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Generalized Kinematics Analysis of Hybrid Mechanisms Based on Screw Theory and Lie Groups Lie Algebras

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ORIGINAL ARTICLE

Generalized Kinematics Analysis of Hybrid Mechanisms Based on Screw Theory and Lie Groups Lie Algebras

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Received June xx, 201x; revised February xx, 201x; accepted March xx, 201x

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Abstract: The advanced mathematical tools are used to conduct research on the kinematics analysis of hybrid mechanisms, and the generalized analysis method and concise kinematics transfer matrix are obtained. First, according to the kinematics analysis of serial mechanisms, the basic principles of Lie groups Lie algebras in dealing with the spatial switching and differential operations of screw vectors are briefly explained. Then, based on the standard ideas of Lie operations, the method for kinematics analysis of parallel mechanisms is derived, and Jacobian matrix and Hessian matrix are formulated both recursively and in closed form. After that, according to the mapping relationship between the parallel joints and the corresponding equivalent series joints, a forward kinematics analysis method and two inverse kinematics analysis methods of hybrid mechanisms are studied. A case study is performed to verify the calculated matrices, in which a humanoid hybrid robotic arm with the parallel-series-parallel configuration is taken as an example. Simulation experiment results show that the obtained formulas are exact and the proposed method for kinematics analysis of hybrid mechanisms is practicable.

Keywords: Hybrid mechanism • Screw theory • Lie groups Lie algebras • Kinematics analysis • Humanoid robotic arm

1 Introduction

As robotic technology evolves, various types of robots are entering in our daily life and increasingly being applied to

assist humans in many different fields [1]. At present, the research hotspots of robotics are still focused on the lightweight design and compliance control [2, 3]. However, although advanced control algorithms and drive technologies have expanded the application of robots, their further application is limited by the inherent characteristics of typical robots [4].

The serial mechanism with a large workspace and flexible movement is the typical configuration of robots [4]. The serial robot with joint actuators mounted has a bulky mechanical structure, large moment of inertia, and low payload to weight ratio. As compared with the serial mechanism, the parallel mechanism has several advantages of greater stiffness, higher payload to weight ratio, reduced inertia, and higher precision [5,6]. Although the parallel mechanism effectively compensates for the shortcomings of serial mechanisms, it also has the disadvantage of small working space. Therefore, the hybrid mechanism which combines the advantages of the serial mechanism and the parallel mechanism has attracted attentions in variety of scientific and industrial applications.

The Denavit-Hartenberg method [7, 8] is commonly used to design and analyse the hybrid mechanism. Li et al. [9] explored a better kinematic performance and design scheme for a novel mechanical leg. Pinski et al. [10] investigated a 4 degrees-of-freedom (DOF) hybrid parallel-serial slave mechanism and developed a bilateral haptic controller to compensate for coupling and assembly errors. Liu et al. [11] studied a bionic flexible manipulator driven by pneumatic muscle actuator and designed a fuzzy torque control algorithm based on the computed torque method. Ling et al. [12] presented a kinetostatic modeling method for flexure-hinge-based compliant mechanisms with hybrid serial-parallel substructures to provide accurate

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and concise solutions by combining the matrix displacement method with the transfer matrix method. Hu [13] proposed a serial-parallel hybrid mechanism formed by two well-known Tricept parallel manipulators connected in serial and derived simple and compact formulae for the forward and inverse acceleration based on vector approach. However, the Denavit-Hartenberg method involves several weaknesses including a singularity problem and difficulty in locating the immediacy of physical meaning in differential kinematics. In addition, the existing studies rarely involves the generalized kinematic analysis method applicable to hybrid mechanisms.

On the other hand, screw theory [14] and Lie groups Lie algebras [15] are the useful mathematical tool and provide simplified symbolic representation that can be used to obtain geometric-intuitive kinematic analyses. Recently, various studies applied screw theory and Lie groups Lie algebras to robotic applications and focused on simplifying the complicated methods involved. Li et al. [16] extended the method to the kinematic analysis and derived the closed-solution of inverse kinematics for serial mechanisms. Dai et al. [17, 18] combined screw theory with Lie group algebra and summarized the related knowledge according to its relation with mechanisms. Huang et al. [19] presented a systematic approach for the kinematic calibration of a 6-DOF hybrid polishing robot and formulated the linearized error model based on screw theory. Li et al. [20, 21] presented the type synthesis of parallel mechanisms according to screw theory and Lie group. Sun et al. [22, 23] proposed a generalized method to solve inverse kinematics of serial and parallel mechanisms using finite screw. Gabardi et al. [24] investigated the kinematics analysis of a 4-UPU fully parallel manipulator and performed the analysis of actuation Jacobian, constraint Jacobian, and singularity configurations by screw theory. Liu et al. [25, 26] analyzed the comprehensive interaction mechanism of motion-force transmissibility to the acceleration capacity of robots and used the performance atlases method to perform the parameters optimization for different types of parallel robots. Hence, it is feasible and tentative to propose a generalized method of kinematic analysis for hybrid mechanisms.

