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H.I. Alrebdi  
Princess Nourah bint Abdulrahman University

Norah Alsaif  
Princess Nourah bint Abdulrahman University

Sanam Suraj  
University of Delhi

Euaggelos E. Zotos (✉ evzotos@physics.auth.gr)  
Aristotle University of Thessaloniki

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Investigating the properties of equilibrium points of the collinear restricted 4-body problem

H. I. Alrebei\textsuperscript{a}, Norah A. M. Alsaif\textsuperscript{a}, Md Sanam Suraj\textsuperscript{b}, Evaggelos E. Zotos\textsuperscript{c,d,*}

\textsuperscript{a}Department of Physics, College of Science, Princess Nora bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
\textsuperscript{b}Department of Mathematics, Sri Aurobindo College, University of Delhi, New Delhi-110017, Delhi, India
\textsuperscript{c}Department of Physics, School of Science, Aristotle University of Thessaloniki, GR-541 24, Thessaloniki, Greece
\textsuperscript{d}S.M. Nikolskii Mathematical Institute of the Peoples’ Friendship University of Russia (RUDN University), Moscow, 117198 Russia

Abstract

The circular version of the planar restricted 4-body problem is considered. We assume that the two peripheral bodies have not spherical shape but they are either prolate or oblate. The dynamical properties of the points of equilibrium of the system are investigated using several types of numerical methods and techniques. In particular, we calculate not only the coordinates of the positions of the libration points but also their linear stability and dynamical types. Our main objective is to reveal the influence of the mass parameter of the system along with the shape parameter on the equilibrium dynamics. Our analysis indicates that in the case where the peripheral bodies are prolate in shape the equilibrium dynamics of the system is more interesting and complex with respect to the case where the two peripheral bodies have oblate shapes.

Keywords: Restricted 4-body problem – Linear stability – Equilibrium points

1. Introduction

The problem of the \( N \) bodies of the celestial mechanics is an issue of paramount importance and inspired many researchers to raise a new problem. One of the most important cases is the so-called restricted \((N + 1)\) bodies regular polygon problem, i.e., the Maxwell-type configuration where the \((N - 1)\) bodies of equal masses \( m \) are situated at the vertices of a regular polygon of \((N - 1)\) sides whereas the \( N_{th} \) body with a different value of mass \( m_0 \) is situated at the mass center of the system of \((N - 1)\) bodies.

The interesting scenario of the ring \((N+1)\) problem was discussed by many authors, e.g., (Arribas & Elipe, 2004). This model was considered to observe the phenomenon of planetary rings, asteroids, some stellar formations, or the motion of the test particles in the vicinity of the planetary rings, etc. It was found that Maxwell (1952) discussed the stability of a discrete particle ring to analyze the rings of Saturn. Further, Maxwell’s analysis was reformulated by Tisserand in 1889 and showed that there is a relation between the mass of each particle in the ring and number of them for the system to be stable in the linear sense. Later on, many researchers have discussed the so called ring problem to explore various aspects, i.e., the computation of periodic orbits of the test particle moving in the Newtonian gravitational influence of the primaries situated on a ring (e.g., (Kalvouridis, 1999, 2001, 2003; Croustaloudi & Kalvouridis, 2011)), the convergence basins in the polygon problem of \((N + 1)\) bodies (e.g., (Aggarwal et al., 2021)). The gravitational ring \((N + 1)\) problem can be reduced to various types of dynamical systems by taking different values of \( N \). In Ref. Arribas et al. (2016b), the authors have revealed the existence and stability properties of equilibria in the particular case of \((N + 1)\)-body problem for \( N = 3 \) where \((N - 1)\) peripheral primaries have equal masses and radiation forces due to these peripheral primaries are also equal whereas in Ref. Arribas et al. (2016b) the out-of-plane equilibria are discussed in the same configuration. When \( N = 4 \), the ring problem reduces to the restricted problem of five bodies of Oll"{e}ngren (1998) where the fourth body is located at the mass center of the system, while the rest of the bodies are placed at the vertices of an equilateral triangle. Further, many authors have discussed the \((N + 1)\) body ring problem with a special class of the potential function referred to as quasi-homogeneous potential (see e.g., (Diaconu, 1996; Arribas et al., 2007, 2008; Aggarwal et al., 2021; Palacios et al., 2019)).

