An Inhomogeneous Model for Laser Welding of Industrial Interest

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An Inhomogeneous Model for Laser Welding of Industrial Interest

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Abstract

An innovative non-homogeneous dynamic model for recovering the absolute temperature during the laser welding process of Al-Si 5\% alloy plates, as per the industrial activity of an Italian company with proven experience operating in the laser welding area, is presented here. Firstly, the model considers that, metallurgically, during welding, the alloy melts with the presence of solid/liquid phases until total melting, and afterwards re-solidifies with the reverse process. Further, a particular polynomial substitute thermal capacity of the alloy has been chosen, according to experimental evidence, so that the volumetric solid state fraction is easily identifiable. Moreover, to the usual radiative/convective boundary conditions, to make the model more realistic, the contribution due to the positioning of the plates on the workbench is considered (endowing the model with Cauchy-Stefan-Boltzmann boundary conditions). Having verified the well-posedness of the problem, a Galerkin-FEM approach has been implemented on MatLab PDE ToolBox to numerically recover the absolute temperature maps, obtained by modeling the heat sources due to the laser with formulations depending on the laser sliding speed. The results achieved have shown good adherence to the experimental evidence, opening up interesting future scenarios for technology transfer.
1 Introduction to The Problem

High-frequency laser welding locally melts metallic elements creating a very strong thin and deep weld [Dal Fabbro 2016, Hong et al. 2017, Yang et al. 2020, Mohr et al. 2020, Rimalc et al. 2020]. The laser beam, flowing at a certain speed \(v\), is conveyed over a small section; ensuring welding precision, power concentration on a limited surface - without additional materials to the element to be welded (avoiding unsightly residues that are often dangerous because they are harmful to the mechanical strength of the weld) [Katayamas 2010, Xiaodong 2010]. Although laser welding is a consolidated technique, there remains a strong need to develop new and more complete physical-mathematical models for recovering the absolute temperature distributions, \(T\), in materials subject to welding. This would identify any “a priori” thermal problems both in the welding area and in its immediate vicinity. Furthermore, if \(T = T(x, t)\), with \(x \in \Omega \subset \mathbb{R}^3\), \(\Omega\) domain, \(t\) time, it is easy to evaluate the elimination of thermal overload due to welding, highlighting any mechanical anomalies of the welded products [Ren et al. 2022, Dos Santos Paes et al. 2022].

In the past, many dynamic models have been studied to recover \(T(x, t)\) in metallic products subjected to laser welding - starting from the following non-linear non-homogeneous parabolic heat equation [Ghosh et al. 2013]

\[
\mathcal{C}(T(x, t)) \frac{\partial T(x, t)}{\partial t} = \nabla \cdot [\lambda(T(x, t))\nabla T(x, t)] + Q_l(x, t),
\]

where \(\mathcal{C}(T(x, t))\) is the volumetric specific heat, \(\lambda(T(x, t))\) is the thermal conductivity, \(\nabla T(x, t)\) is the temperature gradient and \(Q_l\) is the volumetric heat source due to the laser [Ghosh et al. 2013, Kaplan 2011, Unni et al. 2021, Ragavendran et al. 2021, Morosanu et al. 2023]; formulated in parabolic frontiers with suitable boundary and initial conditions [Ghosh et al. 2013, Kaplan 2011, Oussaid et al. 2019, Unni et al. 2021, Ragavendran et al. 2021, Morosanu et al. 2023, Fakir et al. 2018]. When the laser moves, starting from an initial temperature \(T_0\), \(T(x, t)\) raises for which the material, initially solid, begins to melt highlighting the co-presence of solid-liquid (intermediate state) until the total melting of the material occurs [Sarila et al. 2022]. The laser, as it moves, melts new areas of material while the previous ones, due to the reduction of \(T(x, t)\), re-solidify passing from a liquid to an intermediate state (co-presence of solid and liquid) until only solid phase is obtained. Further, many models lack the fact that they do not consider convective terms, terms due to irradiation [Wang et al. 2021, Belyaev et al. 2009] and terms due to
practical purposes (i.e., taking into account that the welding is done on a workbench) [Fakir et al. 2018, Giglio et al. 2021]. In this paper, we present a non-homogeneous parabolic model for the dynamic temperature recovery during laser welding of two Al-Si 5% alloy plates that, usually, compared to pure aluminum, offer high mechanical resistance when subjected to welding, as evidenced by the industrial activity of IRIS s.r.l (leading Italian company in the laser welding sector). Unlike pure Al (which melts at a specific value of the temperature), the binary alloy melts and re-solidifies at a certain range of temperature, in which the material forms a mushy zone, governing $T(x, t)$ in the welding area. The equation is written in terms of the substitute thermal capacity, $C(T(x, t))$, of the binary alloy, here formulated as a polynomial [Majachrzak et al 2007]. This is made to take into account only the presence of liquid, solid, or both - depending on the temperature of the material, obtaining a volumetric solid state fraction strictly dependent on the volumetric latent heat. The equation was used to simulate a thin strip of 3D laser welding of two Al-Si 5% alloy plates with perfectly smooth surfaces (to avoid voids), made up of two portions of material belonging to each plate. For the parts of the plates not affected by the welding, since the laser source is not present on them, a classic Fourier model of heat transmission has been hypothesized. Neumann boundary conditions have also been formulated to make the heat fluxes from the weld strip (at a higher temperature) to the areas not subject to welding (at a lower temperature) compatible. Further, to make the approach more realistic, boundary conditions - due to both contact with the air and contact with the workbench - have been added for the surfaces in question, finally achieving Cauchy-Stefan-Boltzmann boundary conditions. Once verified that the proposed model is well-posed (via hypothesis testing of a well-known result of the recent literature [Miranville et al. 2021]) and reinforced by the fact that it does not allow the explicit recovery of $T(x, t)$, an optimized Galerkin-FEM approach (to reduce the computational load useful for any real-time applications) has been implemented in MatLab R2022 PDE Tool and tested for the resolution of the problem by also selecting appropriate formulations of laser heat source, according with known experimental evidence [Mohan et al. 2022].

The remainder of the paper is structured as follows. Once the governing equations are described (Section 2), specifying both the heat transfer in the pure metal domain and the solidification process based on an interval of temperature for metal alloys, the proposed non-homogeneous parabolic models are presented and discussed referring the specimen under study (Section 3). After presenting the exploited laser heat sources (Subsection 3.5), the Galerkin-FEM procedure is presented and implemented (Section 4), once the well-posedness of the proposed model is verified (Subsection 4.1). The numerical results are presented and discussed (Section 5). Finally, some conclusions and future perspectives conclude this work.
2 Melting-Resolidification Process

As known, neglecting overheating temperature phenomena of liquid metal (for which convective phenomena lose their meaning), the equation describing the cooling and following solidification process in metals is writable as [Ciesielski et al. 2019]

\[ C(T(x,t)) \frac{\partial T(x,t)}{\partial t} = \nabla \cdot [\lambda(T(x,t))\nabla T(x,t)] + Q_l(x,t) + Q_m(x,t) \]  

(2)

where \( T(x,t) \) is assumed to be continuous and \( Q_m(x,t) \) is the capacity of volumetric internal heat sources derived from the melting phase change process. During the melting and re-solidification process, both solid and liquid volumetric fractions, \( f_S(T(x,t)) \) and \( f_L(T(x,t)) \) respectively, such that \( f_S(T(x,t)) + f_L(T(x,t)) = 1 \) \(^1\), coexist in the material at the neighborhood of the points considered. Therefore, an appropriate internal heat source, \( Q_m(x,t) \), is formulatable in terms of \( f_S(T(x,t)) \) or \( f_L(T(x,t)) \). Particularly, if \( L_v \) is the volumetric latent heat of fusion, the heat source due to the solidification becomes [Ciesielski et al. 2019, Nudez del Prado et al. 2021, DebRoy et al. 2021]. Then, equation (2), exploiting (3), becomes

\[ C(T(x,t)) \frac{\partial T(x,t)}{\partial t} = \nabla \cdot [\lambda(T(x,t))\nabla T(x,t)] + Q_l(x,t) + L_v \frac{df_S(T(x,t))}{dt} , \]

that, exploiting \( L_v \frac{df_S(T(x,t))}{dt} = \frac{df_S(T(x,t))}{dT} \frac{dT(x,t)}{dt} \), becomes

\[ \left( C(T(x,t)) - L_v \frac{df_S(T(x,t))}{dT} \right) \frac{dT(x,t)}{dt} = \nabla \cdot [\lambda(T(x,t))\nabla T(x,t)] + Q_l(x,t) , \]

