On Ramanujan's Continued Fractions of Order Twenty-four

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Abstract: Two continued fractions \(U(q)\) and \(V(q)\) of order twenty-four are obtained from a general continued fraction identity of Ramanujan. Some theta-function and modular identities for \(U(q)\) and \(V(q)\) are established to prove general theorems for the explicit evaluations of \(U(\pm q)\) and \(V(\pm q)\). From the theta-function identities of \(U(q)\) and \(V(q)\), three colour partition identities are derived as application to partition theory of integer. Further, \(2^\alpha, 4^\beta\) and \(8^\gamma\) dissection formulas are established for the continued fractions \(U^*(q) := q^{-5/2}U(q)\) and \(V^*(q) = q^{-1/2}V(q)\), and their reciprocals.

Keywords and phrases: Continued fractions of order twenty-four; Ramanujan’s theta-functions; explicit values; colour partition of integer; dissection formulas.

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1. Introduction

For any complex numbers \(a\) and \(q\), define the \(q\)-product \((a; q)_{\infty}\) as
\[
(a; q)_{\infty} := \prod_{t=0}^{\infty} (1 - aq^t), \quad |q| < 1.
\]

(1.1)

For simplicity, we often write
\[
(a_1; q)_{\infty}(a_2; q)_{\infty}(a_3; q)_{\infty}\cdots(a_m; q)_{\infty} = (a_1, a_2, a_3, \cdots, a_m; q)_{\infty}.
\]

Ramanujan’s general theta-function \(f(x, y)\) [4, p. 34] is defined as
\[
f(x, y) = \sum_{t=-\infty}^{\infty} x^{t(t+1)/2}y^{t(t-1)/2}, \quad |xy| < 1.
\]

(1.2)

In terms of \(f(x, y)\), Jacobi’s triple product identity [4, p. 35, Entry 19] can be stated as
\[
f(x, y) = (-x; xy)_{\infty}(-y; xy)_{\infty}(xy; xy)_{\infty} = (-x, -y; xy; xy)_{\infty}.
\]

(1.3)

Also, from [4, p. 34, Entry 18(iv)], we note that, for any integer \(h\),
\[
f(x, y) = x^{h(h+1)/2}y^{h(h-1)/2}f(x(xy)^h, y(xy)^{-h}).
\]

(1.4)

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Three important special cases of \( f(a, b) \) are the theta-functions \( \phi(q) \), \( \psi(q) \) and \( f(-q) \) [4, p. 36, Entry 22(i)-(iii)] given by
\[
\phi(q) := f(q, q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{(-q; q)_\infty}{(q; q)_\infty}, \quad (1.5)
\]
\[
\psi(q) := f(q, q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \quad (1.6)
\]
\[
f(-q) := f(-q, -q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2} = (q; q)_\infty. \quad (1.7)
\]

Ramanujan also defined the function \( \chi(q) \) [4, p. 36, Entry 22(iv)] as
\[
\chi(q) = (-q; q^2)_\infty. \quad (1.8)
\]

One of the Ramanujan’s remarkable contributions is in the field of \( q \)-continued fraction. An interesting continued fraction of Ramanujan is the Ramanujan–Göllnitz–Gordon continued fraction \( H(q) \) recorded on page 299 of Ramanujan’s second notebook [10] and is given by
\[
H(q) := q^{1/2} (q; q^6)_\infty (q^7; q^8)_\infty = q^{1/2} \frac{f(-q, -q^7)}{f(-q^7, -q^5)} = \frac{q^{1/2}}{1 + q + \frac{q^2}{1 + q^3 + \frac{q^4}{1 + q^5 + \ldots}}}, \quad |q| < 1. \quad (1.9)
\]

Second equality of (1.9) easily follows from the first equality and (1.3), and \( H(q) \) can be referred as a continued fraction of order eight. Göllnitz [7] and Gordon [8], independently, rediscovered and proved (1.9). Andrews [1] proved (1.9) as a corollary of a more general result. An alternative proof of (1.9) was also given by Ramanathan [9]. Ramanujan offered two other identities [10, p. 299] for \( H(q) \), namely,
\[
\frac{1}{H(q)} - H(q) = \frac{\phi(q^2)}{q^{1/2} \psi(q^4)}, \quad (1.10)
\]
and
\[
\frac{1}{H(q)} + H(q) = \frac{\phi(q)}{q^{1/2} \psi(q^4)}. \quad (1.11)
\]