In any dynamical system, the study of the orbital properties is one of the important aspects. The periodic orbits
are solutions of fundamental importance in the set of solutions to the restricted problem of few bodies. The libration points and periodic orbits of the restricted few-body system characterize, through the behaviour of neighbouring orbits which explore the motion of the test particle, thus these solutions constitute a very important tool to discuss this problem for theoretical studies and applications (e.g., (Szabóhelyi, 1967)). In order to obtain more insights into the few-body problem, recently, many authors have studied them to analyze their properties, e.g., in restricted 3-body problem (e.g., (Perdios, 1996, 2007; Alzahrani & Abouelmagd, 2017; Abozaid et al., 2020; Abouelmagd et al., 2020; Alshaery & Abouelmagd et al., 2020)), in the planar circular Pluto-Charon system (e.g., (Zotos et al., 2019)), in the pseudo-Newtonian restricted problem of 3 bodies (e.g., (Alrebdi et al., 2022)), in the restricted 4-body problem (e.g., (Papadakis, 2007; Baltagiannis & Papadakis, 2013)) etc.

The main aim of this paper is to shed some light on the important aspects of the dynamics of the collinear restricted 4-body problem when the peripheral primaries are oblate (prolate) spheroid. Additionally, the combined effects of the mass ratio and the prolateness/oblateness parameter on the existence and stability of the equilibria in the three-dimensional collinear 4-body problem (see e.g., (Suraj et al., 2020)), where the shape of the peripheral bodies is not spherical are analyzed in this study. Studying this configuration is of paramount importance since many celestial bodies are not in spherical shape but are oblate(prolate) bodies. Moreover, in present the work the existence and stability of the libration points are discussed by scanning the entire $(\beta, A)$-plane which generalizes the previous study of Ref. (Suraj et al., 2020) which was performed only for two fixed values of $\beta$.

The layout of the manuscript has the following structure: in Section 2, we provide the description of the mathematical setup with its most important properties. In Section 3, the dynamics of the equilibria points are illustrated by scanning the entire $(\beta, A)$-plane where $\beta \in [0, 5]$ and the values of $A$ lie in the permissible range. We close the paper with Section 4 where the summary of our numerical analysis, our findings, and future works are discussed.

2. Mathematical setup of the system

The system is composed of three primary bodies $P_i, i = 0, 1, 2$, in which the two peripheral bodies $P_i, i = 1, 2$ with masses $m_i$ move in a circular orbit around their common centre of mass. Moreover, the central body $P_0$ with mass $m_0 = \beta m$ (where $\beta$ is, in a way, a mass parameter) is placed at the origin of the synodic reference frame $Oxyz$ (see the schematic in Fig. 1). Furthermore, we assume that the primary bodies are in relative equilibrium and therefore rotate with a constant angular velocity $\omega$. In the present problem, we have assumed that the peripheral bodies $P_i$ are non-spherical, more precisely, the peripheral primaries are prolate or oblate bodies and the prolateness/oblateness parameters are $A_i = A, i = 1, 2$, respectively (for details see Suraj et al. (2020)) whereas the central body $P_0$ has a spherical shape.

Therefore, in the synodic reference frame the coordinates of the primaries $P_i, i = 0, 1, 2$ are $(x_i, 0, 0)$, $x_0 = 0, x_1 = -x_2 = \frac{1}{2}$, respectively. We choose the units of masses and time in such a way that $Gm = 1$ and the distance between the peripheral primaries is unity. According to Suraj et al. (2020), the condition to keep the configuration of the peripheral primaries remain invariant on the circular orbit of radius 0.5 and angular velocity $\omega$ is that $\omega^2 = \Lambda = 2(1+4\beta) + 6A(1+8\beta)$. In order for the function $A$ to be positive $A$ must satisfy the condition

$$A > -\frac{(1+4\beta)}{3(1+8\beta)}, \quad (1)$$

for each value of $\beta$ and we refer to the value of $A$ is admissible if it satisfies the given inequality.

A test particle, along with its three-dimensional mo-
tion, follows the set of equations

\[ \begin{align*}
\ddot{x} - 2\dot{y} &= \frac{\partial \Phi}{\partial x}, \\
\ddot{y} + 2\dot{x} &= \frac{\partial \Phi}{\partial y}, \\
\ddot{z} &= \frac{\partial \Phi}{\partial z},
\end{align*} \tag{2a-2c} \]

where

\[ \Phi(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1}{\Lambda} \left( \frac{\beta}{r_0} + \sum_{i=1}^{2} \left( \frac{1}{r_i} + \frac{A_i}{2r_i^3} \right) \right), \tag{3} \]

is the system’s effective potential whereas

\[ r_i = \sqrt{(x-x_i)^2 + y^2 + z^2}, \quad i = 0, 1, 2, \tag{4} \]

are the distances between the test particle and the primary bodies (see Suraj et al. (2020) for details).