(4)

where \( C(T(x,t)) \), represents the substitute thermal capacity of an artificial mushy zone sub-domain. Unlike pure metals where melting (and resolidification) occurs at a particular temperature value, binary alloys (such as Al-Si 5% here considered) melt and resolidify in a temperature range \([T_S, T_L]\) (i.e., the

---

\(^1\)For the solid and liquid state, \( f_S(T(x,t)) \) and \( f_L(T(x,t)) \) are the constant values (1 or 0).
temperature field across the entire conventionally homogeneous melt domain) in which the material forms a mushy zone. Therefore, it makes sense to write:

\[ f_S = 1, \quad T(x, t) < T_S \quad \text{(solid state)}, \]
\[ f_S \in (0, 1), \quad T_S \leq T(x, t) \leq T_L \quad \text{(mushy zone)}, \]
\[ f_S = 0, \quad T(x, t) > T_L \quad \text{(molten metal)}. \]

Recently, important results have been obtained starting from the knowledge of \( f_S \) and then obtaining the behavior of \( C(T(x, t)) \) [Ciesielski et al. 2019]. However, the reverse approach is also feasible: a plausible trend of \( C(T(x, t)) \) can be assumed, from which to obtain \( f_S \) [Majachrzak et al 2007]. Here, we consider the inverse approach and we suppose that a good approximation for \( C(T(x, t)) \) is a polynomial one [Majachrzak et al 2007]:

\[
C(T(x, t)) = a_0 + a_1 T(x, t) + a_2 T^2(x, t) + a_3 T^3(x, t) + a_4 T^4(x, t), \quad T(x, t) \in [T_S, T_L],
\]  

(5)

whose coefficients \( a_i \), with \( i \in \{0, 1, 2, 3, 4\} \), are selected in order that both \( C(T(x, t)) \) and its first derivative are continuous. For this reason, the following physical constraints are met:

\[
C(T_L) = C(T_L) \equiv C_L, \\
C(T_S) = C(T_S) \equiv C_S, \\
\frac{dC(T_L)}{dT} = \frac{dC(T_S)}{dT} = 0,
\]  

(6)

also satisfying

\[
\int_{T_S}^{T_L} C(T(x, t)) dT = C_m \Delta T + L_v,
\]  

(7)

where \( \Delta T = T_L - T_S \) and \( C_m \) is the mushy zone volumetric specific heat (usually, \( C_m = \frac{1}{2}(C_S + C_L) \) but other formulations could be taken into account). We note that conditions (6) and (7) allow construction of a bell-shaped trend for \( C(T(x, t)) \) [Majachrzak et al 2007].
Therefore, the coefficients $a_i$ become:

$$a_0 = \frac{(C_L - C_S)T_LT_S(T_L + T_S)}{(\Delta T)^3} + \frac{30T_L^2T_S^2L_v}{(\Delta T)^5},$$

$$a_1 = -\frac{6(C_L - C_S)T_LT_S}{(\Delta T)^3} - \frac{60T_LT_ST_S(T_L + T_S)L_v}{(\Delta T)^5},$$

$$a_2 = \frac{3(C_L - C_S)(T_L + T_S)}{(\Delta T)^3} + \frac{30(T_L^2 + 4T_LT_S + T_S^2)L_v}{(\Delta T)^5},$$

$$a_3 = -\frac{2(C_L - C_S)}{(\Delta T)^3} + \frac{60(T_L + T_S)L_v}{(\Delta T)^5},$$

$$a_4 = \frac{30L_v}{(\Delta T)^5},$$

depending on $L_v$. Furthermore, from (4) and (5), we can write:

$$C(T(x, t)) - L_v \frac{df_s(T(x, t))}{dT} = a_0 + a_1 T(x, t) + a_2 T^2(x, t) + a_3 T^3(x, t) +$$

$$+ a_4 T^4(x, t),$$

with $T(x, t) \in [T_S, T_L]$. But introducing the following definition of $C(T(x, t))$

$$C(T(x, t)) = \begin{cases} 
C_S & \text{if } T < T_S, \\
C_m & \text{if } T_S \leq T \leq T_L, \\
C_L & \text{if } T > T_L,
\end{cases}$$

we obtain

$$C_m - L_v \frac{df_s(T(x, t))}{dT} = a_0 + a_1 T(x, t) + a_2 T^2(x, t) + a_3 T^3(x, t) +$$

$$+ a_4 T^4(x, t),$$

with $T(x, t) \in [T_S, T_L]$.

From which we obtain

$$f_s(T(x, t)) = \frac{(C_m - a_0)T(x, t)}{L_v} - \frac{a_1 T^2(x, t)}{2L_v} - \frac{a_2 T^3(x, t)}{3L_v} -$$

$$- \frac{a_3 T^4(x, t)}{4L_v} - \frac{a_4 T^5(x, t)}{5L_v} + K,$$

(9)
where the constant of integration $K$ is determined by imposing $f_s(T_L) = 0$. Finally, (9) becomes

$$f_S(T(x,t)) = \left(\frac{a_0 - C_m}{L_v} - T(x,t)\right) + \frac{a_1[T_L^2 - T^2(x,t)]}{2L_v} + \frac{a_2[T_L^3 - T^3(x,t)]}{3L_v} + \frac{a_3[T_L^4 - T^4(x,t)]}{4L_v} + \frac{a_4[T_L^5 - T^5(x,t)]}{5L_v}.$$  

(10)

with $T(x,t) \in [T_S, T_L]$. Furthermore, (10) satisfies $f_S(T_S) = 1$ predicting the solidification kinetics of the casting.

3 Governing Equation

3.1 Material and geometries

The specimen consists of two Al-Si 5% alloy plates (without surface oxides which drastically raise the melting temperature), $P_1$ and $P_2$, of equal size (100 mm × 40 mm × 4 mm)\(^2\), juxtaposed along the largest dimension, such that the respective faces (perfectly smooth) adhere to favor welding. The absence of voids between the plates allows, on the one hand, to simulate a better weld quality and, on the other hand, avoids air between the parts to be welded. For our purposes, we divide the domain $\Omega$ into three subdomains: $\Omega_1$ and $\Omega_2$ consisting of the plates placed side by side net of the $\Omega$ portion subject to melting and then re-solidifying (welding strip); $\Omega_3$ (100 mm × 2 mm × 4 mm) corresponding to the welding strip, such that $\Omega_i = P_i \setminus \Omega_3$ with $i = 1, 2$, in which the heat transfer is modeled according to the melting/resolidification of the material in the interval of temperature $[T_S, T_L]$. Fig. 1 shows the partition of $\Omega$ into $P_1$ and $P_2$ while Fig. 2 displays $\Omega_1$ $\Omega_2$ and $\Omega_3$; moreover, Tab. 1 and Tab. 2 highlight the geometry of each $\Omega_i$. Finally, for implementation aims, we label the sixteen faces of $\partial \Omega$ by $F_i$ ($i = 1, ..., 16$), so that $\partial \Omega = \left(\bigcup_{i=1}^{16} F_i\right) \setminus (F_2 \cup F_7)$ (see Fig. 3) whose dimensions are specified in Tab. 3.

\(^2\)These dimensions have been suggested by IRIS s.r.l.
Concerning both $\Omega_1$ and $\Omega_2$, since the laser beam does not pass over them, a model exploiting (1) without laser heat source is sufficient for recovering
Table 1: Dimensions of $\Omega_1$, $\Omega_2$ and $\Omega_3$.

<table>
<thead>
<tr>
<th>$\Omega_i$</th>
<th>length (mm)</th>
<th>width (mm)</th>
<th>thickness (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1$</td>
<td>39</td>
<td>100</td>
<td>4</td>
</tr>
<tr>
<td>$\Omega_2$</td>
<td>39</td>
<td>100</td>
<td>4</td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>2</td>
<td>100</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 2: Geometric characterizations of $\Omega_1$, $\Omega_2$ and $\Omega_3$.