In our previous work [27], a novel 8-DOF hybrid manipulator is proposed to realize a kinematic function similar to that of the human arm, as shown in Figure 1. Furthermore, a closed-form solution for the inverse displacement problem of the hybrid humanoid robotic arm (HRA) is derived. In this work, we focus on the application of screw theory and Lie groups Lie algebras in the field of kinematics analysis of hybrid mechanisms. This paper is

organized as follows. In Section 2, the method of kinematics analysis for hybrid mechanisms are proposed. In Section 3, the method is illustrated by the example of the humanoid HRA. In Section 4, the accuracy of the proposed method is verified through the simulation experiment. Finally, conclusions are discussed in Section 5.

Figure 1 Three-dimensional model of the HRA (left arm)

2 Kinematics Analysis of Mechanisms

According to the kinematics analysis of serial mechanisms, the basic principles of Lie operations in dealing with the spatial switching and differential operations of screw vectors are briefly explained, and the generalized kinematics analysis of parallel and hybrid mechanisms are established. A detailed description of screw theory and Lie groups Lie algebras are available in previous studies [14, 15, 18].

2.1 Kinematics Analysis of Serial Mechanisms

The Jacobian matrix of series kinematic chains can be obtained as:

$$\begin{cases} \mathbf{J}(\boldsymbol{\theta}) = [\xi_1 \ \xi'_2 \ \mathbf{L} \ \xi'_n] \\ \xi'_i = \mathbf{Ad}_{\exp(\xi_1, \theta_1) \mathbf{L} \exp(\xi_{i-1}, \theta_{i-1})} \xi_i \end{cases}, \quad (1)$$

where ξ_i denotes the initial unit screw vector of the i -th motion pair, ξ'_i denotes the real-time unit screw vector of the i -th motion pair, $\mathbf{Ad}$ denotes the concomitant effect of Lie groups on Lie algebras, $\exp$ denotes exponential product formula.

The forward kinematics analysis of series kinematic chains can be derived as:

$$\mathbf{V}_o = \mathbf{J}_o(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} = \begin{bmatrix} \omega_o \\ \mathbf{v}_o \end{bmatrix}, \quad (2)$$

$$\boldsymbol{\varepsilon}_o = \mathbf{J}_o(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} + \mathbf{J}_o^*(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}}, \quad (3)$$

where $\mathbf{V}_o$ denotes the velocity vector of the point on the

end-platform that coincides with the origin of the base coordinate system, ε_O denotes the acceleration vector of this point, ω_O denotes the orientation velocity vector of this point, ν_O denotes the linear velocity vector of this point.

The differential matrix of Jacobian matrix can be calculated as:

$$\begin{cases} \dot{\mathcal{J}}_O(\theta) = [\dot{\xi}_1 \ \dot{\xi}_2 \ L \ \dot{\xi}_n] \\ \dot{\xi}_i = (\dot{\phi}_1 \xi_1 + L + \dot{\phi}_{i-1} \cdot \xi'_{i-1}) \times \xi'_i \\ \dot{\mathcal{J}}_O(\theta) \dot{\phi} = \dot{\phi}_1 \xi_1 \times (\xi'_2 \cdot \dot{\phi}_2 + L + \xi'_n \cdot \dot{\phi}_n) \\ \quad + \dot{\phi}_2 \xi_2 \times (\xi'_3 \cdot \dot{\phi}_3 + L + \xi'_n \cdot \dot{\phi}_n) \\ \quad + L + \dot{\phi}_{n-1} \cdot \xi'_{n-1} \times \xi'_n \cdot \dot{\phi}_n \end{cases} \quad (4)$$

Thus, Eq. (3) can be expressed as:

$$\varepsilon_O = J_O(\theta) \dot{\phi} + \dot{\mathcal{H}}_O(\theta) \dot{\phi}, \quad (5)$$

where $\dot{\mathcal{H}}_O(\theta) = \begin{cases} \xi'_m \times \xi'_n & m < n \\ \mathbf{0} & \text{other cases} \end{cases}$, m denotes the number of rows of the matrix, n denotes the number of columns of the matrix.

The forward kinematics analysis of series mechanisms can be derived as:

$$V_M = \begin{bmatrix} \omega_M \\ \nu_M \end{bmatrix} = \begin{bmatrix} \omega_O \\ \nu_O + \omega_O \times P_M \end{bmatrix} = J_M(\theta) \dot{\phi}, \quad (6)$$

$$\varepsilon_M = J_M(\theta) \ddot{\phi} + \dot{\mathcal{H}}_M(\theta) \dot{\phi} + c_M, \quad (7)$$

where P_M denotes the position vector of the end-reference point, V_M denotes the velocity vector of this point, ε_M denotes the acceleration vector of this point, ω_M denotes the orientation velocity vector of this point, ν_M denotes the linear velocity vector of this point, $c_M = \begin{bmatrix} \mathbf{0}_{3 \times 1} \\ \omega_M \times \nu_M \end{bmatrix}$.