Proofs of (1.10) and (1.11) can be found in [4, p. 221]. Chan and Huang [6] found many identities involving the continued fraction \( H(q) \) and evaluated explicitly \( H(e^{-\pi \sqrt{1/2}}) \) for several positive integers \( n \). Vasuki and Srivatsa Kumar [12] also established new modular relations for \( H(q) \). Baruah and Saikia [3] established some general theorems for explicit evaluations of \( H(q) \) and evaluated some values.

In his notebooks, Ramanujan also recorded some general continued fractions. For example, Ramanujan recorded the following general continued fraction [4, p. 24, Entry 12]:

Suppose that \( a, b \) and \( q \) are complex numbers with \( |ab| < 1 \) and \( |q| < 1 \), or that \( a = b^{2m+1} \) for some integer \( m \). Then
\[
\frac{(a^2 q^3; q^4)_\infty (b^2 q^3; q^4)_\infty}{(a^2 q^2; q^4)_\infty (b^2 q^4; q^4)_\infty} = \frac{1}{1 - ab + \frac{(a - bq) (b - aq)}{(1 - ab)(q^2 + 1) + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1) + \ldots}}}, \quad (1.12)
\]
Two special cases of the (1.12) and of our interest in this paper are the continued fractions \( U(q) \) and \( V(q) \) defined as

\[
U(q) := q^{5/2} \frac{(q, q^{23}; q^{24})_{\infty}}{(q^{11}, q^{13}; q^{24})_{\infty}} = q^{5/2} \frac{f(-q, -q^{23})}{f(-q^{11}, -q^{13})} = \frac{q^{5/2}(1-q)}{(1-q^6) + \frac{q^6(1-q^6)(1-q^7)}{(1-q^6)(1+q^{12}) + \frac{q^6(1-q^{17})(1-q^{19})}{(1-q^6)(1+q^{24}) + \cdots}}}
\]

(1.13)

and

\[
V(q) := q^{1/2} \frac{(q^5, q^{19}; q^{24})_{\infty}}{(q^7, q^{17}; q^{24})_{\infty}} = q^{1/2} \frac{f(-q^5, -q^{19})}{f(-q^7, -q^{17})} = \frac{q^{1/2}(1-q^5)}{(1-q^6) + \frac{q^6(1-q^5)(1-q^{11})}{(1-q^6)(1+q^{12}) + \frac{q^6(1-q^{15})(1-q^{23})}{(1-q^6)(1+q^{24}) + \cdots}}}
\]

(1.14)

To obtain the \( U(q) \) and \( V(q) \), first replace \( q \) by \( q^6 \) in (1.12), then set \( \{a = q^{5/2}, b = q^{7/2}\} \) and \( \{a = q^{1/2}, b = q^{11/2}\} \), and used the results \((1-q)(q^{25}; q^{24})_{\infty} = (q; q^{24})_{\infty} \) and \((1-q^6)(q^{20}; q^{24})_{\infty} = (q^5; q^{24})_{\infty} \), respectively. From the definitions, \( U(q) \) and \( V(q) \) can be called as continued fractions of order twenty-four.

The objective of this paper is to prove some results of \( U(q) \) and \( V(q) \) analogous to those of \( H(q) \). In Sect. 2, we prove some theta-function representations and modular identities for the continued fractions \( U(q) \) and \( V(q) \). In Sect. 3, we establish some general theorems for explicit evaluations of \( U(q) \) and \( V(q) \). In Sect. 4, we derive three colour partition identities from the theta-function identities of \( U(q) \) and \( V(q) \). In Sect. 5, we establish 2−, 4−, 8-dissection formulas for the continued fractions \( U^*(q) := q^{-5/2}U(q) \) and its reciprocal. Finally, in Sect. 6, we establish 2−, 4−, 8−dissection formulas \( V^*(q) = q^{-1/2}V(q) \) and its reciprocal.