<table>
<thead>
<tr>
<th>$\Omega_i$</th>
<th>length (mm)</th>
<th>$\hat{\Omega}_i$</th>
<th>length (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_1$</td>
<td>[0,100] × [0,39] × [0,4]</td>
<td>$\hat{\Omega}_1$</td>
<td>(0,100) × (0,39) × (0,4)</td>
</tr>
<tr>
<td>$\Omega_2$</td>
<td>[0,100] × [41,80] × [0,4]</td>
<td>$\hat{\Omega}_2$</td>
<td>(0,100) × (41,80) × (0,4)</td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>[0,100] × [39,41] × [0,4]</td>
<td>$\hat{\Omega}_3$</td>
<td>(0,100) × (39,41) × (0,4)</td>
</tr>
</tbody>
</table>

Table 3: Geometric characterizations of $\partial \Omega_1$, $\partial \Omega_2$ and $\partial \Omega_3$.

<table>
<thead>
<tr>
<th>Label</th>
<th>Dimensions</th>
<th>Label</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{12}$</td>
<td>[0,100] × {0} × [0,4]</td>
<td>$F_2$</td>
<td>[0,100] × {41} × [0,4]</td>
</tr>
<tr>
<td>$F_7$</td>
<td>[0,100] × {39} × [0,4]</td>
<td>$F_1$</td>
<td>[0,100] × {80} × [0,4]</td>
</tr>
<tr>
<td>$F_5$</td>
<td>{0} × [41,80] × [0,4]</td>
<td>$F_4$</td>
<td>[0,100] × [41,80] × {4}</td>
</tr>
<tr>
<td>$F_3$</td>
<td>[100] × [41,80] × [0,4]</td>
<td>$F_6$</td>
<td>[0,100] × [41,80] × {0}</td>
</tr>
<tr>
<td>$F_{10}$</td>
<td>{0} × [39,41] × [0,4]</td>
<td>$F_9$</td>
<td>[0,100] × [39,41] × {4}</td>
</tr>
<tr>
<td>$F_8$</td>
<td>[100] × [39,41] × [0,4]</td>
<td>$F_{11}$</td>
<td>[0,100] × [39,41] × {0}</td>
</tr>
<tr>
<td>$F_{15}$</td>
<td>{0} × [0,39] × [0,4]</td>
<td>$F_{14}$</td>
<td>[0,100] × [0,39] × {4}</td>
</tr>
<tr>
<td>$F_{13}$</td>
<td>[100] × [0,39] × [0,4]</td>
<td>$F_{16}$</td>
<td>[0,100] × [0,39] × {0}</td>
</tr>
</tbody>
</table>

$T(x,t)$. Concerning $\Omega_3$, since the presence of the laser beam, we propose a model where the equation considers the melting/re-solidification of the material in its melting range, $\Delta T$, on which to formulate the replacement heat capacity of the alloy according to the known experimental evidence, and from which to easily obtain the volume fraction of the solid state.

3.2 Domains not belonging to the laser welding domain: the model

Starting from (1), with $Q_l(x,t) = 0$ (absence of the laser), and considering thermal conductivity $\lambda_i$ independent on the temperature $T_i(x,t)$ supposed to be continuous (It makes sense because the thermal conductivity of the AlSi 5%, during the laser welding, can be considered, in the first approximation, as a constant.), we write

$$c_s \frac{\partial T_i(x,t)}{\partial t} = \lambda_i \nabla^2 T_i(x,t), \quad \forall x \in \hat{\Omega}_i \text{ with } i = 1, 2, \quad \forall t > 0,$$
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to which associate the initial conditions

\[ T_i(x, 0) = T_0 \quad \forall x \in \Omega_i \text{ with } i = 1, 2. \]

Moreover, since the sides of the surfaces are next to the air conduction heat flux \( h_{air}[T_{air} - T_i(x, t)] \), and radiation flux, \( \epsilon \sigma_B [T^4_{air} - T^4_i(x, t)] \), occur, where \( h_{air} \) is the convection coefficient of air, \( \epsilon \) is the emissivity and \( \sigma_B \) the Boltzmann constant. We identify \( \hat{n} \) the outward unit normal vector to generical faces. Therefore, \( \forall x \in F_j, j \in \{1, 3, 4, 5, 12, 13, 14, 15\} \) and \( \forall t > 0 \)

\[ \lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = h_{air}[T_{air} - T_i(x, t)] + \epsilon \sigma_B [T^4_{air} - T^4_i(x, t)]. \]  

(11)

Furthermore, for the surfaces in contact with the workbench (faces \( F_6 \) and \( F_{16} \)), we introduce the following boundary condition

\[ \lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = h^{bench}[T^{bench} - T_i(x, t)], \quad \forall x \in F_6 \cup F_{16} \quad \forall t > 0. \]

with \( h^{bench} \) the convection coefficient of the workbench. Finally, it is necessary to consider the heat flow coming from \( \Omega_3 \) toward both \( \Omega_1 \) and \( \Omega_2 \), at a certainly higher temperature (the reverse heat flow can be considered as negligible because will not noticeably modify the numerical solution (3))

\[ \lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = \lambda_3 \frac{\partial T_3(x, t)}{\partial \hat{n}} \quad \forall x \in F_2 \cup F_7 \quad \forall t > 0, \]

Therefore, the model for \( \Omega_1 \) and \( \Omega_2 \) can be compactly written as:

\[
\begin{cases}
\frac{\partial T_i(x, t)}{\partial t} = C_S \nabla^2 T_i(x, t), & \forall x \in \Omega_i \text{ with } i = 1, 2 \quad \forall t > 0 \\
T_i(x, 0) = T_0, & \forall x \in \Omega_i \text{ with } i = 1, 2 \\
\lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = h_{air} (T_{air} - T_i) + \epsilon \sigma_B (T^4_{air} - T^4_i), & \forall x \in F_4 \cup F_{14} \quad \forall t > 0, \\
\lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = h_{air} (T_{air} - T_i) + h^{bench} (T^{bench} - T_i), & \forall x \in F_6 \cup F_{16} \quad \forall t > 0, \\
\lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = h_{air} (T_{air} - T_i), & \forall x \in F_1 \cup F_3 \cup F_5 \cup F_{12} \cup F_{13} \cup F_{15} \quad \forall t > 0, \\
\lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}} = \lambda_3 \frac{\partial T_3(x, t)}{\partial \hat{n}}, & \forall x \in F_2 \cup F_7 \quad \forall t > 0.
\end{cases}
\]

(12)

3.3 Laser welding domain: the model

When the laser beam flows on the plates, starting from \( T_0 \), the temperature \( T(x, t) \) increases, so that the material melts rise, obtaining the following phase

\[ \text{Springer Nature 2021 \textit{LaTeX} template} \]

\[ ^3 \text{Considering this heat flow negligible means a software design "for the benefit of safety" because, strictly speaking, the "miniscule" heat dissipation that goes from the plates towards the welding.} \]
transitions:

\[
solid \rightarrow solid + liquid \rightarrow liquid.
\]

Furthermore, the laser beam, moving again, melts further material while the previously melted material, due to the lowering of \(T(x, t)\), re-solidifies as follows:

\[
liquid \rightarrow solid + liquid \rightarrow solid.
\]

To model this process, we start from [Ciesielski et al. 2019] in which melting and re-solidification processes for a metallurgical problem have been considered at a temperature interval \(\Delta T\). Then, in our case, equation (4) is valid, to whose right-hand side we add \(Q_i^{(v)}(x, t)\), which models the volumetric moving laser heat source:

\[
C(T_3(x, t)) \frac{\partial T_3(x, t)}{\partial t} = \lambda_3 \nabla^2 T_3(x, t) + Q_i^{(v)}(x, t), \quad \forall x \in \Omega_3 \quad \forall t > 0, \quad (13)
\]

with the following initial conditions

\[
T_3(x, 0) = T_0 \quad \forall x \in \Omega_3.
\]

Concerning the boundary conditions, a first Robin condition concerns the heat flow which flows from \(\Omega_3\) towards the workbench (face \(F_{11}\)):

\[
\lambda_3 \frac{\partial T_3(x, t)}{\partial \hat{n}} = h_{\text{bench}} [T_{\text{bench}} - T_3(x, t)] \quad \forall x \in F_{11} \quad \forall t > 0.
\]

