Spatial Self-Bending Soliton Phenomenon of (2+1) Dimensional bidirectional Sawada-Kotera Equation

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Research Article

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Posted Date: July 19th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-3167839/v1

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Additional Declarations: No competing interests reported.
Spatial Self-Bending Soliton Phenomenon of (2+1) Dimensional bidirectional Sawada-Kotera Equation

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Abstract

In this paper, we mainly study the spatial self-bending soliton phenomenon of the (2+1) dimensional bidirectional Sawada-Kotera equation. The bilinear form of the (2+1)-dimensional bidirectional Sawada-Kotera equation is obtained by Hirota bilinear method, and then the \(N\)-soliton solution is obtained through special parameter constraints \((e^{i\gamma} = 0)\), the spatial self-bending soliton is derived from the \(N\)-soliton solution, and the curvature of the spatial self-bending soliton is given furthermore, the interaction solution of spatial self-bending soliton and respiratory wave is obtained, and the interaction solution of spatial self-bending soliton and higher-order lump wave is obtained by using the method of long wave limit.

Keywords: spatial self-bending soliton; \(N\)-soliton; breather wave; higher-order lump

1. Introduction

Nonlinear local excitation modes such as isolated waves, breathers, and strange waves play a very important role in various branches of physics such as nonlinear optics, Bose-Einstein condensation, fluid physics, plasma physics, complex systems and complex networks, atmospheric and oceanic physics, and especially in photo physics \([1–8]\). Since the discovery of the soliton, experimental and theoretical approaches have been developed to search for novelty in specific physical systems. localized wave structures in specific physical systems, to investigate the mechanisms of generation of various localized waves, and to investigate the interactions between different localized waves Characteristics \([9–12]\). Since its exact solution is a special kind of solution and exists stably in space, it is of great practical importance for many complex physical phenomena and some nonlinear engineering problems. Aspects of the study of soliton solutions, many scholars have already studied it thoroughly. However, so far for spatial self-bending solitons and the interaction solutions of (2+1) dimensional bidirectional Sawada-Kotera (bSK) are less studied.

This paper is concerned with the study of the spatial self-bending soliton and some interacting waves (e.g., order-higher breath waves and lump waves) of the (2+1)-dimensional bSK equation\([13]\), which is usually written as

\[
5(\partial_x)^{-1}u_t - 45u_x^2u_x - 15u_xu_{xxx} - 15u_x(\partial_x)^{-1}u_t - 5u_{xxx} - u_{xxxx} + 9u_y = 0, \tag{1}
\]

where, \((\partial_x)^{-1} = (\frac{\partial}{\partial x})^{-1}\), \(u = u(x, y, t)\), Eq. (1) is an very important two-way nonlinear evolution equation and can be obtained by the Sawada-Kotera equation\([14]\),

\[
u_t + 45u_x^2u_x - 15u_xu_{xx} - 15u_xu_{xxx} + u_{xxxx} = 0, \tag{2}
\]

This equation is an important one-way nonlinear evolution equation and has very good mathematical properties. As an equation in the Liouville field hierarchy, the SK equation is used to describe many physical nonlinear phenomena in conformal field theory \([14]\), such as gravity, turbulence in fluids, shallow water phenomena, and quantum gravity in standard field theory. Some scholars have studied the bidirectional of Eq. (1) and the relationship with Eq. (2). In addition, in many practical cases, Eq. (1) also has opposite analog wave propagation directions \([15–18]\). Therefore, the study of the bSK equation is of great interest. In recent years, Eq. (1) has also been studied extensively by many scholars, such as Darboux transform, Bäcklund transformation, Lax pair, two-periodic wave solutions, soliton molecules and hybrid solutions \([13, 19, 20]\). However, the spatially bent resonant soliton phenomenon of Eq. (1) has not been studied so far.

