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Article

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Abstract
In this study, a new scrambling response model is introduced to deal with a sensitive variable, and then following the proposed scrambling response model, a generalized estimator for variance estimation of sensitive variable based-on two auxiliary information. The expressions of the bias, the mean square error, and the minimum mean square error are derived up to the first order of approximation. Simulation and empirical results show that the proposed generalized estimator under both scrambling response models attains minimum mean square error as compared to existing estimators. Also, the degree of privacy protection of the respondent for the proposed model is evaluated through simulation and empirical study.

Keywords: Variance estimation; Exponential-type estimator; Scrambled randomized response; Sensitive variable, Privacy protection measure; Mean square error

1. Introduction
Information regarding complex characteristics for example family income, induced abortions, criminal activities, etc. may cause refusal bias, response bias, or both. The randomized response technique (RRT) was introduced by Warner (1965) to overcome such a difficult situation where the response is qualitative in nature. In RRT, the information attained ensures the privacy of the respondent. This work was extended to quantitative response models with scrambling responses. Pollock and Bek (1976) presented the theory of additive and multiplicative scrambling randomized response (SRR) technique. On the choice of scrambling mechanism, many researchers made effort to develop models such as Himmelfarb and Edgell (1980), Eichhorn and Hayre (1983), Gupta et al. (2010), Diana and Perri (2011), Hussain and Khan (2013) and many more. Besides, developments by different survey researchers are made to the estimation stage using additive models. Sousa et al. (2010) first presented ratio-type estimators estimating the population mean of the complex study variable using the non-delicate auxiliary variable. Additionally, Koyuncu et al. (2014), Gupta et al. (2017), Saleem et al. (2019), Sanaullah et al. (2020a, 2020b), Saleem and Sanaullah (2022), and Khalid et al. (2023) presented various estimators for the population mean of the complex variable using different RRT models.

In human regular life, variation remains existent everywhere. Naturally, not the two individuals or things are identical. In all fields, we necessitate estimating the population variance, such as the climate factors from place to place, the degree of blood pressure, etc. The medical researcher needs a suitable understanding of the level of variation of a particular HIV treatment dose curing or affecting from person to person to be able to plan whether to reduce or change the treatment for
a particular person. Practically, several situations can be seen where the estimation of population variance can be observed for complex issues. In survey sampling, the auxiliary variable is used to intensify the precision of population variance estimators at the stage of estimation. The work on the estimation of population variance for the non-sensitive variable of interest was done by countless statisticians such as Gupta and Shabbir (2008), Asghar et al. (2014), Sanaullah et al. (2014), and Niaz et al. (2021). Singh et al. (2015) firstly introduced a new estimator to estimate the population variance of the sensitive variable of interest centered on a multiplicative scrambled response model using auxiliary information. They presented different procedures for estimating variance using Das and Tripathi (1978) and Isaki (1983) estimators. Later, Gupta et al. (2020) presented three variance estimators under Diana and Perri’s (2011) RRT model using two scrambling variables. Aloraini et al. (2022) proposed some separate and combined variance estimators using stratified sampling following the strategy presented by Gupta et al. (2020).

The present study follows the methodology of Gupta et al. (2020) and suggested a new generalized exponential estimator, to estimate the variance of the finite population which is complex in nature. The jargon of the bias and the mean square error of the proposed estimator originated up to the first order of approximation. The outline of the article is organized as follows: In Section 2 the sampling strategy for the scrambled response model presented by Diana and Perri (2011) is discussed. Section 3, displays the proposed generalized estimator for two auxiliary variables under the existing model along with the expressions of bias and MSE. In section 4, we also propose a generalized randomized response model. The unbiased variance estimator, ratio estimator, and proposed generalized estimator are modified under the proposed model in the same section. The privacy protection measure for the models are discussed in section 5. To support the proposed methodology a simulation study is presented in Section 6 and some concluding interpretations are given in section 7.

2. Sampling Strategy for Scrambled response model
Let a simple random sample done without replacement (SRSWOR) of size \( n \) be drawn a finite population of \( U=\{U_1, U_2, ..., U_N\} \). Let \( Y \) be a true response of sensitive quantitative variables and \( Y \) be the non-sensitive auxiliary variable, positively correlated to \( Y \).

Let

\[
\begin{align*}
\bar{X}_1 &= \frac{\sum_{i=1}^{N} X_{1i}}{n}, & \bar{X}_2 &= \frac{\sum_{i=1}^{N} X_{2i}}{n}, & \bar{Y} &= \frac{\sum_{i=1}^{N} Y_i}{n}, & \bar{Z} &= \frac{\sum_{i=1}^{N} Z_i}{n}, \\
\end{align*}
\]