2. Theta-function and modular identities for \( U(q) \) and \( V(q) \)

In this section, we prove some theta-function and modular identities of \( U(q) \) and \( V(q) \).

**Theorem 2.1.** We have

\[
(i) \quad \frac{1}{U(q)} - U(q) = \frac{f(-q^5, -q^7)\phi(q^6)}{q^{3/2}f(-q, -q^{11})\psi(q^{12})} = \frac{\phi(q^6)(\phi(q) + \phi(q^3))}{q^{3/2}\psi(q^{12})(\phi(q) - \phi(q^3))},
\]

\[
(ii) \quad \frac{1}{V(q)} - V(q) = \frac{f(-q, -q^{11})\phi(q^6)}{q^{1/2}f(-q^5, -q^7)\psi(q^{12})} = \frac{\phi(q^6)(\phi(q) - \phi(q^3))}{q^{3/2}\psi(q^{12})(\phi(q) + \phi(q^3))},
\]

\[
(iii) \quad \left( \frac{1}{U(q)} - U(q) \right) \left( \frac{1}{V(q)} - V(q) \right) = \frac{\phi^2(q^6)}{q^{3/2}\psi^2(q^{12})},
\]

\[
(iv) \quad \frac{(U^{-1}(q) - U(q))}{(V^{-1}(q) - V(q))} = \frac{(\phi(q) + \phi(q^3))^2}{(\phi(q) - \phi(q^3))^2},
\]

\[
(v) \quad \frac{1}{U(q)} - U(q) + \frac{1}{V(q)} - V(q) = \frac{2\phi(q^6)(\phi^2(q) + \phi^2(q^3))}{q^{3/2}\psi(q^{12})(\phi^2(q) - \phi^2(q^3))},
\]

\[
(vi) \quad \frac{1}{U(q)} - U(q) - \frac{1}{V(q)} + V(q) = \frac{4\phi(q^6)(\phi(q)\phi(q^3))}{q^{3/2}\psi(q^{12})(\phi^2(q) - \phi^2(q^3))}.
\]
On Ramanujan’s continued fractions of order twenty-four

Proof. From (1.13), we deduce that

\[
\frac{1}{\sqrt{U(q)}} - \sqrt{U(q)} = \frac{f(-q^{11}, -q^{13}) - q^{5/2}f(-q, -q^{23})}{\sqrt{q^{5/2}f(-q, -q^{23})f(-q^{11}, -q^{13})}}.
\]  

(2.1)

From [4, p. 46, Entry 30 (ii) and (iii)], we note that

\[
f(a, b) = f(a^3b, ab^2) + af(b/a, a^5b^3).
\]  

(2.2)

Setting \(a = -q^{5/2}\) and \(b = q^{7/2}\) in (2.2), we obtain

\[
f(-q^{5/2}, q^{7/2}) = f(-q^{11}, -q^{13}) - q^{5/2}f(-q, -q^{23}).
\]  

(2.3)

Employing (2.3) in (2.1), we obtain

\[
\frac{1}{\sqrt{U(q)}} - \sqrt{U(q)} = \frac{f(-q^{5/2}, q^{7/2})}{\sqrt{q^{5/2}f(-q, -q^{23})f(-q^{11}, -q^{13})}}.
\]  

(2.4)

Again, from (1.13), we obtain

\[
\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} = \frac{f(-q^{11}, -q^{13}) + q^{5/2}f(-q, -q^{23})}{\sqrt{q^{5/2}f(-q, -q^{23})f(-q^{11}, -q^{13})}}.
\]  

(2.5)

Employing (2.2) with \(a = q^{5/2}\) and \(b = -q^{7/2}\) in (2.5), we obtain

\[
\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} = \frac{f(q^{5/2}, -q^{7/2})}{\sqrt{q^{5/2}f(-q, -q^{23})f(-q^{11}, -q^{13})}}.
\]  

(2.6)

From (2.4) and (2.6), we obtain

\[
\frac{1}{U(q)} - U(q) = \frac{f(-q^{5/2}, q^{7/2})f(q^{5/2}, -q^{7/2})}{q^{5/2}f(-q, -q^{23})f(-q^{11}, -q^{13})}.
\]  