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In recent years, the Burgers equation and the Sharma-Tasso-Olver equation as two concrete examples, from the Levi spectral problem, two basic Darboux transformations have been obtained. Then from the trivial seed solution, the authors set the multi-kink solutions and soliton fission and fusion solutions into the following form \[21, 22\]:

\[
u = \alpha \ln(1 + \sum_{j=1}^{N} p_j \exp(\xi_j + \sum_{j=1}^{N} A_j \mu_j))_{xx}. \tag{3}\]

However, this approach is too special to get the interaction between fissionable or fusonable waves and other types of waves. In order to generate the hybrid solutions just mentioned, Chen et al. Construct a new form of \(N\)-soliton solution:

\[
f = \alpha \ln \left( \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu \xi_j + \sum_{j=1}^{N} A_j \mu_j + \sum_{j=2}^{N} A_2 \mu_j \right) \right), \tag{4}\]

and, \(\xi_j, A_{ij}\) refer to Ref. [22]. This method can lead to some hybrid solutions, like an interaction between a first-order lump wave and \(N\)-fissionable waves. On the basis of Eq. (1), no matter how constrained it is, a hybrid of fissionable waves and fusonable waves cannot be obtained. That is because Eq. (1) has fewer terms than the classic \(N\)-soliton solution. Recently, Some scholars have already studied a simpler method from \(N\) soliton [23, 24], on the basis of this we have made adjustments to get spatially self-bending soliton of Eq. (1).

This paper focuses on the spatial self-bending solitons phenomenon of the (2+1)-dimensional bidirectional Sawada-Kotera (bSK) equation. The paper is structured as follows: in Section 2, the bilinear form of the (2+1)-dimensional bSK equation is first obtained by means of the Hirota bilinear method, and then the \(N\)-soliton solution is obtained. The \(M\)-spatial self-bending solitons solutions are derived from the \(N\)-soliton solutions by a special parametric constraint \(e^{ix_0} = 0\). The 1 and 2-spatial self-bending solitons are given and the curvature of the 1-spatially self-bending soliton is given. In Section 3, the interaction solutions of the spatial self-bending solitons with the breathing waves are obtained. In Section 4, the interaction solutions of spatial self-bending solitons and lump waves are obtained by applying the method of long-wave limit.

2. \(M\)-spatial self-bending solitons

The \(N\)-soliton solution to Eq. (1) can be easily found by using bilinear method:

\[
u(x, y, t) = 2(\ln f)_{xx}, \tag{5}\]

the Eq. (1) becomes the following form by the Hirota bilinear method

\[
\begin{align*}
S(f \cdot f) &= (D_x D_y - D_y^2 - 5D_x^2 + 5D_x^2)(f \cdot f) \\
&= 18(f_{xx} f - f_{xy} f_x) - 2(f_{xxxx} f^3 - 6f_{xxxx} f_{xx} + 15f_{xxxx} f_{xx} - 10f_{xx}^2) \\
&\quad - 10(f_{xxxx} f - 3f_{xxx} f_x + f_{xxx} f_{xx}) + 10(f_{xx} f - f_x^2) = 0, \tag{6}
\end{align*}
\]

where

\[
f = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu \eta_j A_{j\mu} + \sum_{j=1}^{N} \mu \eta_j \right), \tag{8}\]

which needs to satisfy

\[
\begin{align*}
\eta_j &= \kappa_j x + p_j y + \omega t + \phi_j, \tag{9}
\\
\phi^\nu_{\kappa_j} &= \frac{(k_j - \kappa_j)^5 + 5(\kappa_j - \kappa_j)^5(\omega_j - \omega_j) - \phi_j}{(k_j - \kappa_j)^5 + 5(\kappa_j - \kappa_j)^5(\omega_j + \omega_j) - \phi_j}, \tag{10}
\\
p_j &= \frac{k_j^5 + 5k_j^5\omega_j - 5\omega_j^2}{9k_j}. \tag{11}
\end{align*}
\]