Let us define the following assumptions and expectations to get the bias and mean square error

\[
E(\delta_x) = E(\delta_z) = E(e_z) = 0, \quad E(\delta_x^2) = \theta(\lambda_{400} - 1), \quad E(\delta_z^2) = \theta(\lambda_{400} - 1).
\]
\[ E(\delta_{x2}^2) = \theta(\lambda_{004} - 1), \quad E(e_{x2}^2) = \theta C_{x_z}^2, \quad E(\delta_{x1} e_{x2}) = \theta \lambda_{120} C_z, \quad E(\delta_{x1}^2) = \theta \lambda_{022} C_z, \quad E(\delta_{x2}^2) = \theta \lambda_{102} C_z, \quad E(\delta_{x1} \delta_{x2}) = \theta (\lambda_{022} - 1), \quad E(\delta_{z1} \delta_{x1}) = \theta (\lambda_{220} - 1), \quad \text{and} \quad E(\delta_{z2} \delta_{x2}) = \theta (\lambda_{202} - 1). \]

Based on the Diana and Perri (2011) RRT model \( Z=TY+S \), Gupta et al. (2020) introduced basic variance and some ratio-type estimators. The basic variance estimator is as

\[ t_0 = \sigma_s^2 = \frac{\sigma_z^2 - \sigma_x^2 - (\sigma^2 Z^2)}{\sigma^2 + 1}. \]  

The MSE of \( t_0 \) is as,

\[ \text{MSE}(t_0) = \theta \left( \frac{1}{(\sigma^2_s + 1)^2} \right) (\sigma^2_z (\lambda_{400} - 1) + 4\sigma^2_T Z^2 C_z^2 - 4\sigma^2_x Z^2 \lambda_{300} C_z). \]  

The ratio estimators is given by,

\[ t_{\text{ratio}} = \frac{s_z^2 - s_x^2 - \sigma^2_s + \sigma^2_x}{\sigma^2_s + 1} * \left( \frac{\sigma^2_z}{s^2_s} \right). \]  

The MSE of \( t_{\text{ratio}} \) estimator is as,

\[ \text{MSE}(t_{\text{ratio}}) = \theta \left( \frac{1}{(\sigma^2_s + 1)^2} \right) \left[ \sigma^2_z (\lambda_{400} - 1) - 2\sigma^2_x \sigma^2_{x1} (\lambda_{220} - 1)(\sigma^2_T + 1) 
+ \sigma^2_z (\lambda_{040} - 1)(\sigma^2_s + 1) + 1 \right] 
+ \theta \frac{1}{(\sigma^2_s + 1)^2} 4C_z (\sigma^2_T Z^4 C_z - \sigma^2_x \sigma^2_{x1} Z^2 \lambda_{300} + \sigma^2_T Z^2 \lambda_{120}(\sigma^2_T + 1)). \]  

The generalized ratio estimator is as,

\[ t_{\text{gratio}} = \left( \frac{s_z^2 - s_x^2 - \sigma^2_s + \sigma^2_x}{\sigma^2_s + 1} \right) + (\sigma^2_x - s^2_x) \left( \frac{\sigma^2_z + \beta}{\omega(\alpha s^2_s + \beta) + (1-\omega)(\alpha s^2_s + \beta)} \right)^g. \]  

The MSE of \( t_0 \) is as,

\[ \text{minMSE}(t_{\text{gratio}}) = \theta \left( \frac{\sigma^2_z (\lambda_{400} - 1) + 4\sigma^2_T Z^2 C_z^2 - 4\sigma^2_x \sigma^2_{x1} Z^2 \lambda_{300} C_z}{(\sigma^2_s + 1)^2} \right) - \theta \left( \frac{1}{(\lambda_{040} - 1)} \right) (\sigma^2_x (\lambda_{220} - 1) - 2\sigma^2_T Z^2 \lambda_{120} C_z)^2. \]  

where \( \lambda_{rs} = \frac{\mu_{rsa}}{\mu_{2000} \mu_{2020} \mu_{020} \mu_{022}}, \quad \mu_{rsa} = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - \bar{Y})^r (X_{1i} - \bar{X})^r (X_{2i} - \bar{X})^a, \quad \theta = \frac{1}{n}. \)

3. The proposed Estimator

In this section, a generalized exponential estimator is presented following Koyuncu et al. (2014) The form of the proposed estimator is given by,
where $k_1, k_2, k_3,$ are the three optimizing and unrestricted constants which need to be estimated such that the MSE of the estimator is minimum, and $\lambda_1$ and $\lambda_2$ are the generalization constants which need to be placed with some suitable values, known parameters, or function of known parameters to get different efficient and or existing estimators. A few examples are shown in Table 1 by setting different values to the constants.

**Table 1:** The class of estimators for different choices of constant’s values

<table>
<thead>
<tr>
<th>Class of Estimators</th>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{1D(1)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(2)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(3)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(4)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(5)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(6)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(7)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
<tr>
<td>$t_{1D(8)}$</td>
<td>$k_1$</td>
<td>$k_2$</td>
<td>$k_3$</td>
<td>$\lambda_1$</td>
<td>$\lambda_2$</td>
</tr>
</tbody>
</table>
To obtain the Bias, and the MSE, we define the following error terms, rewriting Eq (7), we have

\[
t_{1D} = k_1 \left( \frac{\sigma_i^2 (1+\delta_i) - \sigma_i^2 Z_i^2 (1+\delta_i)}{\sigma_i^2} + k_2 (\sigma_i^2 - \sigma_i^2 (1+\delta_i)) + k_3 (\sigma_i^2 - \sigma_i^2 (1+\delta_i)) \right) \left[ \exp \left( \frac{\sigma_i^2}{\sigma_i^2 + \sigma_i^2 (1+\delta_i)} \right) \right] \left[ 1 + \frac{\sigma_i^2}{\sigma_i^2 + \sigma_i^2 (1+\delta_i)} \right]^{\gamma_i}
\]