Furthermore, the upper side (face \(F_9\)), is in contact with the air; so, the following Cauchy-Stefan-Boltzmann boundary condition makes sense:

\[
\lambda_3 \frac{\partial T_3(x, t)}{\partial \hat{n}} = h_{\text{air}} [T_{\text{air}} - T_3(x, t)] + \epsilon k_B [T_{\text{air}}^4 - T_3^4(x, t)] + Q_i^{(s)}(x, t) \quad \forall x \in F_9 \quad \forall t > 0,
\]

in which the contribution due to irradiation is present (\(Q_i^{(s)}(x, t)\), laser beam heat source). Finally, lateral surfaces of \(\Omega_3\) in contact with the internal lateral surfaces of \(\Omega_1\) and \(\Omega_2\) are only affected by conduction flows; therefore

\[
\lambda_3 \frac{\partial T_3(x, t)}{\partial \hat{n}} = \lambda_i \frac{\partial T_i(x, t)}{\partial \hat{n}}, \quad \forall x \in F_2 \cup F_7 \quad \forall t > 0.
\]
Finally, for $\Omega$, the model is compactly written as:

$$
\begin{align*}
& C(T_3(x, t)) \frac{\partial T_3(x, t)}{\partial t} = \lambda_3 \nabla^2 T_3(x, t) + Q^{(v)}_l(x, t), \quad \forall x \in \Omega_3 \quad \forall t > 0 \\
& T_3(x, 0) = T_0, \quad \forall x \in \Omega_3 \\
& \lambda_3 \frac{\partial T_3(x, t)}{\partial n} = Q^{(s)}_l(x, t) + h^{air}(T_{air} - T_3) + \epsilon \sigma_B (T^4_{air} - T^4_3), \quad \forall x \in F_9 \quad \forall t > 0, \\
& \lambda_3 \frac{\partial T_3(x, t)}{\partial n} = h^{air}(T_{air} - T_3) + h^{bench}(T_{bench} - T_3), \quad \forall x \in F_{11} \quad \forall t > 0, \\
& \lambda_3 \frac{\partial T_3(x, t)}{\partial n} = h^{air}(T_{air} - T_3), \quad \forall x \in F_8 \cup F_{10} \quad \forall t > 0, \\
& \lambda_3 \frac{\partial T_3(x, t)}{\partial n} = \lambda_1 \frac{\partial T_1(x, t)}{\partial n}, \quad \forall x \in F_2 \cup F_7 \quad \forall t > 0.
\end{align*}
$$

(14)

### 3.4 Full Model in the Domain

Finally, for $\Omega$, the model is compactly written as:

$$
\begin{align*}
& \mu \frac{\partial T(x, t)}{\partial t} = \lambda \nabla^2 T(x, t) + \eta Q^{(v)}_l(x, t), \quad \forall x \in \Omega \quad \forall t > 0 \\
&T(x, 0) = T_0, \quad \forall x \in \Omega \\
& \lambda \frac{\partial T(x, t)}{\partial n} = \eta Q^{(s)}(x, t) + h^{air}(T_{air} - T) + \epsilon \sigma_B (T^4_{air} - T^4), \quad \forall x \in F_4 \cup F_9 \cup F_{14} \quad \forall t > 0, \\
& \lambda \frac{\partial T(x, t)}{\partial n} = h^{air}(T_{air} - T) + h^{bench}(T_{bench} - T), \quad \forall x \in F_6 \cup F_{11} \cup F_{16} \quad \forall t > 0, \\
& \lambda \frac{\partial T(x, t)}{\partial n} = h^{air}(T_{air} - T), \quad \forall x \in F_1 \cup F_3 \cup F_5 \cup F_8 \cup F_{10} \cup F_{12} \cup F_{13} \cup F_{15} \quad \forall t > 0.
\end{align*}
$$

(15)

that compactly assumes the form

$$
\begin{align*}
& \mu \frac{\partial T(x, t)}{\partial t} = \lambda \nabla^2 T(x, t) + \eta Q^{(v)}_l(x, t), \quad \forall x \in \Omega \quad \forall t > 0 \\
&T(x, 0) = T_0, \quad \forall x \in \Omega \\
& \lambda \frac{\partial T(x, t)}{\partial n} + h^{air}(T - T_{air}) + \beta h^{bench}(T - T_{bench}) + \alpha \epsilon \sigma_B (T^4 - T^4_{air}) = \eta Q^{(s)}_l(x, t), \quad \forall x \in \partial \Omega \quad \forall t > 0.
\end{align*}
$$

(16)

where

$$
T(x, t) = \begin{cases} 
T_1(x, t) & \text{if } x \in \Omega_1, \\
T_2(x, t) & \text{if } x \in \Omega_2, \\
T_3(x, t) & \text{if } x \in \Omega_3,
\end{cases}
$$

(17)
\[ \mu = \begin{cases} C_S & \text{if } \mathbf{x} \in \Omega_1 \cup \Omega_2, \\ C(T) & \text{if } \mathbf{x} \in \Omega_3, \end{cases} \]  \tag{18} 

\[ \lambda = \begin{cases} \lambda_1 & \text{if } \mathbf{x} \in \Omega_1, \\ \lambda_2 & \text{if } \mathbf{x} \in \Omega_2, \\ \lambda_3 & \text{if } \mathbf{x} \in \Omega_3, \end{cases} \]  \tag{19} 

\[ \eta = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_1 \cup \Omega_2, \\ 1 & \text{if } \mathbf{x} \in \Omega_3, \end{cases} \]  \tag{20} 

\[ \alpha = \begin{cases} 1 & \text{if } \mathbf{x} \in F_6 \cup F_{16} \cup F_{11}, \\ 0 & \text{if } \mathbf{x} \in \partial \Omega \setminus (F_6 \cup F_{16} \cup F_{11}), \end{cases} \]  \tag{21} 

\[ \beta = \begin{cases} 1 & \text{if } \mathbf{x} \in F_4 \cup F_{14} \cup F_{9}, \\ 0 & \text{if } \mathbf{x} \in \partial \Omega \setminus (F_4 \cup F_{14} \cup F_{9}). \end{cases} \]  \tag{22} 

### 3.5 Realistic Formulations for both Volumetric and Superficial Laser Heat Sources

To recover \( T(\mathbf{x}, t) \), it is necessary to define precisely the shape of the “melting hole” and the subsequent solidification scheme. Then, according to the final mechanical properties of the welding, we should mathematically formalize both \( Q_l^{(v)}(\mathbf{x}, t) \) and \( Q_l^{(s)}(\mathbf{x}, t) \).

**Classical Gaussian laser heat source**

To model the moving heat source, as a first approach, we use the established volumetric and superficial Gaussian formulation [Mohan et al. 2022]:

\[ Q_l^{(v)}(\mathbf{x}, t) = \frac{R_f I_0}{r_U} e^{-\frac{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}{r_U^2}}, \]  \tag{23} 

\[ Q_l^{(s)}(\mathbf{x}, t) = R_f I_0 e^{-\frac{(x-x_0)^2 + (y-y_0)^2}{r_U^2}}, \]

displayed in Fig. 4, where \((x_0, y_0, z_0) = (vt, y_0, z_0)\) are the coordinates of the point where the laser beam starts, \(v\) denoting the laser speed, \(r_U\) is the laser radius, \(I_0\) is the laser intensity (which contributes to give the laser power) and \(R_f\) is the reflexivity. It basically just means that on the surface that it’s interacting with, they define a heat flux proportional to a Gaussian distribution.
An Inhomogeneous Model for Laser Welding of Industrial Interest

Fig. 4: Representation of Gaussian laser heat source.

Conical laser heat source
As a term of comparison, we will also use the conical laser heat source (see Fig. 5) deriving from a Gaussian heat distribution. This allowed the following formulations to be used [Mohan et al. 2022]:

\[
Q_l^{(v)}(x, t) = \frac{R_l I_0}{r_U} e^{-\frac{(x - vt)^2 + (y - y_0)^2 + (z - z_0)^2}{r(z)^2}},
\]

\[
Q_l^{(s)}(x, t) = R_l I_0 e^{-\frac{(x - vt)^2 + (y - y_0)^2}{r_U^2}},
\]

where \(r(z)\), representing the action radius of the laser on \(z\), is formulable as:

\[
r(z) = r_U - (r_U - r_L) \cdot \frac{z_U - z}{z_U - z_L},
\]

in which \(r_U\) and \(r_L\) represent the radius on \(z = z_U\) (upper) and \(z = z_L\) (lower), respectively.
Fig. 5: Representation of the conical laser heat source.

