.31 1.57 1.53 1.61 2.19 1.61 -

Table 16 Test 3 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral on tip B.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.2</td>
<td>0.678</td>
<td>0.611</td>
</tr>
<tr>
<td>0.4</td>
<td>0.646</td>
<td>0.583</td>
</tr>
<tr>
<td>0.6</td>
<td>0.639</td>
<td>0.576</td>
</tr>
</tbody>
</table>

Max. Error % 13.36 2.19 1.62 1.61 1.61 1.61 -

Tables 17 and 18 present the results of $K_{II}$ obtained on tip B.

Table 17 Test 3 - Normalized $K_{II}$ results for different methods of J-Integral evaluation on tip B.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
<td>20 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>0.476</td>
<td>0.542</td>
<td>0.569</td>
<td>0.568</td>
</tr>
<tr>
<td>0.4</td>
<td>0.516</td>
<td>0.588</td>
<td>0.617</td>
<td>0.616</td>
</tr>
<tr>
<td>0.6</td>
<td>0.534</td>
<td>0.608</td>
<td>0.639</td>
<td>0.637</td>
</tr>
</tbody>
</table>

Max. Error % 15.00 3.21 2.21 2.04 1.45 2.04 -
Table 18  Test 3 - Normalized $K_{II}$ results for different number of integration points in direct evaluation of J-Integral on tip B.

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [22]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.498</td>
<td>0.565</td>
<td>0.568</td>
<td>0.568</td>
<td>0.568</td>
<td>0.568</td>
<td>0.557</td>
</tr>
<tr>
<td>0.4</td>
<td>0.541</td>
<td>0.613</td>
<td>0.616</td>
<td>0.616</td>
<td>0.616</td>
<td>0.616</td>
<td>0.607</td>
</tr>
<tr>
<td>0.6</td>
<td>0.560</td>
<td>0.634</td>
<td>0.637</td>
<td>0.638</td>
<td>0.637</td>
<td>0.637</td>
<td>0.627</td>
</tr>
</tbody>
</table>

Max. Error % 10.92 1.45 2.03 2.04 2.04 -

When analyzing $K_{II}$ on crack tip A, even though the difference between the results obtained by the three methods and the expected one is at the third decimal, the maximum error obtained is greater than expected, due to the low reference values, which maximize the relative error.

4.4 Test 4: Compact Tension Specimen

The Compact Tension Specimen (CTS) is analyzed in the fourth example. The specimen has standardized dimensions presented in Tada et al. [23], which also provides the necessary equations to obtain mode I stress intensity factor. The initial mesh is composed by 114 quadratic elements, being 16 elements per side and per hole. The initial crack, with length $a$ equal to 0.2$b$, is divided into 4 equally sized elements per face and, for each increase in the crack length, $a$, two elements per face are added. The mesh of CTS, as well as dimensions and boundary conditions are presented in Fig. 6.

![Fig. 6 Test 4 - Dimensions, mesh and boundary conditions.](image)

The applied load causes mode I fracture on the crack tip. The reference results for $K_I$ is given by:

$$K_I = F\sigma\sqrt{\pi a} = \frac{P}{b}\sqrt{\pi a},$$

(23)
where

\[
F = \frac{2 \left( 2 + \frac{a}{b} \right)}{(1 - \frac{a}{b})^2} \sqrt{\frac{b}{\pi a}} \left[ 0.443 + 2.32 \left( \frac{a}{b} \right)^2 - 6.66 \left( \frac{a}{b} \right)^3 + 7.36 \left( \frac{a}{b} \right)^4 - 2.8 \left( \frac{a}{b} \right)^5 \right].
\]  
(24)

In eq. 23 and 24, \( P \) is the applied force per unit thickness, \( b \) is the dimension shown in Fig. 6 and \( F \) is the geometric factor related with CTS. The results are obtained for different ratios \( a/b \). Such results are presented in table 19 for the three integration approaches. The results obtained only with the direct evaluation of the J-Integral are shown in table 20.