(8)

**Bias**

\[
Bias(t_{1D}) = \sigma_y^2 (k_1 - 1) - k_1 \sigma_y^2 \theta \frac{1}{2} \left[ \lambda_1 \{ \lambda_2 (\mu_{022} - 1) - \lambda_1 (\mu_{040} - 1) \} - 2 \lambda_2 (\mu_{022} - 1) \right] - k_3 \sigma_y^2 \theta \left[ \frac{\lambda_1}{4} (\mu_{040} - 1) - \lambda_2 (\mu_{022} - 1) \right]
\]

(9)

The mean square error of the generalized estimator \( t_{1D} \) is given by

\[
MSE(t_{1D}) = \sigma_y^4 (k_1 - 1)^2 + k_1^2 \sigma_y^2 \theta A + k_2^2 \sigma_y^2 \theta B (\mu_{040} - 1) + k_3^2 \sigma_y^2 \theta C + k_1 k_2 \sigma_y^2 \theta B C + k_2 k_3 \sigma_y^2 \theta C
\]

(10)

Differentiate Eq (10) with respect to \( k_1, k_2, \) and \( k_3, \) and after the simplification optimum values of the constants are given by,

\[
k_1 = \frac{\sigma_y^2}{A_7}, \quad k_2 = \frac{-k_1 A_4}{A_7 A_6}, \quad \text{and} \quad k_3 = \frac{k_1 A_5}{A_6 A_6}
\]

and utilizing the optimum values of the constants into Eq (10), the simplified form of the minim MSE of the estimator is given by

\[
\min MSE(t_{1D}) = \sigma_y^6 \left( 1 - \frac{1}{A_7} \right)
\]

(11)

where,

\[
A_1 = \sigma_y^4 \left[ \frac{\lambda_1}{4} (\mu_{040} - 1) + \lambda_2 (\mu_{022} - 1) - \lambda_1 (\mu_{022} - 1) \right] + \frac{\sigma_y^2 (\mu_{040} - 1)}{(\theta + 1)} + 4 \frac{\sigma_y^4 \sigma_y^2 C_z}{(\theta + 1)^2} + \frac{z^2 \sigma_y^2 \sigma_y^2 \mu_{300} C_z}{(\theta + 1)}
\]

\[
A_2 = \sigma_y^4 \left[ \frac{\lambda_1}{2} (\mu_{040} - 1) - \lambda_2 (\mu_{022} - 1) \right] - \frac{\sigma_y^2 (\mu_{220} - 1)}{(\theta + 1)} C_z
\]

\[
A_3 = \sigma_y^4 \left[ \frac{\lambda_1}{2} (\mu_{022} - 1) - \lambda_2 (\mu_{022} - 1) - \frac{\sigma_y^2 (\mu_{220} - 1)}{(\theta + 1)} C_z
\]

\[
A_4 = A_2 (\mu_{004} - 1) + A_3 (\mu_{022} - 1)
\]

\[
A_5 = A_2 (\mu_{022} - 1) + A_3 (\mu_{040} - 1)
\]

\[
A_6 = (\mu_{040} - 1) (\mu_{004} - 1) - (\mu_{022} - 1)^2
\]

\[
A_7 = \sigma_y^4 + \theta A_1 + \theta (\mu_{040} - 1) \left( \frac{A_4}{A_6} \right)^2 + \theta (\mu_{004} - 1) \left( \frac{A_5}{A_6} \right)^2 + 2 \theta \frac{A_2 A_4}{A_6} - 2 \theta \frac{A_3 A_5}{A_6} + 2 \theta (\mu_{022} - 1) \frac{A_4 A_5}{A_6}
\]
4. The proposed RRT model and Estimator-II
4.1. The proposed RRT Model

Our scrambled randomized response model provides a combination of multiplicative, additive, and subtractive models. Since $Y$ is the sensitive variable of interest and hence subject to social desirability bias. $S$ and $R$ are the two independent scrambling variables and are mutually uncorrelated with $Y$. We assume

$$Z_{NP} = g(Y + aS) + (1 - g)R(Y + aS)$$

4. Rewriting, we get

$$
\sigma^2_{Z_{NP}} = g^2(\sigma_Y^2 + a^2 \sigma_S^2) + (1 - g)^2 \sigma_R^2(\sigma_Y^2 + a^2 \sigma_S^2),
$$

$$
\sigma^2_{Z_{NP}} = g^2(\sigma_Y^2 + a^2 \sigma_S^2) + (1 - g)^2 \sigma_R^2(\sigma_Y^2 + a^2(1 - g)^2 \sigma_{RS}^2),
$$

$$
= g^2(\sigma_Y^2 + a^2 \sigma_S^2) + (1 - g)^2 \left( (\sigma_R^2 \sigma_Y^2 + \sigma_R^2)(E[Y])^2 + \sigma_Y^2 (E[R])^2 \right) + a^2(1 - g)^2 (\sigma_R^2 \sigma_S^2 + \sigma_R^2 (E[S])^2 + \sigma_S^2 (E[R])^2),
$$

$$
= \sigma_Y^2 [g^2 + (1 - g)^2(\sigma_R^2 + 1)] + a^2 \sigma_S^2 [g^2 + (1 - g)^2(\sigma_R^2 + 1)] + (1 - g)^2 \sigma_R^2 \mu_Y^2.
$$