Ellipsoid laser heat source
As a further term of comparison, we exploit this interesting formulation, depicted in Fig. 6, formulated as [Pyo et al. 2022]

\[
Q_l^{(v)}(x, t) = R_f \frac{\sqrt{3}P}{abc\sqrt{\pi}} \cdot e^{- \left[ \frac{(x - vt)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} \right]},
\]

\[
Q_l^{(s)}(x, t) = R_f \frac{\sqrt{3}P}{ab\sqrt{\pi}} \cdot e^{- \left[ \frac{(x - vt)^2}{a^2} + \frac{(y - y_0)^2}{b^2} \right]},
\]

(26)

where \(a, b, c\) are the length, the width, the depth respectively [Pyo et al. 2022, Goldak et al. 2005, Chen et al. 2020].
An Inhomogeneous Model for Laser Welding of Industrial Interest

Fig. 6: Representation of ellipsoidal laser heat source.

Remark 1 The solution $T(x, t)$ of (15) is implicitly linked to $I_0$ which, generating the laser beam, represents the primary cause of distribution of $T(x, t)$ in the plates.

4 The Galerkin-FEM Approach

4.1 Some remarks on existence, uniqueness and regularity of the solution

We focus our attention on numerical techniques for solving (15) after verifying that it is a well-posed one. Thus we recall the following:

**Theorem 1** (Miranville-Morosanu) Let $\Omega \subset \mathbb{R}^n$ bounded domain, with a $C^2$ boundary $\partial \Omega$. For a finite time $\tilde{t} > 0$, we consider the following nonlinear parabolic second-order PDE boundary value problem [Miranville et al. 2021]:

\[
\begin{cases}
\frac{\partial T(x, t)}{\partial t} = \Phi(T(x, t)) \nabla \cdot (K(T(x, t)) \nabla T(x, t)) + \\
+ \Psi(T(x, t)) r^{(v)}(x, t), & \text{in } (0, \tilde{t}] \times \Omega, \\
K(T(x, t)) \frac{\partial T(x, t)}{\partial \hat{n}} + p_1 [T(x, t) - \theta_1] + p_2 [T(x, t) - \theta_2] + \\
+ p_3 [T^4(x, t) - \theta_3^4] = p_4 r^{(s)}(x, t), & \text{on } (0, \tilde{t}] \times \partial \Omega,
\end{cases}
\]

(27)

where $\Phi(T(x, t))$ controls the speed of the diffusion process; $K(T(x, t))$ represent the mobility attached to the solution $T(x, t)$; $r^{(v)}(x, t)$ and $r^{(s)}(x, t)$ are distributed control and boundary control, respectively.

Furthermore, $\nabla$ denotes the gradient, $\hat{n} = \hat{n}(x)$ is the outward unit normal vector to $\Omega$ as a point $x \in \partial \Omega$ and $\frac{\partial}{\partial \hat{n}}$ denotes differentiation along $\hat{n}$.

If the following conditions are satisfied:
1) \( p_1, p_2, p_3, p_4 \), are non-negative constants;
2) \( \Phi(T(x, t)) \), is a positive and bounded real function of class \( C^1((0, \bar{t}] \times \Omega) \), with bounded derivative;
3) \( K(T(x, t)) \) assumed to satisfy the following inequality
   \[ 0 < K_m \leq K(T(x, t)) \leq K_M, \quad \forall (x, t) \in (0, \bar{t}] \times \Omega; \]
   where \( K_m, K_M \) are constants;
4) \( \Psi(T(x, t)) \), is a positive bounded real function;
5) \( r^{(v)}(x, t) \in L^p((0, \bar{t}] \times \Omega) \) with \( p \geq 2 \);
6) \( r^{(s)}(x, t) \in W^{1-p^{-2}, 2-p^{-1}p}((0, \bar{t}] \times \partial \Omega) \);
7) \( T_0(x) \in W^{2-p^{-2}p, \infty}(\Omega) \), verifying

\[
K(T_0(x)) \frac{\partial T_0(x)}{\partial n} + p_1[T_0(x) - \theta_1] + p_2[T_0(x) - \theta_2] + p_3[T_0(x) - \theta_3] = p_4 r^{(s)}(x, 0);
\]
then problem (27) is well-posed.

Proof For the proof of this theorem refer to [Miranville et al. 2021].

Remark 2 It is worth noting that Theorem 1 requires that \( \Phi(T(x, t)) \) is a positive real function, bounded and, above all, of \( C^1((0, \bar{t}] \times \Omega) \) requiring the continuity of \( T(x, t) \).

Theorem 1 can be successfully applied in our laser welding process setting

\[
\Phi(T(x, t)) = \frac{1}{\mu(T(x, t))},
\]

\[
K(T(x, t)) = \lambda, \quad K_m \leq K \leq K_M,
\]

\[
\Psi(T(x, t)) = \frac{\eta}{\mu(T(x, t))},
\]

\[
p_1 = h^{air}, \quad p_2 = \beta h^{bench}, \quad p_3 = \alpha \epsilon \sigma_B, \quad p_4 = \eta, \quad \theta_1 = T^{air}, \quad \theta_2 = T^{bench}, \quad \theta_3 = T^{air},
\]

\[
r^{(v)}(x, t) = Q^{(v)}_l(x, t), \quad r^{(s)}(x, t) = Q^{(s)}_l(x, t).
\]

Thus, the non-linear inhomogeneous parabolic model (16) is achieved. It is easy to prove that \( Q^{(v)}_l(x, t) \in L^p((0, \bar{t}] \times \partial \Omega), Q^{(s)}_l(x, t) \in W^{1-p^{-2}, 2-p^{-1}p}((0, \bar{t}] \times \partial \Omega). So, Theorem 1 guarantees the well-posedness of solutions to the problem (16).
4.2 Galerkin-FEM basics

According to the Subsection 3.4, we rewrite both the equation and boundary conditions of \((15)\) as follows [Quarteroni 2015]:

\[ R_1 : \mu \frac{\partial T(x,t)}{\partial t} - \lambda \nabla^2 T(x,t) - \eta Q_l^{(v)}(x,t) = 0, \]

\[ R_2 : \lambda \frac{\partial T(x,t)}{\partial n} - \beta h_{bench}[T_{bench} - T(x,t)] + h_{air}[T_{air} - T(x,t)] + \alpha \epsilon \sigma_B [T_{air}^4 - T^4(x,t)] - \eta Q_l^{(s)}(x,t) = 0. \] (32)

If \(w_1\) and \(w_2\) are two weight functions, from both \((32)\), we can write

\[ \int_\Omega w_1 R_1 d\Omega + \int_{\partial \Omega} w_2 R_2 d(\partial \Omega) = 0, \] (33)

The first integral in \((33)\), becomes

\[ \int_\Omega w_1 R_1 d\Omega = \int_\Omega w_1 \left( \mu \frac{\partial T(x,t)}{\partial t} \right) d\Omega + \int_\Omega w_1 \lambda \nabla^2 T(x,t) d\Omega - \int_\Omega w_1 \eta Q_l^{(v)}(x,t) d\Omega. \] (34)

Integrating by parts, the second integral of the right side in \((34)\) becomes

\[ \int_\Omega w_1 \lambda \nabla^2 T(x,t) d\Omega = \int_{\partial \Omega} \lambda w_1 \frac{\partial T(x,t)}{\partial n} d(\partial \Omega) - \int_\Omega \lambda \nabla w_1 \cdot \nabla T(x,t) d\Omega. \] (35)