(2.7)

Again, from [4, p. 46, Entry 30 (i),(iv)], we note that

\[
f(a, ab^2)f(b, a^2b) = f(a, b)\psi(ab).
\]  

(2.8)

\[
f(a, b)f(-a, -b) = f(-a^2, -b^2)\phi(-ab).
\]  

(2.9)

Setting \(a = -q^{5/2}\) and \(b = q^{7/2}\) in (2.9) and \(a = -q\) and \(b = -q^{11}\) in (2.8), we obtain

\[
f(-q^{5/2}, q^{7/2})f(q^{5/2}, -q^{7/2}) = f(-q^5, -q^7)\phi(q^6).
\]  

(2.10)

and

\[
f(-q, -q^{23})f(-q^{11}, -q^{13}) = f(-q, -q^{11})\psi(q^{12}),
\]  

(2.11)

respectively. Employing (2.10) and (2.11) in (2.7), we arrive at the first part of (i) as

\[
\frac{1}{U(q)} - U(q) = \frac{f(-q^5, -q^7)\phi(q^6)}{q^{5/2}f(-q, -q^{23})\psi(q^{12})}.
\]  

(2.12)
Proof. We have
\[\phi(q) + \phi(q^3) = 2\chi(q)(-q^5, -q^7). \tag{2.13}\]
\[\phi(q) - \phi(q^3) = 2q\chi(q)(-q, -q^{11}). \tag{2.14}\]
Using (2.13) and (2.14) in (2.12), we arrive at (i). Proof of (ii) is similar to the proof of (i) and proofs of (iii), (iv), (v) and (vi) directly follow from (i) and (ii), so we omit.

Squaring (2.6), we obtain
\[
\frac{1}{U(q)} + U(q) = \frac{f^2(q^{5/2}, -q^{7/2})}{q^{5/2}\chi(-q, -q^{11}\chi(-q, -q^{11})} - 2. \tag{2.15}\]

From [4, p. 46, Entry 30 (iv, vi)], we note that
\[
f^2(a, b) = f(a^2, b^2)\phi(ab) + 2abf(b/a, a^3b)\psi(a^2b^2). \tag{2.16}\]
Setting \(a = q^{5/2}\) and \(b = -q^{7/2}\), we obtain
\[
f^2(q^{5/2}, -q^{7/2}) = f(q^5, q^7)\phi(-q^6) + 2q^{5/2}\chi(-q, -q^{11})\psi(q^{12}). \tag{2.17}\]

Employing (2.11) and (2.17) in (2.15) and simplifying, we arrive at
\[
\frac{1}{U(q)} + U(q) = \frac{f(q^6, q^7)\phi(-q^6)}{q^{5/2}\chi(-q, -q^{11})\psi(q^{12})}. \tag{2.18}\]
Similarly, we have
\[
\frac{1}{V(q)} + V(q) = \frac{f(q, q^{11})\phi(-q^6)}{q^{5/2}\chi(-q^2, -q^{11})\psi(q^{12})}. \tag{2.19}\]
Again, using (2.13) and (2.14) in (2.18) and (2.19), we complete the proof of the second part of (vii) and (viii), respectively. Proofs of (ix) and (x) follow from (2.18), (2.19), (2.13) and (2.14). \(\square\)

**Theorem 2.2.** Let \(n, p\) and \(r\) be positive integers and \(a, b, c\) and \(d\) be odd positive integers. If \(W(q)\) is defined as
\[
W(q) = q^{p/r}\frac{\psi(-q^a, -q^b)}{\chi(-q^a, -q^b); \quad a + b = c + d},
\]
then
\[
W^n(q)W^n(-q) = \begin{cases} \quad W^n(q^2), & \text{if } np \equiv 0 \pmod{2r} \\ -W^n(q^2), & \text{if } np \equiv r \pmod{2r}. \end{cases}
\]