3. Dynamics of the equilibrium points

If one wants to calculate the planar coordinates \((x, y)\) of the system’s equilibrium, then he/she must solve the system \(\dot{x} = \dot{y} = \dot{z} = 0\) (assuming that in the planar problem, we have already \(\dot{z} = \ddot{z} = 0\)). The solutions of this system can only be numerical and this is because the high complexity of the involved equations does not allow any derivation of analytical solutions.

In our analysis, we shall consider both the case of prolate \((A < 0)\) and oblate \((A > 0)\) peripheral primaries. For positive values of \(A\), our preliminary analysis suggests that the equilibrium dynamics of the system remain unperturbed. Therefore, the parameter \(A\) shall have values in the interval \((-1/3, 0.1)\). Moreover, the values of the mass parameter \(\beta\) will mainly lie in the interval \([0, 5]\), without excluding analysis for higher values of \(\beta\).

The bivariate Newton-Raphson (N-R) scheme on the system for the configuration plane reads

\[ \begin{align*}
x_{n+1} &= x_n + \left( \frac{\Phi_x \Phi_{yy} - \Phi_y \Phi_{xy}}{\Phi_{yy} \Phi_{xx} - \Phi_{yy}^2} \right) (x_n, y_n), \\
y_{n+1} &= y_n + \left( \frac{\Phi_y \Phi_{xx} - \Phi_x \Phi_{xy}}{\Phi_{yy} \Phi_{xx} - \Phi_{yy}^2} \right) (x_n, y_n),
\end{align*} \tag{5} \]

where the \(\Phi_{ij}\) terms are the derivatives (first and second order) of the potential (3).

Then, the algorithm for numerically obtaining the equilibria of the system is described as follows:

- **Step 1:** We define a uniform and dense grid of 1024 × 1024 values of \((A, \beta)\).
- **Step 2:** For every pair of \((A, \beta)\) values, we apply the above-described N-R scheme for scanning a 500 × 500 initial conditions on the \((xyz)\) plane.

- **Step 3:** We count the total number of the equilibrium points (that act as numerical attractors of the N-R scheme) and we also determine their linear stability.

In order to discuss the linear stability we shall linearize the equations (2a-2b) (when \(z = 0\)) in the vicinity of the equilibria

\[ \begin{align*}
\dot{x} &= \Xi x, \\
\dot{x} &= (\dot{x}, \dot{y}, x, y)^T,
\end{align*} \]

where \(x\) represents the vector’s state of the test particle with respect to the points of libration. Then, the coefficient matrix \(\Xi\) can be written as

\[ \Xi = \begin{bmatrix} O & I \\ B & C \end{bmatrix}, \tag{6} \]

where

\[ \begin{align*}
O &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
B &= \begin{bmatrix} \Phi_{xx}^0 & \Phi_{xy}^0 \\ \Phi_{xy}^0 & \Phi_{yy}^0 \end{bmatrix}, \\
C &= \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}, \tag{7-10} \end{align*} \]

where the superscript "0" shows the second-order partial derivatives of \(\Phi\), evaluated at the particular libration points.

The characteristic equations of the matrix \(\Xi\), given in Eq. (6) can be written as follows:

\[ \sum_{i=0}^{2} \kappa_{2i} \lambda^{2i} = 0, \tag{11} \]

where

\[ \begin{align*}
\kappa_0 &= \Phi_{xx} \Phi_{yy} - \Phi_{xy}^2, \\
\kappa_2 &= 4 - \Phi_{xx} - \Phi_{yy}, \\
\kappa_4 &= 1. \tag{12a-12c} \end{align*} \]

An equilibrium point is considered to be linearly stable (LS) if the Eq. (11), evaluated at the specific equilibrium point, has four roots that are pure imaginary numbers. This happens only when all the conditions given in Eq. (13) are satisfied simultaneously

\[ \kappa_2 > 0, \quad \kappa_0 > 0, \quad \kappa_2^2 - 4 \kappa_0 > 0. \tag{13} \]

Indeed, if \(\lambda^2 = \Xi\), then the characteristic equation given in Eq. (11) becomes

\[ \sum_{i=0}^{2} \kappa_{2i} \lambda^{i} = 0, \tag{14} \]
and this equation has two negative real roots namely $\Xi_1$ and $\Xi_2$, and consequently, Eq. (11) will have four pure imaginary roots $\lambda_i, i = 1, 2, 3, 4$, when the conditions in Eq. (13) are satisfied.