Therefore, the equation \((34)\), by means of \((35)\), is writable as

\[ \int_\Omega w_1 R_1 d\Omega = \int_\Omega w_1 \mu \frac{\partial T(x,t)}{\partial t} d\Omega - \int_{\partial \Omega} \lambda w_1 \frac{T(x,t)}{\partial n} d(\partial \Omega) + \int_\Omega \lambda \nabla w_1 \cdot \nabla T(x,t) d\Omega - \int_\Omega w_1 \eta Q_l^{(v)}(x,t) d\Omega = 0, \]

from which

\[ \int_{\partial \Omega} \lambda w_1 \frac{\partial T(x,t)}{\partial n} d(\partial \Omega) = \int_\Omega w_1 \left( \mu \frac{\partial T(x,t)}{\partial t} - \eta Q_l^{(v)}(x,t) \right) d\Omega + \int_\Omega \lambda \nabla w_1 \cdot \nabla T(x,t) d\Omega. \] (36)
The second integral in (33), \( \forall i \in \{1, 2, 3\} \) and \( \forall j \in \{1, 2, ..., 16\} \), becomes
\[
\int_{\partial \Omega} w_2 R_2 d(\partial \Omega) = \int_{\partial \Omega} w_2 \left\{ \lambda \frac{\partial T(x, t)}{\partial n} - \beta h^{\text{bench}} [T_{\text{bench}} - T(x, t)] + h^{\text{air}} [T_{\text{air}} - T(x, t)] - \alpha \epsilon \sigma B \left[ T^4_{\text{air}} - T^4(x, t) \right] + \eta Q^I(s)(x, t) \right\} d(\partial \Omega) = 0,
\]
from which
\[
\int_{\partial \Omega} \lambda w_2 \frac{\partial T(x, t)}{\partial n} d(\partial \Omega) = \int_{\partial \Omega} w_2 \eta Q^I(s)(x, t) d(\partial \Omega) + \int_{\partial \Omega} w_2 \beta h^{\text{bench}} [T_{\text{bench}} - T(x, t)] d(\partial \Omega) + \int_{\partial \Omega} w_2 h^{\text{air}} [T_{\text{air}} - T(x, t)] d(\partial \Omega) + \int_{\partial \Omega} w_2 \alpha \epsilon \sigma B \left[ T^4_{\text{air}} - T^4(x, t) \right] d(\partial \Omega).
\]
For \( w_1 = w_2 = w \), both (36) and (4.2) have the same left side, so that, subtracting side-by-side, we can write:
\[
\int_{\Omega} w \left( \mu \frac{\partial T(x, t)}{\partial t} - \eta Q^I(v)(x, t) \right) d\Omega + \int_{\Omega} \lambda w \nabla w \cdot \nabla T(x, t) d\Omega + \int_{\partial \Omega} w \eta Q^I(s)(x, t) d(\partial \Omega) - \int_{\partial \Omega} w \beta h^{\text{bench}} [T_{\text{bench}} - T(x, t)] d(\partial \Omega)
- \int_{\partial \Omega} w h^{\text{air}} [T_{\text{air}} - T(x, t)] d(\partial \Omega) - \int_{\partial \Omega} w \alpha \epsilon \sigma B \left[ T^4_{\text{air}} - T^4(x, t) \right] d(\partial \Omega) = 0.
\]
(37)

We discretize \( \Omega \) into \( n \) nodes, on each of which the temperature is indicated with \( T_k \) \( (k = 1, ..., n) \). If \( N_k \) are the shape functions, then
\[
T = \sum_{k=1}^{n} N_k T_k = N_1 T_1 + N_2 T_2 + ... + N_n T_n,
\]
\[
T = [N]\{T\} = \sum_{i=1}^{n} N_i T_i, \quad T^4 = [N]\{T^4\}, \quad \frac{\partial T}{\partial t} = [N]\{\dot{T}\},
\]
\[
\frac{\partial T}{\partial x} = [N_x]\{T\}, \quad \frac{\partial T}{\partial y} = [N_y]\{T\}, \quad \frac{\partial T}{\partial z} = [N_z]\{T\}.
\]
Assuming that the weight functions are equal to the shape functions, the following makes sense
\[
\frac{\partial w}{\partial x} = [N_x], \quad \frac{\partial w}{\partial y} = [N_y], \quad \frac{\partial w}{\partial z} = [N_z],
\]
so that (37) becomes

\[
\int_{\Omega} \mu(N \{ T \}) N \{ \dot{T} \} d\Omega + \int_{\Omega} \lambda \left( N_x[N_x] + N_y[N_y] + N_z[N_z] \right) \{ T \} d\Omega = \\
= \int_{\Omega} \eta Q_{i}^{(v)}(x, t) d\Omega + \int_{\partial \Omega} \eta Q_{i}^{(s)}(x, t) d(\partial \Omega) + \\
+ \int_{\partial \Omega} [N] \beta h_{\text{bench}} T_{\text{bench}} d(\partial \Omega) + \int_{\partial \Omega} [N] h_{\text{air}} T_{\text{air}} d(\partial \Omega) + \\
+ \int_{\partial \Omega} [N] \alpha \varepsilon \sigma B T_{\text{air}}^4 d(\partial \Omega) - \int_{\partial \Omega} [N] \alpha \varepsilon \sigma B [N] \{ T^4 \} d(\partial \Omega) + \\
\int_{\partial \Omega} [N] \alpha \varepsilon \sigma B [N] \{ T \} d(\partial \Omega) - \int_{\partial \Omega} [N] \alpha \varepsilon \sigma B [N] \{ T^4 \} d(\partial \Omega).
\]

(38) is writable as

\[
[Z] \{ \dot{T} \} + [\tilde{K}] \{ T \} + [H] \{ T^4 \} = \{ F \},
\]

whose integrals are computed by the Crank-Nicholson procedure. The FEM approach, according to (39), has been numerically implemented on an Intel Core 2 CPU 1.45 GHz machine and MatLab R2022 PDE tool, testing them on different kinds of laser sources as described in Subsection 3.5.

### 5 Results of Computations

Here, the results obtained using the laser heat source as above detailed are presented and discussed. The alloy Al-Si 5% physical parameters, kindly provided by IRIS s.r.l., are listed in Table 4 and the laser physical parameters related to a typical welding process for alloy Al-Si 5% are listed in Table 5 (the laser intensity is computed by \( I_0 = \frac{P}{\pi r_U^2} \)).
Table 4: Physical parameters of Al-Si 5% alloy.

<table>
<thead>
<tr>
<th>parameter</th>
<th>value</th>
<th>unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_S$</td>
<td>$2.943 \cdot 10^6$</td>
<td>$J/(m^3K)$</td>
</tr>
<tr>
<td>$C_L$</td>
<td>$3.07 \cdot 10^6$</td>
<td>$J/(m^3K)$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>290</td>
<td>$W/(mK)$</td>
</tr>
<tr>
<td>$L_v$</td>
<td>$990.6 \cdot 10^6$</td>
<td>$J/(m^3)$</td>
</tr>
<tr>
<td>$T_S$</td>
<td>850.15</td>
<td>$K$</td>
</tr>
<tr>
<td>$T_L$</td>
<td>923.15</td>
<td>$K$</td>
</tr>
<tr>
<td>$\epsilon$</td>
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<td></td>
</tr>
</tbody>
</table>

Table 5: Laser parameters.

<table>
<thead>
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<th>parameter</th>
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<tr>
<td>$v$</td>
<td>velocity</td>
<td>40</td>
<td>$mm/s$</td>
</tr>
<tr>
<td>$P$</td>
<td>power</td>
<td>2400</td>
<td>$W$</td>
</tr>
<tr>
<td>$r_U$</td>
<td>radius</td>
<td>1.5</td>
<td>$mm$</td>
</tr>
<tr>
<td>$R_f$</td>
<td>reflexivity</td>
<td>0.9</td>
<td></td>
</tr>
<tr>
<td>$I_0$</td>
<td>intensity</td>
<td>340</td>
<td>$W/mm^2$</td>
</tr>
</tbody>
</table>

Furthermore, we set $\sigma_B = 1.35 \cdot 10^{-23} J/K$, $T_0 = T_{air} = T_{bench} = 298K$ (typical environmental temperature in welding forge [Escribano-Garcia et al. 2022, D’Ostuni et al. 2017]); $h_{air} = 15 \cdot 10^{-6} W/(mm^2K)$ and $h_{bench} = 20 h_{air}$ (as experimentally suggested from metallurgical experiments [Tsirkas et al. 2010, Park et al. 2008] to guarantee both the correct penetration without favoring the thermal degradation of the alloy structure and is such that it produces significant effects even at depth).