Rewriting, we get

$$
\sigma_Y^2 = \frac{\sigma^2_{Z_{NP}} - (1 - g)^2 \sigma_R^2 \mu_Y^2}{[g^2 + (1 - g)^2(\sigma_R^2 + 1)]} - a^2 \sigma_S^2.
$$

(i) Estimating $\sigma^2_{Z_{NP}}$ by its unbiased estimator $s_Z^2$,

$$
t_{NP_1} = \frac{s_Z^2 - (1 - g)^2 \sigma_R^2 \mu_Y^2}{[g^2 + (1 - g)^2(\sigma_R^2 + 1)]} - a^2 \sigma_S^2.
$$

(ii) Ratio estimator under the proposed randomized model:

$$
t_{NP_2} = \frac{s_Z^2 - (1 - g)^2 \sigma_R^2 \mu_Y^2}{[g^2 + (1 - g)^2(\sigma_R^2 + 1)]} - \frac{a^2 \sigma_S^2}{s_{X_1}^2}.
$$

Rewrite (22) we get
By pertaining expectations together on (24), the Bias and mean square error we obtain are as

\[
\text{Bias} (t_{NP2}) = \frac{\theta}{G} \left[ (\lambda_{040} - 1)G - (\sigma_2^2 \lambda_{120} C_Z + (1 - g)^2 \sigma_R^2 \tilde{Z}^2 (C_Z - 2 \lambda_{120}))C_Z \right],
\]

\[
\text{MSE} (t_{NP2}) = \frac{\theta}{G} \left[ \frac{1}{G} \{\sigma_2^2 (\lambda_{400} - 1)G - (1 - g)^2 \sigma_R^2 \tilde{Z}^2 (2 \lambda_{300} - (1 - g)^2 \sigma_R^2 \tilde{Z}^2)C_Z \} - \sigma_2^2 \{\sigma_2^2 (\lambda_{220} - 1) + (1 - g)^2 \sigma_R^2 \tilde{Z}^2 \lambda_{120} C_Z - \sigma_2^2 (\lambda_{040} - 1) \} \right],
\]

where

\[ G = [g^2 + (1 - g)^2 (\sigma_R^2 + 1)]. \]

### 4.2. The Proposed Estimator under the proposed RRT model

The exponential estimator expressed in (7) can be generalized in the situation of two auxiliary non-sensitive variables as,

\[
t_{NP2} = \left[ \frac{\sigma_2^2 (1 - \delta_Z) - (1 - g)^2 \sigma_2^2 \tilde{Z}^2 (1 + e_0)^2}{[g^2 + (1 - g)^2 (\sigma_R^2 + 1)]} - a^2 \sigma_5^2 \right] + w_2 (\sigma_2^2 - s^2_{x1}) + w_3 (\sigma_2^2 - s^2_{x2}),
\]

\[
\text{Rewrite (26) we get,}
\]

\[
t_1 = \left[ \frac{\sigma_2^2 (1 - \delta_Z) - (1 - g)^2 \sigma_2^2 \tilde{Z}^2 (1 + e_0)^2}{[g^2 + (1 - g)^2 (\sigma_R^2 + 1)]} - a^2 \sigma_5^2 \right] + w_2 (\sigma_2^2 - s^2_{x1}) + w_3 (\sigma_2^2 - s^2_{x2}(1 - \delta_{x2})),
\]

\[
\text{The expression of the bias and MSE of } t_1 \text{ to the first order of approximation is given by,}
\]

\[
\text{Bias} (t_1) = \sigma_2^2 w_1 - w_1 \theta \left[ (1 - g)^2 \tilde{Z}^2 \sigma_R^2 C_Z \left( \frac{\sigma_x}{g^2 + (1 - g)^2 (\sigma_R^2 + 1)} + \mu_{120} \right) + v_1 \left( \sigma_2^2 \mu_{022} - 1 \right) + v_1 \left( \frac{1}{4} \sigma_2^2 \mu_{040} - 1 \right) \right] + \theta \left( \frac{1}{2} \sigma_2^2 \mu_{004} - 1 \right) + \theta \left[ \sigma_2^2 \mu_{004} - 1 \right] + \theta \left[ \sigma_2^2 \mu_{004} - 1 \right] + v_2 \left( v_2 - 1 \right) \sigma_2^2 - k_3 \sigma_2^2 \theta \left( \frac{1}{2} \mu_{022} - 1 \right).
\]

\[
\text{MSE} (t_1) = \sigma_2^4 w_1 (1 - \delta_Z) + w_1 \theta B_1 + k_2 \sigma_2^4 \theta (\mu_{040} - 1) + \frac{1}{4} k_2 \sigma_2^4 \theta (\mu_{004} - 1) + 2 k_1 k_2 \sigma_2^2 \theta B_2 - 2 k_1 k_2 \sigma_2^2 \theta B_3 + k_2 k_3 \sigma_2^2 \theta (\mu_{022} - 1).
\]