Obviously, in the non-singular configurations, the inverse kinematics analysis of series mechanisms can be calculated as:

$$\dot{\phi} = J(\theta)^{-1} V_M, \quad (8)$$

$$\ddot{\phi} = J(\theta)^{-1} (\varepsilon_M - \dot{\mathcal{H}}_M(\theta) \dot{\phi} - c_M). \quad (9)$$

2.2 Kinematics Analysis of Parallel Mechanisms

Due to the coupling effect of kinematic chains on the active platform, the kinematics transfer matrices (J and H) cannot be directly obtained.

2.2.1 Velocity Analysis of Parallel Mechanisms

The forward velocity analysis of parallel mechanisms can be obtained as:

$$V_M = J_M^{\phi} \dot{\phi} = [J_M^{\phi}]^{(b)} \dot{\phi}^{(b)}, \quad (10)$$

where J_M^{ϕ} denotes the Jacobian matrix of the end-platform based on active pairs, $\dot{\phi}^{(b)}$ denotes the velocity vector of

active pairs, $[J_M^{\phi}]^{(b)}$ denotes the Jacobian matrix of the b -th kinematic chain, $\dot{\phi}^{(b)}$ denotes the velocity vector of motion pairs of the b -th kinematic chain.

Meanwhile, in the non-singular configurations, the inverse velocity analysis of parallel mechanisms can be derived as:

$$\begin{cases} \dot{\phi} = J_M^{\phi} V_M \\ \dot{\phi}^{(b)} = [J_M^{\phi}]^{(b)} V_M = [J_M^{\phi}]^{-1(b)} V_M \end{cases} \quad (11)$$

Assuming that the first motion pair of each kinematic chain acts as the active pairs, $\dot{\phi}^{(b)}$ can be expressed as:

$$\dot{\phi}^{(b)} = [\dot{\phi}_1^{(b)} \ \dot{\phi}_2^{(b)} \ L \ \dot{\phi}_n^{(b)}]^T. \quad (12)$$

According to Eq. (11), J_M^{ϕ} and J_M^{θ} can be calculated as:

$$J_M^{\phi} = [J_M^{\phi}]^{-1}, \quad (13)$$

$$J_M^{\theta} = \begin{bmatrix} [J_M^{\theta}]^{(1)} & [J_M^{\theta}]^{(2)} & L & [J_M^{\theta}]^{(b)} \end{bmatrix}^T, \quad (14)$$

where $(1 :)$ denotes the first row of matrices.

2.2.2 Acceleration Analysis of Parallel Mechanisms

Obviously, the forward acceleration analysis of parallel mechanisms can be obtained as:

$$\begin{aligned} \varepsilon_M &= J_M^{\phi} \ddot{\phi} + \dot{\mathcal{H}}_M^{\phi} \dot{\phi} + c_M \\ &= [J_M^{\phi} \ddot{\phi} + \dot{\mathcal{H}}_M^{\phi} \dot{\phi}]^{(b)} + c_M \end{aligned} \quad (15)$$

Meanwhile, the inverse acceleration analysis of parallel mechanisms can be derived as:

$$\begin{cases} \ddot{\phi} = J_M^{\phi} (\varepsilon_M - \dot{\mathcal{H}}_M^{\phi} \dot{\phi} - c_M) \\ = J_M^{\phi} \varepsilon_M - \dot{\mathcal{H}}_M^{\phi} \dot{\phi} - J_M^{\phi} (c_M) \\ \ddot{\phi}^{(b)} = [J_M^{\phi} (\varepsilon_M - \dot{\mathcal{H}}_M^{\phi} \dot{\phi} - c_M)]^{(b)} \end{cases} \quad (16)$$

According to Eqs. (12), (14) and (16), $\ddot{\phi}^{(b)}$ also can be calculated as:

$$\ddot{\phi}^{(b)} = \begin{bmatrix} [J_M^{\theta}]^{(1)} \\ [J_M^{\theta}]^{(2)} \\ \vdots \\ [J_M^{\theta}]^{(b)} \end{bmatrix}_M \varepsilon_M - \begin{bmatrix} \dot{\phi}^{(1)} [L_1]^{(1)} \dot{\phi}^{(1)} \\ \dot{\phi}^{(2)} [L_1]^{(2)} \dot{\phi}^{(2)} \\ \vdots \\ \dot{\phi}^{(b)} [L_1]^{(b)} \dot{\phi}^{(b)} \end{bmatrix}_M - \begin{bmatrix} [J_M^{\theta}]^{(1)} \\ [J_M^{\theta}]^{(2)} \\ \vdots \\ [J_M^{\theta}]^{(b)} \end{bmatrix}_M c_M, \quad (17)$$

where $[L_1]^{(b)} = [J_M^{\theta}]^T [[J_M^{\theta}]^{(1)}]^T * H_M^{\phi} [J_M^{\theta}]^{(b)}$,

$[J_M^{\theta}]^{(b)} = [J_M^{\theta}]^{(b)} J_M^{\phi}$, $*$ indicates the generalized scalar product of matrices, the matrix before $*$ is equivalent to a constant.