Moreover, we further classify the equilibrium by evaluating the number of non-negative eigenvalues of the matrix of the second-order partial derivatives of the effective potential functions given in Eq. (9). Specifically, in a system of two-degree of freedom, if we denote $\lambda_+$ the number of total positive eigenvalues, then a particular equilibrium point can be categorized as:

† It is an index-I (ID1) saddle point when $\lambda_+ = 1$, i.e., only one positive eigenvalue then.

† It is an index-II (ID2) saddle point or minimum of the effective potential when $\lambda_+ = 2$, i.e., both the eigenvalues are positive.

† It is a local maximum of the potential when $\lambda_+ = 0$, i.e., none of the eigenvalues are positive.

In what follows, we present our analysis of the equilibrium dynamics of the system by distinguishing between different total numbers and types of libration points. For depicting our results we will use the graphical approach of Nagler (2004, 2005), based on color-coded two-dimensional diagrams.

The basins on the $(A, \beta)$ plane corresponding to a different total number of equilibria are depicted in Fig. 2(a-b). One can see that the different regions are well-defined, while there is no indication of “chaotic” or fractal-like structures. Our computations suggest that when $\beta > 0$ the system can have either 6, 10, or 14 equilibrium points, depending of course on the specific values of the parameters $A$ and $\beta$. Furthermore, from the diagram of Fig. 2 we can observe that when $A > 0$ (the case with oblate peripheral primaries) there are always 6 equilibria, regardless of...
Figure 4: Diagrams showing the locations of the equilibria (red dots), through the intersections of the curves of the iso-contours $\Phi_x = 0$ (green) and $\Phi_y = 0$ (blue), when the system has 6 libration points in total. The positions of the two primaries correspond to the black dots.

the values of $A$ and $\beta$. On the other hand, when $A < 0$ (the case with prolate peripheral primaries) the dynamics of the system exhibit higher complexity and thus it is more interesting.

Our calculations also revealed that when $\beta = 0$ the system has either 5, 9, 11, or 13 equilibrium points. However, when $\beta = 0$ the system degenerates to the restricted version of the 3-body problem which has already been investigated in detail in Zotos et al. (2020). On this basis, in the present study, we shall consider only the cases with $\beta > 0$.

3.1. Scenario I: 6 equilibria

The case with 6 libration points in total is the most probable scenario of the dynamical system. As it was explained above when the two peripheral bodies are oblate in shape the system has always 6 equilibria, regardless of the particular values of $A$ and $\beta$. However, also in the case of
Figure 5: Similar to Fig. 3, when 10 equilibria exist: 2 LS + 4 CL (green), and 4 LS + 4 CL (red).

Figure 6: Similar to Fig. 4, when 10 equilibria exist.

Prolate peripheral bodies, the scenario with 6 equilibrium points is again the most probable one.

In Fig. 3 we present the sub-cases corresponding to a different number of collinear (CL) and linearly stable (LS) points. As a matter of fact, there are four sub-cases in which the system has either 0 or 2 linearly stable points and 0 or 4 CL points. It should be noted that for $A > 0$ the system has always 4 non-CL points which are all unstable.

Moreover, all CL points are ID1 saddles, while the two equilibria located on the vertical $y$ axis are ID2 saddles.

The diagrams in Fig. 4(a-d) provide the positions of the 6 equilibria of the system, through the intersection points of the iso-contours of the curves $\Phi_x = 0$ and $\Phi_y = 0$. In panels (a) and (b) we see that there are sub-cases with 6 non-CL points. In these cases, the libration points $L_3$ and $L_4$ are either LS or unstable and always ID2 saddles, while
the rest four of them are always unstable ID1 saddles of the effective potential. Furthermore, there are two more sub-cases (see panels (c) and (d) of Fig. 4) where the system has 4 CL points and two more on the vertical $y$ axis. These two points $L_5$ and $L_6$ are either LS or unstable ID2 saddles, while all the four CL points are always unstable ID1 saddles.