Differentiate with respect to \( w_1, w_2 \) and \( w_3 \), the optimum values attained are as

\[
w_1 = \frac{\sigma_2^2}{\sigma_2^2 + \theta B_8}, \quad w_2 = \frac{-w_1 B_4}{\sigma_{x1} B_5}, \quad \text{and} \quad w_3 = \frac{w_2 B_6}{\sigma_{x2} B_7}.
\]

The MSE imputing this optimum value is given as.
\[
\min \text{MSE}(t_{1N}) = 1 - \frac{\sigma^6_y}{\sigma^6_y + \theta B_B}, \quad (30)
\]

where,
\[
D_1 = \frac{1}{[g^2 + (1-g)^2(\sigma_R^2 + 1)]^2}[\sigma_x^2(\mu_{400} - 1) + 4(1-g)^4\sigma_x^4 Z^4 C_x^2 - 2(1-g)^2 \sigma_x^2 \sigma_R^2 Z^2 \mu_{300} C_x],
\]
\[
D_2 = \frac{1}{[g^2 + (1-g)^2(\sigma_R^2 + 1)]^2}[\sigma_x^2 \{v_1(\mu_{220} - 1) + v_2(\mu_{202} - 1)\} - 2(1-g)^2 \sigma_x^2 Z^2 \{v_1 \mu_{120} + v_2 \mu_{102}\}],
\]
\[
D_3 = v_1^2(\mu_{040} - 1) + v_2^2(\mu_{004} - 1) - 2v_1 v_2(\mu_{022} - 1),
\]
\[
D_4 = \theta(D_1 + \sigma_x^2 D_2 - 2\sigma_x^2 D_3),
\]
\[
B_1 = D_1 + \sigma_x^4 \left[\frac{v_2^2}{2}(\mu_{040} - 1) + \frac{v_2^2}{2}(\mu_{004} - 1) - 2v_1 v_2(\mu_{022} - 1)\right] - \sigma_x^2[v_1 D_2 - v_2 D_3 + 2v_1 v_2(\mu_{022} - 1)],
\]
\[
B_2 = D_2 + [\sigma_x^2(\mu_{040} - 1) + 2(\mu_{022} - 1)]v_1,
\]
\[
B_3 = D_3 + [\sigma_x^2(\mu_{022} - 1) + 2(\mu_{004} - 1)]v_2,
\]
\[
B_4 = B_2(\mu_{004} - 1) - B_3(\mu_{022} - 1),
\]
\[
B_5 = \sigma_x^2(\mu_{040} - 1)(\mu_{004} - 1) - (\mu_{022} - 1)^2,
\]
\[
B_6 = B_2(\mu_{022} - 1) - B_3(\mu_{040} - 1),
\]
\[
B_7 = \sigma_x^2(\mu_{040} - 1)(\mu_{004} - 1) - (\mu_{022} - 1)^2,
\]

and
\[
B_8 = B_1 + (\mu_{040} - 1) \left[\frac{B_4}{B_5}\right]^2 + (\mu_{004} - 1) \left[\frac{B_6}{B_7}\right]^2 - 2 \left[\frac{B_4 B_4 - B_3 B_6}{B_5 B_7} + (\mu_{022} - 1) \frac{B_4 B_6}{B_5 B_7}\right].
\]

5. Privacy levels

In the literature, many privacy protection measures are presented by different authors. For our study, privacy measure due to Yan et al. (2008) is used to compute privacy for Diana and Perri’s (2011) model, and the proposed randomized response model.

The privacy protection measure presented by Yan et al. is given by,
\[
\Delta = E(Z_i - Y_i)^2 \quad (31)
\]

a. Diana and Perri (2011) model is given by,
\[
Z = RY + S, \quad (32)
\]

The privacy protection level is given by
\[
\Delta_D = E(Z_i - Y_i)^2, \quad (33)
\]
\[
= \sigma_R^2(\mu_Y^2 + \sigma_Y^2) + \sigma_S^2.
\]

b. The proposed Randomized response model’s privacy protection mlevel is as,
\[
\Delta_PN = E(Z_i - Y_i)^2, \quad (34)
\]
\[
= (\mu_Y^2 + \sigma_Y^2 + a^2 \sigma_S^2)(1 + (1-g)^2 \sigma_R^2) - (\mu_Y^2 + \sigma_Y^2)
\]

c. Comparison of privacy protection levels for Diana and Perri’s (2011) model and proposed model:

Using (33) and (34), we have
\[
\Delta_{PN} - \Delta_D = (\mu_Y^2 + \sigma_Y^2)g(2 - 2g) - \sigma_S^2[1 - a^2(1 + (1-g)^2 \sigma_R^2)] > 0, \quad (35)
\]
iff \[ g(g - 2) > \frac{\sigma^2[1-a^2(1+\sigma^2)]}{(\sigma^2(\mu_0^2+\sigma_0^2)-a^2\sigma_0^2\sigma^2)} \].