**Proof.** We have
\[
W^n(q)W^n(-q) = q^{np/r}f^{n}(q^a, -q^b)\chi(-q^a, -q^b)q^{np/r}\frac{f^n(q^a, q^b)}{\chi(q^a, q^b)} \tag{2.20}\]
Employing (2.9) for \(\{a = q^a, b = q^b\}\) and \(\{c = q^c, d = q^d\}\) in (2.20), we have
\[
W^n(q)W^n(-q) = (-1)^{np/r}q^{2np/r}f^{n}(q^a, -q^b)\chi(-q^{a+b})q^{np/r}\frac{f^n(q^a, q^b)}{\chi(q^a, q^b)}
= (-1)^{np/r}(q^2)^{np/r}f^{n}(q^a, -q^b)\chi(-q^{a+b})q^{np/r}\frac{f^n(q^a, q^b)}{\chi(q^a, q^b)}
= (-1)^{np/r}W^n(q^2) \tag{2.21}\]
Now the desired result follows from (2.21) and noting the fact that \(np/r\) is even if \(np \equiv 0 \pmod{2r}\) and odd if \(np \equiv r \pmod{2r}\). \(\square\)
Corollary 2.3. We have

(i) \( U^n(q)U^n(-q) = \begin{cases} U^n(q^2), & \text{if } n \equiv 0 \pmod{4} \\ -U^n(q^2), & \text{if } n \equiv 2 \pmod{4}, \end{cases} \)

(ii) \( V^n(q)V^n(-q) = \begin{cases} V^n(q^2), & \text{if } n \equiv 0 \pmod{4} \\ -V^n(q^2), & \text{if } n \equiv 2 \pmod{4}. \end{cases} \)

Proof. We set \( a = 1, b = 23, c = 11, d = 13, p = 5 \) and \( r = 2 \) in Theorem 2.2 to arrive at (i). Similarly, (ii) follows from Theorem 2.2 with \( a = 5, b = 19, c = 7, d = 17, p = 1 \) and \( r = 2. \)

3. Explicit Evaluations of \( U(\pm q) \) and \( V(\pm q) \)

In this section, we prove general theorems for the explicit evaluations of \( U(q), V(q), U^2(-q) \) and \( V^2(-q) \) by parametrization of theta-functions \( f(-q) \) and \( \phi(q) \). For, we consider the parameters \( s_{4,n} \) and \( h_{3,n} \) defined respectively as

\[
s_{4,n} = \frac{f(q)}{\sqrt{2q^{1/8}f(-q^4)}}, \quad q = e^{-\pi \sqrt{n}/2}
\]

and

\[
h_{3,n} = \frac{\phi(q)}{3^{1/4}\phi(q^3)}, \quad q = e^{-\pi \sqrt{n}/3},
\]

where \( n \) is a positive real number. The parameter \( s_{4,n} \) is the particular case \( k = 4 \) of the parameter \( s_{k,n} \) defined by Berndt [5, p. 9, (4.7)], and the parameter \( h_{3,n} \) is the particular case \( k = 3 \) of the parameter \( h_{k,n} \) defined by Yi [13, p. 384], Yi [13, p. 391] and Saikia [11, p. 3] evaluated many values of \( h_{k,n} \). It is useful to note that Baruah and Saikia [3] proved the following formula for the explicit evaluation of \( H(q) \) [3, p. 275, (3.5)]:

\[
\frac{1}{H(e^{-\pi \sqrt{n}/4})} - H(e^{-\pi \sqrt{n}/4}) = 2s_{4,n}^2.
\]

Baruah and Saikia [3] calculated many explicit values of the parameter \( s_{4,n} \) to evaluate explicit values of \( H(q) \) by appealing to (3.3). To find explicit values of \( U(q) \) and \( V(q) \), we offer following theorems:

Theorem 3.1. We have

(i) \( \frac{1}{U(e^{-\pi \sqrt{n}/12})} - U(e^{-\pi \sqrt{n}/12}) = 2s_{4,n}^2 \left( \frac{3^{1/4}h_{3,n/48} + 1}{3^{1/4}h_{3,n/48} - 1} \right), \)

(ii) \( \frac{1}{V(e^{-\pi \sqrt{n}/12})} - V(e^{-\pi \sqrt{n}/12}) = 2s_{4,n}^2 \left( \frac{3^{1/4}h_{3,n/48} - 1}{3^{1/4}h_{3,n/48} + 1} \right). \)