### Table 19  
Test 4 - Normalized \( K_I \) results for different methods of J-Integral evaluation.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>5.453</td>
<td>5.451</td>
<td>5.441</td>
</tr>
<tr>
<td>0.3</td>
<td>5.782</td>
<td>5.794</td>
<td>5.788</td>
</tr>
<tr>
<td>0.5</td>
<td>7.591</td>
<td>7.605</td>
<td>7.594</td>
</tr>
</tbody>
</table>

Max. Error % 1.97 1.79 1.93 1.84 1.58 1.84 -

### Table 20  
Test 4 - Normalized \( K_I \) results for different number of integration points in direct evaluation of J-Integral.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
</tr>
<tr>
<td>0.2</td>
<td>6.094</td>
</tr>
<tr>
<td>0.3</td>
<td>6.474</td>
</tr>
<tr>
<td>0.5</td>
<td>8.498</td>
</tr>
</tbody>
</table>

Max. Error % 13.01 1.58 1.83 1.84 1.84 1.84 -

### 4.5 Test 5: Double Edge Notch Test Specimen under tensile loading

The fifth test refers to a Double Edge Notch Test (DENT) specimen, under tensile loading, which causes mode I fracture at the cracks tips. The initial mesh is composed by 32 elements, being 4 elements in width \( 2b \), 8 elements in height \( 2h \) and 2 elements per crack face, when \( a = 0.2b \). The test is performed with 5 different ratios \( a/b \) and,
for each increment in this ratio, one element is added per crack face, in order to keep the crack elements with a constant length. Mesh, dimensions and boundary conditions for the DENT specimen are shown in Fig. 7.

![Diagram of DENT specimen](image)

**Fig. 7** Test 5 - Dimensions, mesh and boundary conditions.

The reference value of $K_I$ is given by Tada et al. [23], according to the equation:

$$K_I = F\sigma\sqrt{\pi a},$$

where

$$F = \frac{1.122 - 0.561 \left(\frac{a}{b}\right) - 0.205 \left(\frac{a}{b}\right)^2 + 0.471 \left(\frac{a}{b}\right)^3 - 0.190 \left(\frac{a}{b}\right)^4}{\sqrt{1 - \frac{a}{b}}}.$$  

In these equations, $b$ is the semi-width of the plate, shown in Fig. 7, and $F$ is the geometric factor. Due to symmetry, $K_I$ results are the same in both crack tips. For that reason, only the result computed on the left crack is presented. Tables 21 and 22 respectively contain the results obtained with the different methods and with direct integration only. The reference results, given by eqs. 25 and 26, are also presented in these tables.
Table 21 Test 5 - Normalized $K_I$ results for different methods of J-Integral evaluation.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
<td>20 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>1.128</td>
<td>1.118</td>
<td>1.113</td>
<td>1.114</td>
</tr>
<tr>
<td>0.3</td>
<td>1.130</td>
<td>1.123</td>
<td>1.120</td>
<td>1.121</td>
</tr>
<tr>
<td>0.4</td>
<td>1.144</td>
<td>1.139</td>
<td>1.136</td>
<td>1.137</td>
</tr>
<tr>
<td>0.5</td>
<td>1.179</td>
<td>1.175</td>
<td>1.172</td>
<td>1.173</td>
</tr>
<tr>
<td>0.6</td>
<td>1.245</td>
<td>1.242</td>
<td>1.240</td>
<td>1.240</td>
</tr>
<tr>
<td></td>
<td>Max. Error %</td>
<td>0.40</td>
<td>0.89</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Table 22 Test 5 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>Number of Integration Points for Gaussian Quadrature</th>
<th>Reference [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>0.2</td>
<td>1.236</td>
<td>1.120</td>
</tr>
<tr>
<td>0.3</td>
<td>1.245</td>
<td>1.127</td>
</tr>
<tr>
<td>0.4</td>
<td>1.263</td>
<td>1.143</td>
</tr>
<tr>
<td>0.5</td>
<td>1.304</td>
<td>1.180</td>
</tr>
<tr>
<td>0.6</td>
<td>1.379</td>
<td>1.247</td>
</tr>
<tr>
<td></td>
<td>Max. Error %</td>
<td>10.59</td>
</tr>
</tbody>
</table>