Galerkin-FEM procedure applied to (15), described above and implemented in MatLab, optimizes a mesh with tetrahedral elements. To obtain reliable and superimposable results with experimental evidence, $\Omega$ has been discretized by a mesh with 3500 finite elements (4522 nodes, 3924 edges) which thicken in the vicinity and especially in correspondence of $\Omega_3$ (welding area) where the melting process takes place (Fig. 7a). Moreover, the mesh refinement at $\Omega_3$ has also been slightly extended in $\Omega_1$ and $\Omega_2$ to perform the temperature reduction in passing from the welding area to the remaining part of $\Omega$. Furthermore, the quality of the mesh was confirmed by computing the indices presented in Subsection 5.1 whose values obtained fall within the respective ranges of admissible values for good quality meshes.
The lasting of the considered welding process is 2.5s, as is the usual practice for laser welding of plates of dimensions compatible with those fixed in this work [Mascenik et al. 2020] (see Fig. 7b, where the red point represents the impact point of the laser beam). Once the laser has melted the material, advancing further along the joining line of the plates, the molten material re-solidifies thanks to the significant drop in temperature (blue area, see Fig. 7b).

5.1 Mesh creation

We create the mesh $\overline{T} = \{E_k\}$ where $E_k$ is the generic finite element and such that $E_k \cap E_{k'} = \emptyset$, $k \neq k', \forall k \neq k'$ (is either empty or consists of exactly one node or of one edge), obtained with triangulation techniques to obtain tetrahedral finite elements. We have discretized $\Omega$ in order that $\Omega = \bigcup_{k=1}^{N_T} E_k$, with $N_T = |\overline{T}|$. Moreover, the size of the mesh $h$, $h(E_k)$, is quantifiable as [Quarteroni 2015]

$$h(E_k) = \sup_{x, y \in E_k} \|x - y\|,$$

so that $h = \max_{E_k \in \overline{T}} h(E_k)$. Particularly, the volume of each $E_k$, indicated by $V_k$, is computable as

$$V_k = \frac{1}{6} \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ y_2 - y_1 & y_3 - y_1 & y_4 - y_1 \\ z_2 - z_1 & z_3 - z_1 & z_4 - z_1 \end{vmatrix},$$

Fig. 7: (a) Mesh creation: 3500 finite element (4522 nodes, 3924 edges); (b) Fusion zone and welding direction.
where \(x_i, y_i, z_i\) are the coordinates of the vertices \(P_i\) of \(E_k\). Moreover, for each triangle of \(E_k\), the surface is

\[
S_k = 0.5 \left| \begin{array}{ccc}
x_2 - x_1 & x_3 - x_1 \\
y_2 - y_1 & y_3 - y_1 \\
z_2 - z_1 & z_3 - z_1
\end{array} \right|.
\] (42)

Therefore, a face of \(E_k\) is an ordered triad (allowing all possible combinations), while the edges are considered, not as elements of the faces, but as separate entities and are implicitly defined in terms of ordered pairs of vertices. To each \(E_k\) we associate the sphere circumscribed at its vertices whose radius, \(C_k\), and its center can be obtained by solving the equation:

\[
C_k = \frac{1}{\sqrt{3}} \left( \sum_{i=1}^{4} S_i \right)
\] (43)

where \(l_i^2 = x_i^2 + y_i^2 + z_i^2\) and \(l_i^2 = x_i^2 + y_i^2 + z_i^2\) (\(i = 1, ..., 4\)), or solving a system of linear equation for achieving the center. As in two dimensions, there is a formula that allows you to calculate the radius and that allows you to avoid calculating \(C_k\):

\[
r_k = (24V_k)^{-1}((m + n + s)(m + n - s)(n + s - m)(m - n + s))^{0.5}
\] (44)

where \(m, n\) and \(s\) are the products of the lengths of two opposite edges. Finally, the radius of the inscribed sphere can be evaluated as

\[
\gamma_k = \frac{3V_k}{S_1 + S_2 + S_3 + S_4},
\] (45)

where \(S_i\) is the surface of the triangle \(i\) of \(E_k\). It is worth noting that the quality of any \(E_k\) is a value that measures the shape of the element itself and is essential for the accuracy of the results to be achieved. In particular, for an element \(E_k\), a good measure of quality is given by

\[
Q_k = \sigma(3V_k)^{-1}h_{max}S_k,
\] (46)

where \(\sigma\) is a normalization factor in order that for an equilateral triangle \(Q_k = 1\) (i.e., \(\beta = \sqrt{3}/12\)), \(h_{max}\) is the longer edge and \(S_k\) is the sum of \(S_i\).

In this paper, in addition to (46), we evaluate the quality of the meshes also exploiting the following indices [Quarteroni 2015]: \(r_k(\gamma_k)^{-1}\), \(r_k(h_{max})^{-1}\), \(V_k^2\left( \sum_{i=1}^{4} S_i^2 \right)^{-3}\). Each of these (computationally expensive) indices offers advantages and disadvantages, and none by itself classifies the element exhaustively.
5.2 Exploiting classical Gaussian laser heat source

As already specified, the duration of the welding process is equal to 2.5s, in accordance with the executive practice of laser welding for plates whose dimensions are compatible with those specified in this [Mascenik et al. 2020] paper. Fig. 8 displays the welding process implemented in MatLab; once the material has melted, the laser advances along the joint line of the slabs, melting further material while the previously melted material solidifies as the temperature drops drastically. Fig. 9, relating to the final point of the weld, offers greater evidence of this phenomenon, emphasizing that, already in the numerical simulation phase, (15) well models the processes underlying the proposed approach. As highlighted in Fig. 7a, the mesh is not capillary over $\Omega$ but only over $\Omega_3$ and its immediate vicinity because (15) does not consider any delay times for the distribution of $T(x, t)$ during the passage of the laser beam. Therefore, considering this distribution throughout $\Omega$ instantaneous, the cooling in both $\Omega_1$ and $\Omega_2$ is immediate also considering that the thermal conduction between the plates and the workbench has also been considered.

![Welding process](image)

**Fig. 8:** Welding process. The red point represents the impact area of the laser beam. Initial zone: (a)-(b); central zone: (c)-(d)-(e) and final zone: (f).

Both Figg. 10a and 10b depict this aspect because show the distribution of $T(x, t)$ both in the initial point and in five different points of the welding wire from which it can be seen that, once the temperature peak has been reached,
as the laser beam advances, the cooling is locally evident. Furthermore, the final point of the welding path is affected both by the presence of the laser beam and by the conduction of the other points where the welding has already taken place. In fact, the peaks increase as the weld progresses toward the endpoint. However, we observe that this remark does not affect the quality of the proposed model because, regardless of any delay times in the distribution of $T(x, t)$, $\Omega_3$ remains the most thermally stressed area where, after the laser welding process, checks the mechanical properties. In confirmation of the above, the thermal cycles were obtained transversally to the welding line (in correspondence with its midpoint) from which it is once again highlighted how, moving away from the welding line, the cooling of the plates is drastic (see Fig. 11a). On the other hand, moving along the welding line, the thermal cycles show behaviors that are qualitatively/quantitatively similar to each other (see Fig. 11b). Particularly, during the welding process, the temperature peaks are well above the melting range of the alloy as required by the welding execution practice to guarantee the melting of the material along the entire depth of the plates.

Fig. 9: Distribution of $T(x, t)$ in $\Omega$ when the laser beam, modeled using Gaussian formulation, has reached the final point of the welding path.
Fig. 10: Gaussian 3D laser heat source: distribution of $T(x, t)$ (a) in five different points on the welding wire ($P_0 = (0, 40.5, 4), P_1 = (25, 40.5, 4), P_2 = (50, 40.5, 4), P_3 = (75, 40.5, 4), P_4 = (100, 40.5, 4)$) and (b) in the middle point $P_1$ of the welding wire.

Fig. 11: Thermal cycles in: (a) transverse direction, (b) longitudinal direction.