In order to obtain the spatial self-bending soliton, we added the \(\exp(x)\) range to remove some items in Eq. (8). The \(\exp(x) = 0\) is true if and only if \(x = \ln(0)\), \(\exp(x + \ln(0)) = 0 \exp(x) = 0\), if all \(A_{j\mu} = \ln(0)\), then Eq. (8) can be converted into Eq. (3). If \(A_{j\mu} = \ln(0), 3 \leq j < s \leq N\), then Eq.(4) by Eq. (8) is derived. Which, of course, leads to this very interesting conclusion.
Based on the $N$-soliton solutions, the $M$-spatial self-bending solitons can be derived through the following constraints:

$$ e^{\lambda_{1,j}} = 0, \quad (1 \leq j < s \leq 2M, M < j < s \leq N, N = 2M), $$

and

$$ \omega_r = \frac{q}{2} \pm \sqrt{q^2 + \left(\frac{p}{3}\right)^3} + \frac{r}{2} \pm \sqrt{r^2 + \left(\frac{p}{3}\right)^3} - \frac{b}{3a}, $$

where

$$ p = \frac{3ac - b^2}{3a^2}, \quad q = \frac{27a^2b - 9abc + 2b^3}{27a^3}, \quad a = 5\kappa, \quad b = -5\kappa^3 - 5\kappa \kappa - 5\kappa \omega_1, $$

$$ c = k_0^2 - k_0^4 - 5k_0^{4} + 5k_0^{4} + 5k_0^{4} + 5k_0^{4} + 10k_0^{4} + 10k_0^{4} - 5k_0^{4}. $$

3. 1-Spatial self-bending soliton

From with $M = 1$, the 1-spatial self-bending soliton can be obtained

$$ u(x, y, t) = 2(\ln f)_{xx}, $$

with

$$ f = 1 + e^{x_1} + e^{x_2}. $$

select the following parameters

$$ \kappa_1 = \frac{1}{2}, \quad \kappa_2 = \frac{1}{2}, \quad p_1 = 1, \quad p_2 = -\frac{3}{8}, \quad \phi_1 = 0, \quad \phi_2 = 0, $$

1-spatial self-bending soliton can be obtained by Eq. (17) into the Eq. (15).

Form Figure 1: (a),(b) and (c) show 3D figures of 1-spatial self-bending soliton, as can be seen from the (a) to (c), as time $t$ changes, the bending solitons gradually constantly moves without changing its shape.

In order to obtain the bend of the spatial self-bending soliton, take $\kappa_1 = \frac{1}{2}, \kappa_2 = \frac{1}{2}, \omega_1 = 1$, the solution is:

$$ u = \frac{\exp(a_0 + 1 \frac{1}{8} x + \frac{1}{2} y - \frac{1}{4} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^2 + \frac{1}{2} x y) + \exp(-1 \frac{1}{8} x - \frac{1}{2} y + \frac{1}{4} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^2 + \frac{1}{2} x y)}{2(1 + t\omega_2 + \frac{1}{2} t x + \frac{1}{2} t y - \frac{1}{4} t y^2 - \frac{1}{2} t x^2 + \frac{1}{2} t x y + \frac{1}{4} t x^2)} \frac{\exp(a_0 + 1 \frac{1}{8} x + \frac{1}{2} y - \frac{1}{4} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^2 + \frac{1}{2} x y) + \exp(-1 \frac{1}{8} x - \frac{1}{2} y + \frac{1}{4} y^2 + \frac{1}{2} x^2 + \frac{1}{4} x^2 + \frac{1}{2} x y)}{2(1 + t\omega_2 + \frac{1}{2} t x + \frac{1}{2} t y - \frac{1}{4} t y^2 - \frac{1}{2} t x^2 + \frac{1}{2} t x y + \frac{1}{4} t x^2)} $$

from solve $u + \frac{1}{2} = 0$.

$$ x = \frac{1}{16} y + 2 \ln \frac{5 \exp x - 3 \sqrt{5} \exp y + 10 \exp x + 5 + 5}{2(\exp x + 1) \sqrt{5} \exp x + 10 \exp y + 5 + 5} $$

where

$$ X = t + \frac{1}{36} y + t\omega_2 - \frac{10}{9} t\omega_2 + \frac{5}{36} t\omega_2 $$

bending equation

$$ K = \left| \frac{y_{xx}}{(1 + y_x)^2} \right| $$

and

$$ y_x = \frac{144(\exp x + 1)}{320 e^x \omega_2^2 + 40 e^x \omega_2 + e^x + 279}, $$

$$ y_{xx} = \frac{115200(64 e^x \omega_2^4 + 16 e^x \omega_2^2 + 16 e^x \omega_2 + 49)}{(320 e^x \omega_2^2 - 40 e^x \omega_2 + e^x + 279)^3}, $$

$$ Y = \frac{1}{36} (1 - \omega_2)(20 y y \omega_2 + 35 y - 36 t), $$

when $t = 0$, can get the curvature $K$ of 1-spatial self-bending soliton from choose different $\omega_2$ follow the Table 1:

The curvature figures of the 1-spatial self-bending soliton can be obtained based on different $t$ and $\omega$ as follows:
Figure 1: (Color online) (a), (b), (c) 1-spatial self-bending soliton in three dimensions; (d) 1-spatial self-bending soliton in cross-section

Table 1: Curvature

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-5$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$-\frac{3}{5}$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{2} \ln y - 4 \ln 2 + 2 \ln (\sqrt{5} - 3)$</td>
</tr>
<tr>
<td>$3$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
<tr>
<td>$5$</td>
<td>$\frac{1}{2} \ln y - 2 \ln 2 + \frac{2 \ln 5}{31^{16} y - 2 \ln 2 + 2 \ln 5^{14}/2^{11} + 5}$</td>
</tr>
</tbody>
</table>
From the figures, it can be observed that when time $t$ is taken as a certain value, the degree of soliton bending also changes with the variation of $\omega$. It is easy to observe that when $\omega$ is negative, the degree of bending gradually decreases as $\omega$ increases. When $\omega$ is positive, the degree of bending gradually increases as $\omega$ increases.

When $M = 2$, a expression described an interaction between 2-spatial self-bending soliton have the following form:

$$u(\mathbf{x}, t) = 2(\ln f)_{x x}, \quad (23)$$

with

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_4 + \xi_3 + A_{15} + \xi_3 + A_{16} + \xi_3 + A_{17}}. \quad (24)$$

The relevant parameters $A_{js}, P_x$ are given in the Eq. (12) and Eq. (13) respectively. With appropriate constraints, the Eq. (23) can describe 2-spatial self-bending solitons. The specific parameters are as follows:

$$\kappa_1 = \frac{1}{2}, \quad \kappa_2 = \frac{1}{2}, \quad \omega_1 = 1, \quad \omega_2 = -\frac{3}{8}, \quad \phi_1 = 0, \quad \phi_2 = 0, \quad (25)$$

$$\kappa_3 = 1, \quad \kappa_4 = 1, \quad \omega_3 = \frac{1}{2}, \quad \omega_4 = \frac{3}{2}, \quad \phi_3 = 25, \quad \phi_4 = 25.$$ 2-spatial self-bending solitons can be obtained by Eq. (25) into the Eq. (23), the figures as follows

If $N = 3$, the $N$-soliton solutions can only be simplified to a $N$-spatial self-bending solitons. Most studies have only obtained pure self-bending solitons. By selecting the following parameters, Two mixed self-bending solitons
can be obtained, which first undergo fission and then fusion:

\[
\kappa_1 = -\frac{1}{2}, \quad \kappa_2 = \frac{1}{2}, \quad p_1 = 1, \quad p_2 = \frac{9}{4}, \quad \kappa_3 = \frac{1}{2}, \quad p_3 = \frac{7}{6}, \quad \phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 = 0,
\]

an interaction self-bending soliton between fission first and then fusion can be obtained by Eq. (26) into the Eq. (23).

\[
\kappa_1 = 2, \quad \kappa_2 = \frac{1}{2}, \quad \omega_1 = 1, \quad \omega_2 = -\frac{3}{8}, \quad \phi_1 = 0, \quad \phi_2 = -3,
\]

\[
\kappa_3 = 1 - i, \quad \kappa_4 = 1 + i, \quad p_3 = \frac{1}{3} + i, \quad p_4 = \frac{1}{3} - i, \quad \phi_3 = 10, \quad \phi_4 = 10,
\]

the interactions between 1-spatial self-bending soliton and 1-breath wave we can obtained by Eq. (28) into the Eq. (5).