4.6 Test 6: Double Edge Notch Test (DENT) Specimen under shear loading

The sixth test refers to the same specimen as the previous example but subjected to shear stress. To improve the evaluation of mode II stress intensity factors, a finer mesh than the one used in the previous example was required. The initial mesh is divided into 128 elements, being 20 elements in width $2b$, 40 elements in height $2h$ and 2 elements per crack face, when $a/b = 0.2$. Tests are performed for different ratios $a/b$ and, therefore, for each increment in this ratio, one element is added per crack face. Figure 8 shows the mesh used, the dimensions and boundary conditions.
At this test, the applied load causes pure mode II on crack faces. The reference values for $K_{II}$ are obtained in Tada et al. [23] and are given by:

$$K_{II} = \tau \sqrt{\pi a F}$$  \hspace{1cm} (27)

where $\tau$ is shear stress applied, shown in Fig. 8, and $F$ is the same geometric factor presented in eq. 26. Due to symmetry, both crack tips present the same result for mode II SIF. Because of that, only the results obtained on the left crack tip are presented. Table 23 shows the results with the different evaluation methods of J-Integral, while table 24 presents the results given by direct integration only, using different amounts of integration points.

**Table 23** Test 6 - Normalized $K_{II}$ results for different methods of J-Integral evaluation.

<table>
<thead>
<tr>
<th>a/b</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>0.970</td>
<td>1.075</td>
<td>1.119</td>
</tr>
<tr>
<td>0.3</td>
<td>0.966</td>
<td>1.081</td>
<td>1.128</td>
</tr>
<tr>
<td>0.4</td>
<td>0.977</td>
<td>1.097</td>
<td>1.147</td>
</tr>
<tr>
<td>0.5</td>
<td>1.066</td>
<td>1.131</td>
<td>1.184</td>
</tr>
<tr>
<td>0.6</td>
<td>1.062</td>
<td>1.193</td>
<td>1.249</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>15.02</td>
<td>4.50</td>
<td>0.44</td>
</tr>
</tbody>
</table>
Table 24 Test 6 - Normalized $K_{II}$ results for different number of integration points in direct evaluation of J-Integral.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.986</td>
<td>1.111</td>
<td>1.117</td>
<td>1.117</td>
<td>1.117</td>
<td>1.124</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.901</td>
<td>1.120</td>
<td>1.126</td>
<td>1.127</td>
<td>1.127</td>
<td>1.127</td>
<td>1.131</td>
</tr>
<tr>
<td>0.4</td>
<td>1.005</td>
<td>1.139</td>
<td>1.145</td>
<td>1.145</td>
<td>1.145</td>
<td>1.145</td>
<td>1.149</td>
</tr>
<tr>
<td>0.5</td>
<td>1.035</td>
<td>1.175</td>
<td>1.182</td>
<td>1.182</td>
<td>1.182</td>
<td>1.182</td>
<td>1.184</td>
</tr>
<tr>
<td>0.6</td>
<td>1.090</td>
<td>1.239</td>
<td>1.246</td>
<td>1.246</td>
<td>1.246</td>
<td>1.246</td>
<td>1.247</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>12.58</td>
<td>1.15</td>
<td>0.60</td>
<td>0.58</td>
<td>0.59</td>
<td>0.59</td>
<td>-</td>
</tr>
</tbody>
</table>

It is worth to notice, one more time, the limitation of the trapezoidal rule in mode II analysis (Portela et al. [8]).

4.7 Test 7: Pure Bending Specimen

In test 7, a rectangular plate subjected to a unitary bending moment is analyzed. The plate presents an edge crack, under mode I fracture, due to the loading combination. The initial mesh is composed by 58 quadratic elements, being 3 along the width $b$, 24 along height $2h$ and 2 elements per crack face, when $a = 0.2h$. Tests are performed for 5 different ratios $a/b$ and, for each increment in this ratio, one element is added per crack face, in order to maintain crack elements constant in length. The dimensions involved in the problem, the mesh used and the boundary conditions imposed are shown in Fig. 9.

![Test 7 - Dimensions, mesh and boundary conditions.](image)
The equations to obtain reference values for mode I SIF are presented in Tada et al. [23] and given by:

\[ K_I = F \sigma \sqrt{\pi a} = F \frac{6M}{b^2} \sqrt{\pi a}, \]  
\[(28)\]

where

\[ F = 1.122 - 1.40 \left( \frac{a}{b} \right) + 7.33 \left( \frac{a}{b} \right)^2 - 13.08 \left( \frac{a}{b} \right)^3 + 14.0 \left( \frac{a}{b} \right)^4. \]
\[(29)\]

In eqs. 28 and 29, \( M \) is the applied bending moment per unity of thickness, \( b \) is the width of the plate, shown in Fig. 9, and \( F \) is the geometric factor. The reference values are then calculated by eq. 28 and presented with the results obtained with the three different integration methodologies in table 25 and with the results obtained by direct integration with different number of integration points in table 26.