Thus H_M^{ϕ} and H_M^{θ} can be expressed as:

$$H_M^{\phi} = J_M^{\phi} * H_M^{\theta}, \quad (18)$$

$$\mathbf{H}_M^\phi = \left[\begin{bmatrix} \mathbf{L} \end{bmatrix}_l^{(1)} \quad \begin{bmatrix} \mathbf{L} \end{bmatrix}_l^{(2)} \quad \mathbf{L} \quad \begin{bmatrix} \mathbf{L} \end{bmatrix}_l^{(b)} \right]^T. \quad (19)$$

2.3 Kinematics Analysis of Hybrid Mechanisms

The realization of the conversion of parallel joints and equivalent series joints is the key to the kinematics analysis of hybrid mechanisms. In our previous work [26], the inverse displacement problem of hybrid mechanisms was solved based on the equivalent series mechanism. In general, the equivalent series manipulator can be obtained according to the DOF of hybrid mechanisms. Therefore, there is no doubt that the equivalent series joint has the same kinematics characteristics as the parallel joint:

$$\mathbf{V}_M = \mathbf{J}_\psi^M \dot{\psi} = \mathbf{J}_\phi^M \dot{\phi} = \left[\mathbf{J}_\theta^M \dot{\theta} \right]^{(b)}, \quad (20)$$

$$\begin{aligned} \boldsymbol{\varepsilon}_M &= \mathbf{J}_\psi^M \ddot{\psi} + \dot{\psi}^T \mathbf{H}_\psi^M \dot{\psi} + \mathbf{c}_M \\ &= \mathbf{J}_\phi^M \ddot{\phi} + \dot{\phi}^T \mathbf{H}_\phi^M \dot{\phi} + \mathbf{c}_M, \quad (21) \\ &= \left[\mathbf{J}_\theta^M \ddot{\theta} + \dot{\theta}^T \mathbf{H}_\theta^M \dot{\theta} \right]^{(b)} + \mathbf{c}_M \end{aligned}$$

where $\dot{\psi} = \left[\dot{\phi}_{E1} \quad \dot{\phi}_{E2} \quad \mathbf{L} \quad \dot{\phi}_{Ei} \right]^T$ indicates the motion pairs of the equivalent series manipulator.

The velocity and acceleration vectors of motion pairs of the equivalent series joints can be obtained as:

$$\dot{\psi} = \mathbf{J}_M^\psi \mathbf{J}_\phi^M \dot{\phi} \quad (22)$$

$$\ddot{\psi} = \mathbf{J}_M^\psi \left(\mathbf{J}_\phi^M \ddot{\phi} + \dot{\phi}^T \mathbf{H}_\phi^M \dot{\phi} + \dot{\psi}^T \mathbf{H}_\psi^M \dot{\psi} \right) \quad (23)$$

2.3.1 Forward Kinematics Analysis of Hybrid Mechanisms

According to the kinematics analysis of serial and parallel mechanisms, the generalized method for forward kinematics analysis of hybrid mechanisms is proposed, whose flow diagram is shown in Figure 2. Firstly, the velocity and acceleration vectors of motion pairs of the equivalent series joints can be calculated based on Eqs. (22) and (23). Then, by applying Eqs. (6) and (7), the kinematics characteristics of the end-platform can be obtained. In this paper, we take the HRA as an example to demonstrate the method in Section 3.

Figure 2 Forward kinematics analysis of hybrid mechanisms

2.3.2 Inverse Kinematics Analysis of Hybrid Mechanisms

There are two similar methods for the inverse kinematics analysis of hybrid mechanisms. The flow diagram of the first method is shown in Figure 3. Firstly, the velocity and acceleration vectors of motion pairs of the equivalent series manipulator can be calculated based on Eqs. (8) and (9). Then, by applying Eqs. (20) and (21), the velocity and acceleration vectors of moving platform of the parallel joints can be obtained. Finally, the kinematics characteristics of all the motion pairs of the hybrid manipulator can be derived by applying Eqs. (11) and (16).

When a certain kinematic chain of a parallel joint has the same DOF as itself, this chain can replace the equivalent series joint. The flow diagram of this method is shown in Figure 4, and its solution procedure is similar to that of the first method. In this paper, we take the HRA as an example to demonstrate the two methods in Section 3.

Figure 3 First method for the inverse kinematics analysis of hybrid mechanisms

Figure 4 Second method for the inverse kinematics analysis of hybrid mechanisms

3 Kinematics Analysis of Humanoid Robotic Arm

3.1 Structural Configuration

The humanoid shoulder joint (HSJ) is based on a spherical 5R parallel mechanism, as shown in Figure 5(a). The initial unit axis vectors of all the revolute pairs in the HSJ are shown in Figure 5(b).

(a) Three-dimensional model of the HSJ

(b) Mechanism diagram of the HSJ

Figure 5 Structural configuration of the HSJ

The humanoid elbow joint (HEJ) is based on a series 3-DOF kinematic chain RRR, as shown in Figure 6(a). The initial unit axis vectors of all the revolute pairs in the HEJ are shown in Figure 6(b).