6. An application of the proposed Model:

In this section, motivated by Saleem and Sanaullah (2022) a real-life application is presented to analyze the efficiency of the proposed RRT model compared to the existing models.

A survey is organized to collect real data for the problem of the estimation of the true variance of the Grade Point Average (GPA) of the students of the Department of Statistics, in Forman Christian College University Lahore, who have studied the Course: STATISTICAL METHODS in Spring 2023. Ninety students registered in three sections in this statistics course are considered as our population. In this application, the variable of interest Y is the CGPA of students, and the two auxiliary variables i-e, \(X_1\) is the weekly study hours, and \(X_2\) is the number of courses studied in recent semesters. For the scrambling variables, \(S\) is a normal random variable with a mean equal to zero and a standard deviation equal to 2, and \(R\) is a normal random variable with a mean equal to 1 and a standard deviation equal to 0.02. The following are some characteristics of the population:

\[ N=90, \mu_{X_1} = 27.61; \mu_{X_2} = 19.88; \sigma_{X_1} = 8.66; \sigma_{X_2} = 18.83; \]

For model, \(Z = RY + S\)

\[ \mu_Z = 3.889; \sigma_Z = 2.59; \rho_{ZX_1} = -0.053; \rho_{ZX_2} = 0.017 \]

For Model \(Z_{NP} = g(Y + aS) + (1 - g)R(Y + aS)\)

\[ \mu_{Z_{NP}} = 52.31; \sigma_{Z_{NP}} = 6.78; \rho_{Z_{NP}X_1} = -0.044; \rho_{Z_{NP}X_2} = 0.140 \]

Table 2: The MSEs of the estimators for real population

<table>
<thead>
<tr>
<th>(n)</th>
<th>(Z=R*Y+S)</th>
<th>(Z_{NP} = g(Y + aS) + (1 - g)R(Y + aS))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimators</td>
<td>Mean</td>
</tr>
<tr>
<td>20</td>
<td>(t_0)</td>
<td>5.3726</td>
</tr>
<tr>
<td></td>
<td>(t_{ratio})</td>
<td>7.9481</td>
</tr>
<tr>
<td></td>
<td>(t_{1D})</td>
<td>3.4357</td>
</tr>
<tr>
<td>38</td>
<td>(t_0)</td>
<td>5.3279</td>
</tr>
<tr>
<td></td>
<td>(t_{ratio})</td>
<td>6.4387</td>
</tr>
<tr>
<td></td>
<td>(t_{1D})</td>
<td>3.4072</td>
</tr>
</tbody>
</table>

Table 2 shows the results for MSE estimates of the model given by Diana and Perri (2011) and the proposed model. The results are obtained by using two different sample sizes \(n=20\) and \(38\). One can notice that the proposed estimator provides minimum and better results as compared to the other estimators under both models.

7. Simulation Study
In this section, we conduct a simulation study to evaluate the performance of the proposed generalized exponential-type estimators by comparing some existing variance estimators.

**Population I:**
\[
\Sigma = \begin{bmatrix} 10 & 3 & 2.9 \\ 3 & 2 & 1.1 \\ 2.9 & 1.1 & 2 \end{bmatrix}, \rho_{x1y} = 0.6817, \rho_{x2y} = 0.6705.
\]

**Population II:**
\[
\Sigma = \begin{bmatrix} 6 & 3 & 2.9 \\ 3 & 2 & 1.1 \\ 2.9 & 1.1 & 2 \end{bmatrix}, \rho_{x1y} = 0.8706, \rho_{x2y} = 0.8706.
\]

For both populations, we ruminate three different samples of sizes 200, 300, and 500. The variance of \( S \) i.e. \( \text{Var}(S) \) and variance of \( T \), i.e. \( \text{Var}(T) \) choose different values for simulation.

Table 3 provides the privacy protection level of the RRT models discussed in this study, we follow Gupta et al. (2018) unified measure of the estimator and is given by

\[
\theta = \frac{MSE(t_i)}{\Delta_j} \times 100 \quad \text{where} \ i=0, \text{ratio}, \text{ID, NP1, NP2, 1}; \ j=\text{D} \text{and PN}.
\]

where, \( MSE(t_i) \) is the theoretical MSE of the various estimators and \( \Delta_j \) is the privacy level for Diana and Perri’s (2011) model and the proposed model as discussed in section 5.