### Table 25 Test 7 - Normalized \( K_I \) results for different methods of J-Integral evaluation.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 elem.</td>
<td>20 elem.</td>
<td>6 elem.</td>
<td>20 elem.</td>
</tr>
<tr>
<td>0.2</td>
<td>1.060</td>
<td>1.061</td>
<td>1.060</td>
</tr>
<tr>
<td>0.3</td>
<td>1.127</td>
<td>1.129</td>
<td>1.129</td>
</tr>
<tr>
<td>0.4</td>
<td>1.262</td>
<td>1.266</td>
<td>1.265</td>
</tr>
<tr>
<td>0.5</td>
<td>1.499</td>
<td>1.504</td>
<td>1.503</td>
</tr>
<tr>
<td>0.6</td>
<td>1.917</td>
<td>1.924</td>
<td>1.924</td>
</tr>
<tr>
<td>Max. Error %</td>
<td>0.71</td>
<td>0.77</td>
<td>0.75</td>
</tr>
</tbody>
</table>

### Table 26 Test 7 - Normalized \( K_I \) results for different number of integration points in direct evaluation of J-Integral.

| \( a/b \) | Number of Integration Points for Gaussian Quadrature |
|---|---|---|---|---|---|
| 4 | 6 | 8 | 10 | 12 | 20 | Reference [23] |
| 0.2 | 1.181 | 1.067 | 1.061 | 1.061 | 1.061 | 1.061 | 1.053 |
| 0.3 | 1.259 | 1.136 | 1.129 | 1.129 | 1.129 | 1.129 | 1.122 |
| 0.4 | 1.413 | 1.273 | 1.266 | 1.266 | 1.266 | 1.266 | 1.256 |
| 0.5 | 1.679 | 1.533 | 1.504 | 1.504 | 1.504 | 1.504 | 1.495 |
| 0.6 | 2.150 | 1.936 | 1.925 | 1.925 | 1.925 | 1.925 | 1.910 |
| Max. Error % | 12.57 | 1.38 | 0.80 | 0.79 | 0.79 | 0.79 | - |

### 4.8 Test 8: Three-Point Bend Test Specimen

The eighth and final test refers to a three-point bend (TPB) test specimen. The specimen is divided initially in 105 quadratic elements, being 40 elements in length \( s \), 10 elements in height \( b \), 1 element for load application and 2 elements per crack face.
The test occurs for different ratios \(a/b\) and, for each increase in this ratio, one element is added per crack face, in order to keep crack elements with the same length. Figure 10 illustrates the discretization used in the specimen, dimensions and boundary conditions.

![Fig. 10 Test 8 - Dimensions, mesh and boundary conditions.](image)

Due to the alignment between the load application and the crack, only mode I fracture is generated at the crack tip. Reference results for \(K_I\) are given in Tada et al. [23], according to equation:

\[
K_I = F \sigma \sqrt{\pi a} = F \frac{3P_s}{2b^2} \sqrt{\pi a}, \tag{30}
\]

where

\[
F = \frac{1}{\sqrt{\pi}} 1.99 - \left( \frac{a}{b} \right) \left( 1 - \frac{a}{b} \right) \left[ 2.15 - 3.93 \left( \frac{a}{b} \right) + 2.7 \left( \frac{a}{b} \right)^2 \right] \left( 1 + 2 \frac{a}{b} \right) \left( 1 - \frac{a}{b} \right)^2. \tag{31}
\]

In these equations, \(P\) is the applied force per unity of thickness, \(s\) and \(b\) are dimensions of the specimen, as shown in Fig. 10, and \(F\) is the geometric factor related to the TPB specimen. The reference values are then obtained by eq. 30 and shown in the tables below. Table 27 shows the results obtained with the different evaluation methodologies, while table 28 contains the results obtained with direct integration using different number of integration points.