(a) Three-dimensional model of the HEJ

the fixed platform of HWJ

(b) Mechanism diagram of the HEJ

Figure 6 Structural configuration of the HEJ

The humanoid wrist joint (HWJ) is based on a spherical 3-RRP parallel mechanism, as shown in Figure 7(a). The initial unit axis vectors of all the motion pairs in the HWJ are shown in Figure 7(b).

Figure 7 Structural configuration of the HWJ

The HRA can be equivalent to a series robotic arm, as shown in Figure 8.

Figure 8 Mechanism diagram of the equivalent series robotic arm

3.2 Pre-processing of Parallel Joints

In order to make the kinematics analysis of the hybrid HRA clearer, the Jacobian matrices of the parallel joints and the corresponding equivalent series joints are calculated first.

3.2.1 Pre-processing of Humanoid Shoulder Joint

The HSJ has only 2 degrees of rotational freedom, so the Jacobian matrix of the kinematics chain 1 can be obtained according to Eqs. (6) and (8):

$$\begin{cases} \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]^1 = \left[S_{A_i} \ S_{B_i} \ S_{C_i} \right] \\ \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]^1 = \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]^1 \end{cases} \quad (24)$$

where S'_i denotes the real-time unit axis vector of the i -th motion pair.

However, for the kinematic chain 2, a virtual revolute pair D_2 is added to make the Jacobian matrix a square

matrix, as the form:

$$\begin{cases} \xi_{D_2} = \left[S_{D_2} \ S_{D_2}^0 \right]^T \\ \theta_{D_2} = \phi_{D_2} = \phi_{D_2}^0 = 0 \end{cases} \quad (25)$$

where $S_{D_2} = [0 \ 0 \ 1]^T$, $S_{D_2}^0 = [0 \ 0 \ 0]^T$.

Similarly, the Jacobian matrix of the kinematic chain 2 can be obtained according to Eqs. (6) and (8):

$$\begin{cases} \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]^2 = \left[S_{A_2} \ S_{C_2}' \ S_{D_2}' \right] \\ \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]^2 = \left[S_{A_2} \ S_{C_2}' \right] \\ \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]^2 = \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]^2^{-1} \\ \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]^2 = \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]^2_1 \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]^2_2 \end{cases} \quad (26)$$

where (:1) denotes the first column of matrices.

According to Eqs. (13) and (14), the Jacobian matrix of the HSJ based on the active pairs can be obtained,

$$\begin{cases} \left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{SJ} = \left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj}^{-1} \\ \left[J_{\omega} \right]_{\theta_{Sj}}^{SJ} = \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]_1 \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{SJ} \right]_2 \\ \left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} = \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]_1 \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]_2 \left[\left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]_3^T \\ \left[J_{\omega} \right]_{\theta_{Sj}}^{Sj} = \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]_1 \left[\left[\bar{J}_{\omega} \right]_{\theta_{Sj}}^{Sj} \right]_2 \end{cases} \quad (27)$$

For the corresponding equivalent series joint, the Jacobian matrix can be obtained according to Eqs. (6) and (8):

$$\begin{cases} \left[J_{\omega} \right]_{\psi_{Sj}}^{SJ} = \left[S_{Y_i} \ S_{X_i}' \right] \\ \left[J_{\omega} \right]_{\psi_{Sj}}^{Sj} = \left[J_{\omega} \right]_{\psi_{Sj}}^{SJ}^T \end{cases} \quad (28)$$

3.2.2 Pre-processing of Humanoid Wrist Joint

The HWJ has 3 degrees of rotational freedom, so the Jacobian matrix of the kinematic chains can be obtained according to Eqs. (6) and (8):

$$\begin{cases} \left[\left[J_{\omega} \right]_{\theta_{Wj}}^{WJ} \right]^b = \left[{}^3S_{H_b} \ {}^3S_{K_b}' \ {}^3S_{P_b}' \right] \\ \left[\left[J_{\omega} \right]_{\theta_{Wj}}^{Wj} \right]^b = \left[\left[J_{\omega} \right]_{\theta_{Wj}}^{WJ} \right]^b^{-1} \end{cases} \quad (29)$$

where label 3 in the upper left corner indicates that the calculation is performed in the coordinate system O_3 - $X_3Y_3Z_3$.