5.3 Exploiting conical 3D laser heat source

Here, exploiting the same mesh used for the simulations with the classical Gaussian 3D laser heat source ($^4$), we set all parameters as in Subsection 5.2. Moreover, here $z_U = 4mm$ and $z_L = 0mm$, as often happens in many aluminum alloy plate welding processes. Therefore, the $T(x, t)$ distribution in $\Omega$ as highlighted in Fig. 12 (when the laser beam reaches the final position of the welding path) simulating a laser welding of the same duration as the one

$^4$Further refinements significantly increase the computational complexity without appreciable performance improvements.
simulated in subsection 5.2. Further in this case, in $\Omega_3$, $T(x, t)$ throughout the welding process, settles on values ensuring the fusion of the aluminum even in depth, although the temperatures reached are higher than when a classical Gaussian 3D laser head source is considered, risking carrying out a melting and re-solidification process that does not meet the required quality standards. As regards the thermal cycles both transversely and longitudinally of the welding line, they appear qualitatively superimposable to those obtained in Figs. 11a and 11b. Thus, (15) does not consider any thermal losses (as well as any time-lag that slows down the temperature distribution), the Conical 3D laser head source could not be suitable for modeling the laser beam because the real $T(x, t)$ in $\Omega_3$, being oversized, could not guarantee the total melting of the alloy without compromising the internal structure of the material. This phenomenon is more evident by analyzing Fig. 12 where the aforementioned drop in temperatures is most evident. However, as highlighted in Subsection 5.2, the weld bead still has a truncated cone shape whose geometric parameters are still potentially compatible with those required by current legislation on laser welding. Furthermore, the surface temperature during the welding process is characterized by peaks that largely exceed the melting range of the alloy (see Figs. 13a and 13b) with consequent marked degradation of the surface itself. It follows that the (15), together with the joint use of the 3D heat source conic and the values chosen for both $P$ and $I_0$, does not represent a reliable tool for dynamic mapping of the temperature in the process of laser welding under study, as metallurgically recently proved [D’Ostuni et al. 2017, Escribano-Garcia et al. 2022].

(a)

(b)

Fig. 12: Distribution of $T(x, t)$ in $\Omega$ with conical 3D heat source.
Fig. 13: Conical 3D laser heat source: distribution of $T(x, t)$ (a) in the same points as in Fig. 10a and (b) in the middle point $P_1$ of the welding wire.

5.4 Exploiting ellipsoid laser heat source

As further confirmation of the goodness of using the classical 3D Gaussian formulation for laser beam modeling, we present here the results obtained using the ellipsoid formulation as specified in Subsection 3.5. Here too, as for the previous Subsections, we have simulated the welding of the same duration using the same physical parameters for the alloy (Tab. 4) while, concerning the ellipsoid laser heat source parameters, as the literature suggests, we set $a_f = 1\text{mm}$, $a_r = 5\text{mm}$ and $b = c = 1\text{mm}$ (5). As depicted in Fig. 14, the distribution of $T(x, t)$ in $\Omega_3$ highlights high values of temperature that melting the material only on the surface (6) but without producing deep fusion. This is also evidenced in both Figs. 15b and 15a which show the same qualitative behavior highlighted using the other laser sources, but with different peaks of temperature. We finally observe that, also in this case, both transversal and longitudinal thermal cycles are qualitatively superimposable with those obtained using the other formulations of heat laser sources. Therefore, the proposed model, assisted by this formulation of the heat source laser and with appropriate power output, appears suitable for industrial and iron and steel applications which require superficial and sub-surface fusions of the material [Sarila et al. 2022, Balbaa et al. 2022, Xu et al. 2022, Ma et al. 2022, Ramiarison et al. 2022].

\(^5\)In this case we also used the same mesh exploited in the previous cases since its refinement did not produce appreciable improvements (compared to a conspicuous increase in computational complexity).

\(^6\)Without eliminating the risk of local damage in the structure.
**Fig. 14**: Temperature distribution in Ω when the laser beam, modeled using the ellipsoid formulation, has reached the final point of the welding path.

**Fig. 15**: Ellipsoid 3D laser heat source: distribution of $T(x, t)$ (a) in the same points as in Fig. 10a and (b) in the middle point $P_1$ of the welding wire.
The research proposed in this paper had established preliminary guidelines to evaluate the feasibility of the process of laser welding for juxtaposed Al-Si 5% alloy plates without adding additional material, as per the protocol adopted by a well-known Italian company operating in the field of laser welding at an industrial level. Particularly, an innovative non-homogeneous parabolic dynamic model with Cauchy-Stefan-Boltzmann boundary conditions is proposed taking into account that, metallurgically, the co-presence of solid and liquid fractions occurs during the melting (and consequent re-solidification) of the alloy. Furthermore, to make the approach more realistic, and to consider further boundary conditions for simulating the presence of the workbench where the specimen is to be welded, the approach proposes, according to experimental evidence, a polynomial-shaped substitute thermal capacity of the alloy from which to achieve the solid state fraction. Based on these assumptions, and once the well-posedness of the problem has been verified, the offline numerical recovering of the absolute temperature has been performed by a Galerkin-FEM approach whose mesh quality has been verified through specific indices. These results, in accordance with experimental tests known in the literature, have demonstrated the following peculiarities:
- the proposed model is quite realistic since it simulates the welding process of Al-Si 5% alloy plates, used on an industrial level with appreciable results, placed side by side which takes the form of the creation of a fusion/re-solidification bead, in compliance with regulatory standards, without the addition of additional material which, usually, causes unwanted thickening;
- the process here implemented, based on a temperature melting range of the alloy fits well with the needs of modern metallurgy;
- the further boundary condition, taking into account the fact that the plates are placed on a workbench, lays the first foundations for more accurate modeling, which begins to take into account any thermal losses;
- as further highlighted in Fig. 16, the laser beam modeled by the classic 3D Gaussian formulation fits well with the proposed analytical approach, leaving the operator the right choice of laser beam power (strictly linked to the laser electrical current) to avoid damage to the material structure;
- the proposed polynomial-shaped substitute thermal capacity has made it possible to obtain maps highlighting gradual variations in temperature which agree well with the fact that the analytical solution is unique and regular (also providing the volumetric solid state fraction which, as is known, defines the mechanical properties of the weld).

Obviously, the topic is far from being fully studied and several more studies need to be conducted to optimize the proposed process and then evaluate with standardized characterization tests the results achieved. Particularly, the ongoing research foresees the following future developments:

1. although the results obtained appear promising, it is worth making an effort to consider the non-linearity in the thermal conductivity, i.e., the thermal conductivity can depend on the temperature;
2. the proposed model requires an upgrading concerning the effects due to the significant thermal expansion of the alloy which increases the solidification shrinkage and the consequent risk of cracks;
3. to better simulate the real behavior of laser welding (for possible comparisons with real data obtained from laboratory experiments at the company that provided the data), a further upgrading is due to take into account the fact that during welding vapor bubbles trapped in the material (keyholes) can be produced as well as phenomena of surface tension imbalances, due to temperature gradients, which by generating imbalances in mechanical forces induce liquid migrations (Marangoni effect);
4. the promoted approach needs to model the problem in order to take into account both the fact that the temperature distribution is not instantaneous, and that the laser beam, flowing over restricted areas of material, favors microstructural disturbances of the material;
5. taking advantage of the versatility of laser sources in terms of management of power and power density can be done on one single workstation for both fusion and welding operations local heating to favor certain metallurgical processes;
6. the localized treatment phase is essentially governed by the power density of the laser beam, the interaction time between the material, and the total energy of the process;
7. Although the power of the laser is high, it must be taken into account that it takes some time for the electrical power, converted into the thermal equivalent, to penetrate the material causing it to melt. Then it would be
appropriate consider the response time of the material, before the heat input from the laser penetrates deeply, and introduce one/two relaxation times, considering an Maxwell-Cattaneo-Vernotte/Dual-Phase-Lag type model;

8. the dynamic problem studied, if framed in the context of non-homogeneous materials in contact with each other, requires highly non-uniform meshes (to be appropriately refined and de-refined) with preconditions that are difficult to formulate. Then, we find it interesting to reformulate the problem in this framework to obtain optimized numerical solutions using two-level approaches;

9. Finally, it would be desirable for the proposed model to explain the presence of the electric current generating the laser beam so as to clearly highlight the cause-effect link on which to implement specific control actions.

Obviously, it is necessary to design an experimental campaign of measurements based on high-performance techniques (i.e., focused on the Latin hypercubes) through which to identify feasibility areas related to physical parameters to allow the identification of the process conditions that lead to the creation of welds with high mechanical resistance, and are compliant with current legislation.

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Declarations

Conflict of interest On behalf of all the authors, the corresponding author states that there is no conflict of interest.

References


Inhomogeneous Model for Laser Welding of Industrial Interest


An Inhomogeneous Model for Laser Welding of Industrial Interest