<p>| Table 27 Test 8 - Normalized (K_I) results for different methods of J-Integral evaluation. |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|</p>
<table>
<thead>
<tr>
<th>(a/b)</th>
<th>Trapezoidal rule</th>
<th>Quadratic elements</th>
<th>Direct integration</th>
<th>Reference [23]</th>
<th>Max. Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.974</td>
<td>0.974</td>
<td>0.979</td>
<td>0.974</td>
<td>1.47</td>
</tr>
<tr>
<td>0.3</td>
<td>1.034</td>
<td>1.035</td>
<td>1.036</td>
<td>1.041</td>
<td>1.45</td>
</tr>
<tr>
<td>0.4</td>
<td>1.165</td>
<td>1.168</td>
<td>1.168</td>
<td>1.175</td>
<td>1.50</td>
</tr>
<tr>
<td>0.5</td>
<td>1.399</td>
<td>1.402</td>
<td>1.403</td>
<td>1.411</td>
<td>1.44</td>
</tr>
<tr>
<td>0.6</td>
<td>1.813</td>
<td>1.819</td>
<td>1.820</td>
<td>1.830</td>
<td>0.88</td>
</tr>
</tbody>
</table>

Max. Error %
Table 28 Test 8 - Normalized $K_I$ results for different number of integration points in direct evaluation of J-Integral.

<table>
<thead>
<tr>
<th>$a/b$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>20</th>
<th>Reference [23]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.084</td>
<td>0.979</td>
<td>0.974</td>
<td>0.974</td>
<td>0.974</td>
<td>0.987</td>
<td>0.988</td>
</tr>
<tr>
<td>0.3</td>
<td>1.154</td>
<td>1.041</td>
<td>1.036</td>
<td>1.036</td>
<td>1.036</td>
<td>1.036</td>
<td>1.045</td>
</tr>
<tr>
<td>0.4</td>
<td>1.302</td>
<td>1.175</td>
<td>1.168</td>
<td>1.168</td>
<td>1.168</td>
<td>1.168</td>
<td>1.179</td>
</tr>
<tr>
<td>0.5</td>
<td>1.565</td>
<td>1.411</td>
<td>1.403</td>
<td>1.403</td>
<td>1.403</td>
<td>1.403</td>
<td>1.416</td>
</tr>
<tr>
<td>0.6</td>
<td>2.031</td>
<td>1.830</td>
<td>1.820</td>
<td>1.820</td>
<td>1.820</td>
<td>1.820</td>
<td>1.831</td>
</tr>
</tbody>
</table>

Max. Error % 10.91 0.88 1.43 1.44 1.44 -

5 Discussion

According to the results presented by the eight tests, the direct integration of J-Integrals by Gauss-Legendre quadrature is capable to return results with similar accuracy (or even higher) than more complex integration techniques, which require discretization of the integration path. Throughout the tests, it is possible to notice that contours composed of 6 or 8 integration points are already enough to provide reliable results. In fact, considering all tests performed, the maximum error converges to a stable value if 8 or more integration points are used, presenting a maximum error average of 1.20% in mode I analysis and 1.87% in mode II analysis, when all tests are considered.

6 Conclusion

In this paper, direct integration by Gaussian quadrature was proposed as an approach to evaluate J-Integrals in the Dual Boundary Element context. In order to evaluate the reliability of this approach, eight tests were performed, where the stress intensity factors were determined by J-Integrals, which were evaluated by three different numerical integration procedures: trapezoidal rule, quadratic polynomial interpolation over elements and the proposed direct integration. The results obtained with the three different approaches were compared with reference results and the maximum error was calculated.

Analyzing the results obtained by the eight tests, direct evaluation of J-Integrals by Gauss-Legendre quadrature provides equally good or even better results than the other two methodologies used to evaluate the stress intensity factors, when the contour is discretized into 6 or 20 elements or points. Combining this with the fact that direct integration presents a simpler and faster implementation, the method presents advantages in the analysis of fracture modes I, II and mixed-mode fracture.

Focusing on the direct evaluation method, the contours composed by 6 and 8 integration points provides sufficiently reliable results. In addition, the use of contours discretized into more than 8 integration points is not cost-effective, since the results stabilize after this discretization.
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Declarations

Conflicts of interests/Competing interests. The authors have no conflict to disclose.

References