According to Eqs. (13) and (14), the Jacobian matrix of the HWJ based on the active pairs can be obtained:

$$\begin{cases} {}^3[J_\omega]_{\phi_{WJ}}^{WJ} = [{}^3[J_\omega]_{\phi_{WJ}}^{\phi_{WJ}}]^{-1} \\ {}^3[J_\omega]_{\phi_{WJ}}^{\phi_{WJ}} = \begin{bmatrix} [{}^3[J_\omega]_{\phi_{WJ}}^{\phi_{WJ}}]_1^T & [{}^3[J_\omega]_{\phi_{WJ}}^{\phi_{WJ}}]_2^T & [{}^3[J_\omega]_{\phi_{WJ}}^{\phi_{WJ}}]_3^T \end{bmatrix}^T \end{cases} \quad (30)$$

For the corresponding equivalent series joint, the Jacobian matrix can be obtained according to Eqs. (6) and (8):

$$\begin{cases} {}^3[J_\omega]_{\psi_{WJ}}^{WJ} = [{}^3S_{Z_4} \quad {}^3S'_{Y_4} \quad {}^3S'_{X_4}] \\ {}^3[J_\omega]_{\phi_{WJ}}^{WJ} = [{}^3[J_\omega]_{\psi_{WJ}}^{WJ}]^{-1} \end{cases} \quad (31)$$

3.3 Forward Kinematics Analysis of Humanoid Robotic Arm

According to Eq. (22), the velocity vector of motion pairs of the corresponding equivalent series joint can be calculated as:

$$\begin{cases} \dot{\phi}_{SJ} = [J_\omega]_{\phi_{SJ}}^{\psi_{SJ}} [J_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} \\ \dot{\phi}_{WJ} = {}^3[J_\omega]_{\phi_{WJ}}^{\psi_{WJ}} {}^3[J_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} \end{cases} \quad (32)$$

Meanwhile, according to Eq. (11), the velocity vectors of passive motion pairs of the corresponding parallel joint can be calculated as:

$$\begin{cases} \dot{\phi}_{SJ}^{(b)} = [J_\omega]_{\phi_{SJ}}^{\theta_{SJ}} [J_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} \\ \dot{\phi}_{WJ}^{(b)} = [{}^3[J_\omega]_{\phi_{WJ}}^{\theta_{WJ}}]^{(b)} {}^3[J_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} \end{cases} \quad (33)$$

Thus, according to Eqs. (18) and (19), $[H_\omega]_{\phi_{SJ}}^{SJ}$ and ${}^3[H_\omega]_{\phi_{WJ}}^{WJ}$ can be calculated. Then, according to Eq. (23), the acceleration vector of the corresponding equivalent series joint can be calculated as:

$$\begin{cases} \ddot{\phi}_{SJ} = [J_\omega]_{\phi_{SJ}}^{\psi_{SJ}} \left([J_\omega]_{\phi_{SJ}}^{SJ} \ddot{\phi}_{SJ} + \dot{\phi}_{SJ}^T [H_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} - \dot{\phi}_{SJ}^T [H_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} \right) \\ \ddot{\phi}_{WJ} = {}^3[J_\omega]_{\phi_{WJ}}^{\psi_{WJ}} \left({}^3[J_\omega]_{\phi_{WJ}}^{WJ} \ddot{\phi}_{WJ} + \dot{\phi}_{WJ}^T {}^3[H_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} - \dot{\phi}_{WJ}^T {}^3[H_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} \right) \end{cases} \quad (34)$$

Finally, according to Eqs. (6) and (7), the forward kinematics analysis of the HRA can be obtained as:

$$\begin{cases} V_{HRA} = J_\psi^{HRA} \dot{\psi}_{HRA} \\ \varepsilon_{HRA} = J_\psi^{HRA} \ddot{\psi}_{HRA} + \dot{\psi}_{HRA}^T H_\psi^{HRA} \dot{\psi}_{HRA} + c_{HRA} \end{cases}, \quad (35)$$

where $J_\psi^{HRA} = [\xi'_{Y_1} \quad \xi'_{X_1} \quad \xi'_{D_1} \quad \xi'_{E_1} \quad \xi'_{F_1} \quad \xi'_{Z_4} \quad \xi'_{Y_4} \quad \xi'_{X_4}]$,
 $\psi_{HRA} = [\psi_{Y_1} \quad \psi_{X_1} \quad \theta_D \quad \theta_E \quad \theta_F \quad \psi_{Z_4} \quad \psi_{Y_4} \quad \psi_{X_4}]^T$,
 $H_\psi^{HRA} = \begin{cases} \xi'_m \times \xi'_n & m < n \\ 0 & \text{other cases} \end{cases}$,

$$m, n = Y_1, X_1, D, E, F, Z_4, Y_4, X_4.$$

3.4 Inverse Kinematics Analysis of Humanoid Robotic Arm

3.4.1 The First Method

According to Eqs. (8) and (9), the velocity and acceleration vectors of the equivalent series manipulator can be calculated as:

$$\begin{cases} \dot{\psi}_{HRA} = [J_\psi^{HRA}]^{-1} V_{HRA} \\ \ddot{\psi}_{HRA} = [J_\psi^{HRA}]^{-1} [\varepsilon_{HRA} - \dot{\psi}_{HRA}^T H_\psi^{HRA} \dot{\psi}_{HRA} - c_{HRA}] \end{cases} \quad (36)$$