3.2. Scenario II: 10 equilibria

According to the next probable scenario, the system has 10 equilibria in total. Then, as we can see in Fig. 5(a-b) there are two sub-cases regarding the types of the libration points. In the first case, corresponding to the green basins, there are two LS and four CL points, while in the second case (corresponding to the red basins) the system again has four CL points but the total number of LS points is now

Figure 7: Similar to Fig. 3, when 14 equilibria exist: 4 LS + 4 CL (red), and 4 LS + 8 CL (green).

Figure 8: Similar to Fig. 4, when 14 equilibria exist.
four. It is seen, the second case appears only for relatively low values of $A$ ($A < -0.25$) and for extremely low values of the mass parameter $\beta$ ($\beta < 0.1$).

In Fig. 6(a-b) we provide the diagrams with the iso-contours for the two sub-cases. In the first case, the two CL points $L_2$ and $L_3$ are LS and maxima of the effective potential, the two points $L_7$ and $L_8$ on the vertical axis are unstable ID2 saddles, while all the rest of them are unstable ID1 saddle points. On the other hand, in the second case, there are six libration points on the $y$ axis. Our analysis suggests, that $L_5$ and $L_6$ are LS and ID2 saddles, $L_7$ and $L_8$ are LS and maxima of the effective potential, the rest six equilibria are all unstable and ID1 saddles. It should be noted, that in this sub-case six of the total 10 equilibria are located in the near vicinity of the three primary bodies, while the outer libration points $L_1$, $L_4$, $L_5$, and $L_6$ are located relatively far away from the primaries.

3.3. Scenario III: 14 equilibria

The last case under investigation involves the scenario according to which the system has 14 equilibria in total. In this case, there are two sub-cases regarding the specific types of the libration points (see Fig. 7(a-b)). In both sub-cases, the system has four LS points, while the number of CL points is either 4 or 8.

The locations of all the equilibrium points are illustrated in the diagrams of Fig. 8(a-b). For the first sub-case (see panel (a)), the CL points $L_2$ and $L_3$ as well as the equilibria $L_9$ and $L_{10}$ are LS and maxima, $L_7$ and $L_8$ are unstable ID2 saddles, while all the rest of them are unstable ID1 saddle points. For the second sub-case, our analysis indicates that the CL points $L_2$, $L_3$, $L_6$, and $L_7$ are LS and maxima, $L_{11}$ and $L_{12}$ unstable and ID2 saddles, while all the others are unstable and ID1 saddles.

4. Concluding Remarks

In this work, we considered the restricted version of the collinear 4-body problem with peripheral bodies that have either prolate or oblate shapes and equal masses. For simplicity reasons, we assumed that both peripheral primaries have the same shape ($A_1 = A_2 = A$) so as to reduce the number of the involved parameters. Our mission was to reveal the dynamics of the equilibrium points of the system. For this purpose, we deployed standard root-finding numerical methods of high accuracy for determining the coordinates of the equilibria and also for computing their linear stability.

We managed to unveil how the two parameters of the system, that is the oblateness coefficient $A$ and the mass parameter $\beta$, affect the equilibria dynamics through a systematic and thorough scan of the parameters space. Our analysis reported the following important conclusions:

1. When $\beta > 0$ the number of the total libration points is always even, that is 6, 10, or 14.

2. When the two peripheral primaries are oblate spheroids ($A > 0$) there exist always 6 equilibria, while for $A < 0$ their total number depends on the specific values of $A$ and $\beta$.

3. The libration points of the system can be either maxima of the effective potential, ID1, or ID2 saddles.

Before closing, we would like to emphasize that the problem of the collinear restricted 4-body problem has several physical applications in real-life systems. To begin with, the particular mathematical model can describe the motion of particles around large (exo)-planets with many moons, such as the system of Saturn-Janus-Epimetheus. Moreover, it can also predict the motion of test particles around a system of a host star and two peripheral exoplanets of about equal masses, or even the motion around a triple system of stars where the two peripherals have about the same mass.

For the application of the N-R scheme, we used FORTRAN 77 (see e.g., (Press et al., 1992)) routines that allowed us to determine with high accuracy the values of the coordinates of the system’s libration points. For the graphics, we utilized the latest version of Mathematica® software (Wolfram, 2003).

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