**Table 3**: Privacy level for two populations

<table>
<thead>
<tr>
<th>Model</th>
<th>Population I</th>
<th>Population II</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diana and Perri (2011)</td>
<td>10.1472</td>
<td>7.1942</td>
</tr>
<tr>
<td>Proposed Model</td>
<td>5.6528</td>
<td>4.9476</td>
</tr>
</tbody>
</table>

Tables 4-6 give the MSE and percent relative efficiency (PRE) results for the proposed estimator and existing estimators discussed in this article. The following expression is implied to get the PRE,

\[
PRE = \frac{MSE(t_i)}{MSE(t_0)} \times 100, \quad \text{where,} \ i=\text{ratio}, \text{gratio}, \text{and gep}.
\]

The results are presented in Table 4-6. The Table 4-5 provides the numerical results of estimators discussed in Section 2 and 3 whereas the Table 6 presented the results of estimators discussed in Section 4 based on the proposed model. The values from Table 4-6 confirm that the existing estimators presented by Gupta et al. (2020) are less efficient as compared to the generalized estimator. Also while comparing the proposed model and existing model estimator results in these tables on may observe that the proposed model provides more efficient MSE values as compared to the model presented by Diana and Perri (2010). As we can see as the variance of \( S \) increases the MSE decreases.
Table 4: The MSEs and PREs of the estimators for Population I with $\sigma^2 = 0.5$ using $Z = YT + S$

<table>
<thead>
<tr>
<th>$Var(S)$</th>
<th>$N$</th>
<th>Estimators</th>
<th>Mean ($\hat{\sigma}^2$)</th>
<th>MSE</th>
<th>PRE</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>200</td>
<td>$t_0$</td>
<td>12.6563</td>
<td>4.1872</td>
<td>100.00</td>
<td>0.4126</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>12.7547</td>
<td>4.1573</td>
<td>100.72</td>
<td>0.4097</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>12.6568</td>
<td>4.1013</td>
<td>102.09</td>
<td>0.4042</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>12.0520</td>
<td>3.9767</td>
<td>105.29</td>
<td>0.3919</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>$t_0$</td>
<td>12.6540</td>
<td>1.2985</td>
<td>100.00</td>
<td>0.1280</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>12.6910</td>
<td>1.0253</td>
<td>126.65</td>
<td>0.1010</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>12.6539</td>
<td>1.0065</td>
<td>129.01</td>
<td>0.0980</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>12.0586</td>
<td>0.2382</td>
<td>545.13</td>
<td>0.0235</td>
</tr>
<tr>
<td>0.5</td>
<td>200</td>
<td>$t_0$</td>
<td>14.6764</td>
<td>4.1979</td>
<td>100.00</td>
<td>0.4137</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>14.8676</td>
<td>4.0264</td>
<td>104.26</td>
<td>0.3968</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>14.6795</td>
<td>3.9598</td>
<td>106.01</td>
<td>0.3902</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>13.0802</td>
<td>2.9901</td>
<td>140.39</td>
<td>0.0976</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>$t_0$</td>
<td>14.7046</td>
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<td>0.1386</td>
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<td></td>
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<td>14.7376</td>
<td>1.0221</td>
<td>137.59</td>
<td>0.1007</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>14.7047</td>
<td>0.9945</td>
<td>141.41</td>
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<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>13.0697</td>
<td>0.2428</td>
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</tr>
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<td>200</td>
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<td>13.1945</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>13.3712</td>
<td>3.2163</td>
<td>100.66</td>
<td>0.3170</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>13.1975</td>
<td>3.1964</td>
<td>101.29</td>
<td>0.3150</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>12.0749</td>
<td>0.9605</td>
<td>337.07</td>
<td>0.0947</td>
</tr>
<tr>
<td>500</td>
<td></td>
<td>$t_0$</td>
<td>13.1984</td>
<td>1.3192</td>
<td>100.00</td>
<td>0.1300</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>13.2229</td>
<td>0.8057</td>
<td>163.73</td>
<td>0.0794</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>13.1979</td>
<td>0.7942</td>
<td>166.10</td>
<td>0.0783</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>12.0569</td>
<td>0.2420</td>
<td>545.12</td>
<td>0.0238</td>
</tr>
</tbody>
</table>
Table 5: The MSEs and PREs of the estimators for Population II with $\sigma_f^2 = 0.5$ using $Z = YT + S$