According to Eqs. (20) and (21), the velocity and acceleration vectors of moving platforms of the parallel joints can be calculated as:

$$\begin{cases} V_{SJ} = [J_\omega]_{\psi_{SJ}}^{SJ} \dot{\psi}_{SJ} \\ \varepsilon_{SJ} = [J_\omega]_{\psi_{SJ}}^{SJ} \ddot{\psi}_{SJ} + \dot{\psi}_{SJ}^T [H_\omega]_{\psi_{SJ}}^{SJ} \dot{\psi}_{SJ} + c_{SJ} \end{cases}, \quad (37)$$

$$\begin{cases} {}^3V_{WJ} = {}^3[J_\omega]_{\psi_{WJ}}^{WJ} \dot{\psi}_{WJ} \\ {}^3\varepsilon_{WJ} = {}^3[J_\omega]_{\psi_{WJ}}^{WJ} \ddot{\psi}_{WJ} + \dot{\psi}_{WJ}^T {}^3[H_\omega]_{\psi_{WJ}}^{WJ} \dot{\psi}_{WJ} + {}^3c_{WJ} \end{cases} \quad (38)$$

According to Eqs. (11) and (16), the velocity and acceleration vectors of all the motion pairs of the parallel joints can be obtained as:

$$\begin{cases} \dot{\phi}_{SJ}^{(b)} = [J_\omega]_{\phi_{SJ}}^{\theta_{SJ}} [J_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} \\ \dot{\phi}_{WJ}^{(b)} = [{}^3[J_\omega]_{\phi_{WJ}}^{\theta_{WJ}}]^{(b)} {}^3[J_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} \end{cases}, \quad (39)$$

$$\begin{cases} \ddot{\phi}_{SJ}^{(b)} = [J_\omega]_{\phi_{SJ}}^{\theta_{SJ}} \left(\varepsilon_{SJ} - \dot{\phi}_{SJ}^T [H_\omega]_{\phi_{SJ}}^{SJ} \dot{\phi}_{SJ} - c_{SJ} \right) \\ \ddot{\phi}_{WJ}^{(b)} = [{}^3[J_\omega]_{\phi_{WJ}}^{\theta_{WJ}}]^{(b)} \left({}^3\varepsilon_{WJ} - \dot{\phi}_{WJ}^T {}^3[H_\omega]_{\phi_{WJ}}^{WJ} \dot{\phi}_{WJ} - {}^3c_{WJ} \right) \end{cases} \quad (40)$$

3.4.2 The Second Method

According to the mobility analysis of the hybrid HRA, the kinematic chain 2 of the HSJ and the kinematic chain 1 (or 2, or 3) of the HWJ are selected to form the corresponding branch series robotic arm.

According to Eqs. (8) and (9), the velocity and acceleration vectors of the branch series manipulator can be calculated as:

$$\begin{cases} \dot{\psi}_{B-HRA} = [J_\psi^{B-HRA}]^{-1} V_{HRA} \\ \ddot{\psi}_{B-HRA} = [J_\psi^{B-HRA}]^{-1} [\varepsilon_{HRA} - \dot{\psi}_{B-HRA}^T H_\psi^{B-HRA} \dot{\psi}_{B-HRA} - c_{HRA}] \end{cases} \quad (41)$$

where $J_\psi^{B-HRA} = [\xi'_{A_2} \quad \xi'_{C_2} \quad \xi'_{D_2} \quad \xi'_{E_2} \quad \xi'_{F_2} \quad \xi'_{H_1} \quad \xi'_{K_1} \quad \xi'_{P_1}]$,
 $\psi_{B-HRAHRA} = [\theta_{A_2} \quad \theta_{C_2} \quad \theta_D \quad \theta_E \quad \theta_F \quad \theta_{H_1} \quad \theta_{K_1} \quad \theta_{P_1}]^T$,
 $H_\psi^{B-HRA} = \begin{cases} \xi'_m \times \xi'_n & m < n \\ 0 & \text{other cases} \end{cases}$,

$$m, n = A_2, C_2, D, E, F, H_1, K_1, P_1$$

According to Eqs. (20) and (21), the velocity and acceleration vectors of moving platforms of the parallel joints can be calculated as:

$$\begin{cases} \mathbf{V}_{SJ} = [\mathbf{J}_{\omega}^{SJ} \dot{\theta}_{SJ}]^2 \\ \mathbf{e}_{SJ} = [\mathbf{J}_{\omega}^{SJ} \ddot{\theta}_{SJ} + \ddot{\theta}_{SJ} [\mathbf{H}_{\omega}^{SJ} \dot{\theta}_{SJ} + \mathbf{c}_{SJ}]]^2 \end{cases}, \quad (42)$$

$$\begin{cases} {}^3\mathbf{V}_{WJ} = [{}^3\mathbf{J}_{\omega}^{WJ} \dot{\theta}_{WJ}]^1 \\ {}^3\mathbf{e}_{WJ} = [{}^3\mathbf{J}_{\omega}^{WJ} \ddot{\theta}_{WJ} + \ddot{\theta}_{WJ} [{}^3\mathbf{H}_{\omega}^{WJ} \dot{\theta}_{WJ} + {}^3\mathbf{c}_{WJ}]]^1 \end{cases}. \quad (43)$$

Similarly, according to Eq. (39) and (40), the velocity and acceleration vectors of all the motion pairs of the parallel joints can be obtained as.