When \( M = 1, L = 2 \), taking parameters as follows

\[
\kappa_1 = \frac{1}{2}, \quad \kappa_2 = \frac{1}{2}, \quad \omega_1 = 1, \quad \omega_2 = -\frac{3}{8}, \quad \phi_1 = 0, \quad \phi_2 = -3,
\]

\[
\kappa_3 = 1 - i, \quad \kappa_4 = 1 + i, \quad \omega_3 = \frac{1}{3} + i, \quad \omega_4 = \frac{1}{3} - i, \quad \phi_3 = 10, \quad \phi_4 = 10,
\]

we can obtained the interactions between 1-spatial self-bending solitons and 2-order breather waves.

**5. Interactions between spatial self-bending solitons and higher-order lump waves**

A nonlinear superposition between a \( M \)-spatial self-bending solitons and \( L \)-order lump waves can be derived if the following constraint is applied to the \( N \)-soliton solution:

\[
\kappa_{2m-1} = \kappa_{2m} = \frac{K_{2m-1} \epsilon}{2}, \quad p_{2m-1} = p_{2m} = \frac{P_{2m-1} \epsilon}{2}, \quad \phi_{2m-1} = \phi_{2m} = \pi i, \quad (1 \leq m \leq M), \quad \epsilon \rightarrow 0,
\]

\[
N = 2M + 2L, \quad e^{A x} = 0, \quad (2M < j < s \leq N),
\]
among them, the lump wave controlled by the parameters $K_{2m}, P_{2m}, K_{2m-1}, P_{2m-1}$. Through reasonable parameter constraints and long wave limit method, the interaction between $M$-spatial self-bending solitons and $L$-lumps can be obtained.

In particular, if $M = 1, L = 1$, we can find the expression for a hybrid of 1-spatial self-bending solitons and 1-order lump wave:

$$u(x, y, t) = 2(\ln f)_{xx},$$  \hspace{1cm} (30)

and

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + e^{\xi_1 + \xi_2 + \xi_3} + e^{\xi_1 + \xi_2 + \xi_3 + \xi_4} + e^{\xi_1 + \xi_2 + \xi_3 + \xi_4} + e^{\xi_1 + \xi_2 + \xi_3 + \xi_4}.$$ \hspace{1cm} (31)

taking the following parameters into Eq. (30):

$$\kappa_1 = \frac{1}{2}, \quad \kappa_2 = \frac{1}{2}, \quad \omega_1 = 1, \quad \omega_2 = \frac{3}{8}, \quad \phi_1 = 30, \quad \phi_2 = 10,$$

$$\kappa_3 = \frac{1}{3} - i, \quad \kappa_4 = \frac{1}{3} + i, \quad \omega_3 = 1 - \frac{i}{2}, \quad \omega_4 = 1 + \frac{i}{2}, \quad \phi_3 = i\pi, \quad \phi_4 = i\pi.$$ \hspace{1cm} (32)

with the long wave limit method, we can obtain the hybrid wave of 1-spatial self-bending solitons and 1-order lump wave by Eq. (32) into the Eq. (30).

6. Conclusion

This paper focuses on the spatially curved resonant soliton phenomenon of the $(2+1)$-dimensional bidirectional Sawada-Kotera equation. The paper is structured as follows: in Section 2, the bilinear form of the $(2+1)$-dimensional bidirectional Sawada-Kotera equation is first obtained by means of the Hirota bilinear method, and then the $N$-soliton solution is obtained. The spatial self-bending soliton is derived from the $N$-soliton solutions by a special parametric constraint ($e^{\lambda_i} = 0$) and the curvature of the spatially curved soliton is given. In Section 3, the interaction solutions of $M$-spatial self-bending soliton with the $L$-order breath waves are obtained. In Section 4, the interaction solutions of $M$-spatial self-bending soliton and $L$-order lump waves are obtained by applying
the method of long-wave limit. The physical interpretation of fission-fusion dynamics is also explained graphically through the above interaction solutions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgment

This work is supported by National Natural Sciences Foundation of Anhui Province, China under Grant Nos. 2022AH052570, School-level scientific research projects, Grant Nos. 2021KYXM08.

Reference