<table>
<thead>
<tr>
<th>Var(S)</th>
<th>N</th>
<th>Estimators</th>
<th>$\text{Mean} \left( \hat{\sigma}^2 \right)$</th>
<th>MSE</th>
<th>PRE</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>200</td>
<td>$t_0$</td>
<td>8.1658</td>
<td>1.5367</td>
<td>100.00</td>
<td>0.2136</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>8.2248</td>
<td>1.4245</td>
<td>107.88</td>
<td>0.1980</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>8.1710</td>
<td>1.3286</td>
<td>115.66</td>
<td>0.1847</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>8.0373</td>
<td>1.1215</td>
<td>137.02</td>
<td>0.1559</td>
</tr>
<tr>
<td></td>
<td>500</td>
<td>$t_0$</td>
<td>8.1555</td>
<td>0.3809</td>
<td>100.00</td>
<td>0.0529</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>8.1822</td>
<td>0.3570</td>
<td>106.69</td>
<td>0.0496</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>8.1582</td>
<td>0.3324</td>
<td>114.59</td>
<td>0.0462</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>8.0517</td>
<td>0.2871</td>
<td>132.67</td>
<td>0.0399</td>
</tr>
<tr>
<td>0.5</td>
<td>200</td>
<td>$t_0$</td>
<td>8.9991</td>
<td>2.1105</td>
<td>100.00</td>
<td>0.2934</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>9.0607</td>
<td>1.8331</td>
<td>115.13</td>
<td>0.2548</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>9.0032</td>
<td>1.7227</td>
<td>122.51</td>
<td>0.2395</td>
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<td></td>
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<td>$t_{1D}$</td>
<td>8.0269</td>
<td>1.1276</td>
<td>187.17</td>
<td>0.1567</td>
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<tr>
<td></td>
<td>500</td>
<td>$t_0$</td>
<td>8.9905</td>
<td>0.5165</td>
<td>100.00</td>
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<tr>
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<td></td>
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<td>0.4564</td>
<td>113.17</td>
<td>0.0634</td>
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<tr>
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<td></td>
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<td>8.9923</td>
<td>0.4291</td>
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<td>0.2798</td>
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<td>2.7373</td>
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</tr>
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</tr>
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<td>2.5569</td>
<td>107.06</td>
<td>0.3554</td>
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<td>1.1238</td>
<td>243.58</td>
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<tr>
<td></td>
<td>500</td>
<td>$t_0$</td>
<td>8.7367</td>
<td>0.6678</td>
<td>100.00</td>
<td>0.0928</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{ratio}$</td>
<td>8.7636</td>
<td>0.6414</td>
<td>104.12</td>
<td>0.0892</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{gratio}$</td>
<td>8.7390</td>
<td>0.6144</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1D}$</td>
<td>8.0495</td>
<td>0.2794</td>
<td>239.01</td>
<td>0.0388</td>
</tr>
</tbody>
</table>
Table 6: The MSEs of the estimators for Population I & II with $\sigma^2=0.5$ using the proposed model

<table>
<thead>
<tr>
<th>Var(S)</th>
<th>N</th>
<th>Estimators</th>
<th>Mean $\left(\hat{\sigma}^2_\theta\right)$</th>
<th>MSE</th>
<th>PRE</th>
<th>$\theta$</th>
<th>Mean $\left(\hat{\sigma}^2_\theta\right)$</th>
<th>MSE</th>
<th>PRE</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>200</td>
<td>$t_{NP1}$</td>
<td>12.8756</td>
<td>3.3689</td>
<td>100.0</td>
<td>0.5960</td>
<td>7.9867</td>
<td>1.3649</td>
<td>100.0</td>
<td>0.2759</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{NP2}$</td>
<td>13.0140</td>
<td>3.5686</td>
<td>94.40</td>
<td>0.6313</td>
<td>8.0386</td>
<td>1.2930</td>
<td>105.56</td>
<td>0.2613</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$t_{1N}$</td>
<td>11.0777</td>
<td>1.0035</td>
<td>335.71</td>
<td>0.1775</td>
<td>7.0332</td>
<td>1.1664</td>
<td>117.02</td>
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</tr>
<tr>
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<td>500</td>
<td>$t_{NP1}$</td>
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<td>0.8166</td>
<td>100.0</td>
<td>0.1445</td>
<td>7.5139</td>
<td>0.3972</td>
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<td>$t_{NP2}$</td>
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<td>0.1975</td>
<td>7.5327</td>
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<tr>
<td>0.5</td>
<td>200</td>
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<td>$t_{NP2}$</td>
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<td>1.5241</td>
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A smaller value of $\theta$ is to be preferred. Table 3-5 presents the unified measure along with the PRE of the estimators. It is observed that the proposed generalized estimator using two auxiliary variables efficiently performs either using Diana and Perri’s model or the proposed RRT model. One can notice that the values of $\theta$ are smaller for the proposed generalized model.

8. Conclusion
The present article reflects the problem of estimating population variance $S^2_z$ of the sensitive study variable using a non-sensitive auxiliary variable. Here, a generalized exponential-type estimator based on Diana and Perri’s (2011) randomized response model has been suggested. The comparison of the proposed estimator along with the estimators proposed by Gupta et al. (2020) has been presented in Table 4-6. It is observed that the proposed estimator performs more efficiently. Other than this, a new generalized scrambled response model. The usual variance estimator, ratio estimator, and proposed estimator have been modified using the proposed model. An application of a real survey-based study based on the proposed RRT model has been presented in section 6. The results are obtained using the usual variance, ratio, and proposed estimators under the proposed model and the model presented by Diana and Perri (2011). One can notice that the proposed estimator provides better and minimum MSE results as compared to the usual mean and ratio estimator using either of the models. As the sample size increases the estimator performs more efficiently. The simulation study results are presented in Table 3-6 in which expected variances, MSE and PRE have been compared. The generalized proposed estimator under the proposed randomized response model provides minimum mean square error for both populations as compared to the estimator MSE results using Daiana and Perri’s (2011) model.

Conflicts of Interest: The authors declare no conflict of interest.

Data availability: The data set used and/or analysed during the current study available from the corresponding author on reasonable request.

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References