4 Simulation Experiment

In order to verify the method of kinematics analysis, a verification scheme is proposed as shown in Figure. 9. Firstly, according to the kinematics information of the target trajectory, the velocity and acceleration vectors of the active pairs are obtained based on the inverse kinematics analysis. Then, the kinematics information of the end-moving platform is obtained according to the forward kinematics analysis.

Figure 9 Verification scheme

In this paper, the same target trajectory and given redundant inputs as our previous work [26] are selected, and the first set of solutions for the inverse displacement problem are used as the position-orientation information of the HRA.

The target trajectory:

$$\mathbf{g}_{st} = \begin{bmatrix} 1 & 0 & 0 & X_{st}(t) \\ 0 & 0 & 1 & -r_5 - r_6 \\ 0 & -1 & 0 & -r_4 + Z_{st}(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{cases} X_{st}(t) = h_1 \sin(\frac{2\pi t}{T}) \\ Z_{st}(t) = h_1 - h_1 \cos(\frac{2\pi t}{T}) \end{cases}. \quad (44)$$

$h_1 = 80\text{mm}$
 $T = 10\text{s}$

β_1 and F are selected as the given input variables:

$$\begin{cases} \beta_1 = -\frac{h_2}{2\pi} \sin(\frac{2\pi t}{T}) + h_2 \frac{t}{T}, h_2 = 10^\circ \\ \theta_F = 0 \end{cases}. \quad (45)$$

4.1 Inverse Kinematics Analysis

According to the first method of inverse kinematics analysis, the velocity and acceleration vectors of the active inputs can be calculated, as shown in Figure 10.

Figure 10 The first method of inverse kinematics analysis

Meanwhile, according to the second method of inverse kinematics analysis, the velocity and acceleration vectors of the active inputs can also be calculated, as shown in Figure 11.

(a) Velocity of active pairs

(b) Acceleration of active pairs

Figure 11 The second method of the inverse kinematics analysis

In order to verify whether the two methods are equivalent, the calculation errors between the two methods are obtained, as shown in Figure 12. Obviously, the calculation errors are almost equal to 0, the two methods of inverse kinematics analysis are equivalent.

(a) Errors in velocity

(b) Errors in acceleration

Figure 12 Errors between the two methods of the inverse kinematics analysis

4.2 Forward Kinematics Analysis

The first set of velocity and acceleration vectors of the active pairs are selected for the forward kinematic analysis. According to the method of forward kinematics analysis, the velocity and acceleration vectors of the end-moving platform can be calculated, as shown in Figure 13 and 14.

(a) Angular velocity

(b) Angular acceleration

Figure 13 Rotation of the end-moving platform

(a) Translational velocity

(b) Translational acceleration

Figure 14 Translation of the end-moving platform

Obviously, the orientation of the end-moving platform remains unchanged. In order to verify the accuracy of the method for forward kinematics analysis, the calculation errors between the forward kinematics and the target trajectory are obtained, as shown in Figure 15.

(a) Errors in translational velocity

(b) Errors in translational acceleration

Figure 15 Translational motion error between the forward kinematics and the target trajectory

Clearly, the calculation errors are almost equal to 0, the motion of the end-moving platform are exactly the same as the target trajectory.

Therefore, the accuracy of the methods for forward kinematics analysis and inverse kinematics analysis are all verified.

6 Conclusions

- (1) Compared to other methods for the kinematics analysis of hybrid mechanisms, the proposed methods based on screw theory and Lie groups Lie algebras has the obvious physical significance, and the kinematic transfer matrices ($\mathbf{J}$ and $\mathbf{H}$) can be expressed concisely and uniformly.
- (2) The effect of active pairs on the end-moving platform can be clearly shown. In addition, the velocity and acceleration vectors of all the motion pairs can be easily obtained, which lays the groundwork for the establishment of dynamics model.
- (3) The equivalent series mechanism and branch series mechanism are equivalent in dealing with the inverse kinematics analysis. Although the physical significance of transfer matrices base on the two methods are inconsistent, they do not change the transfer properties of the hybrid mechanism.

7 Declaration

Acknowledgements

The authors sincerely thanks to Professor Shi-Ming Ji of Zhejiang University of Technology for his critical discussion and reading during manuscript preparation.

Funding

Supported by Zhejiang Province Foundation for Distinguished Young Scholars of China (Grant No. LR18E050003), National Natural Science Foundation of China (Grant Nos. 51975523, 51475424, 51905481), and Open Foundation of the State Key Laboratory of Fluid Power and Mechatronic Systems (Grant No. GZKF-201906)

Availability of data and materials

The datasets supporting the conclusions of this article are included within the article.

Authors' contributions

The author's contributions are as follows: Y-BL was in charge of the whole trial; PS wrote the manuscript; KC, W-TZ, QZ and BC assisted with sampling and laboratory analyses.

Competing interests

The authors declare no competing financial interests.

Consent for publication

Not applicable

Ethics approval and consent to participate

Not applicable

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