Proof. Replacing \( q \) by \( q^{1/6} \) in Theorem 2.1(iii) and then employing (1.10), we arrive at

\[
\left( \frac{1}{U(q^{1/6})} + U(q^{1/6}) \right) \left( \frac{1}{V(q^{1/6})} + V(q^{1/6}) \right) = \left( \frac{1}{H(q^{1/2})} - H(q^{1/2}) \right)^2.
\]

Setting \( q = e^{-\pi \sqrt{n}/2} \) in (3.4) and employing (3.3), we obtain

\[
\left( \frac{1}{U(e^{-\pi \sqrt{n}/12})} + U(e^{-\pi \sqrt{n}/12}) \right) \left( \frac{1}{V(e^{-\pi \sqrt{n}/12})} + V(e^{-\pi \sqrt{n}/12}) \right) = 4s_{4,n}^4.
\]
Again, from Theorem 2.1 (iv), we have
\[
\frac{(U^{-1}(q) + U(q))}{(V^{-1}(q) + V(q))} = \left(\frac{\phi(q)}{\phi(q^2)} + 1\right)^2. \tag{3.6}
\]

Setting \( q = e^{-\pi \sqrt{3}/12} \) in (3.6) and employing the definition of the parameter \( h_{3,n} \) from (3.2), we obtain
\[
\frac{(U^{-1}(e^{-\pi \sqrt{3}/12}) + U(e^{-\pi \sqrt{3}/12}))}{(V^{-1}(e^{-\pi \sqrt{3}/12}) + V(e^{-\pi \sqrt{3}/12}))} = \left(\frac{31/4h_{3,n/48} - 1}{31/4h_{3,n/48} + 1}\right)^2. \tag{3.7}
\]

Multiply (3.5) and (3.7), we arrive at (i). Dividing (3.5) by (3.7), we arrive at (ii).

\[\square\]

**Remark 3.2.** From Theorem 3.1, one can easily find the explicit values of \( U(e^{-\pi \sqrt{3}/12}) \) and \( V(e^{-\pi \sqrt{3}/12}) \) with the help of the parameters \( s_{4,n} \) and \( h_{3,n/48} \). For example, employing the values \( s_{4,16} = \sqrt{1 + \sqrt{2}} \) from [3] and \( h_{3,1/3} = 3^{-1/4}(2\sqrt{3} + 3)^{1/4} \) from [13] in Theorem 3.1 (i) and (ii), we obtain
\[
U(e^{-\pi/3}) = \frac{-1 + \alpha^{1/4}}{1 + \sqrt{2} + (1 + \sqrt{2})\alpha^{1/4} + \sqrt{2(2 + \sqrt{2} + 2(1 + \sqrt{2})\alpha^{1/4} + 2\sqrt{\alpha + \sqrt{2}\alpha}}}, \tag{3.8}
\]
and
\[
V(e^{-\pi/3}) = \frac{1 + \sqrt{2} - (1 + \sqrt{2})\alpha^{1/4} + \sqrt{2(2 + \sqrt{2} - 2(1 + \sqrt{2})\alpha^{1/4} + 2\sqrt{\alpha + \sqrt{2}\alpha}}}{1 + \alpha^{1/4}}, \tag{3.9}
\]
where
\[
\alpha = 3 + 2\sqrt{3},
\]
respectively. Again, employing the values \( s_{4,8} = (1 + \sqrt{2})^{-1/4} \) from [3] and \( h_{3,1/6} = 2^{-1/8}3^{-1/2}(3\sqrt{2} + \sqrt{12} + 6\sqrt{2} + \sqrt{3})^{1/2} \) from [11] in Theorem 3.1(i) and (ii), we deduce that
\[
U(e^{-\pi/3\sqrt{2}}) = \frac{m_1 \left(\sqrt{1 + \sqrt{2}} - \sqrt{m_2^2 + (1 + \sqrt{2})}\right)}{m_2}, \tag{3.10}
\]
and
\[
V(e^{-\pi/3\sqrt{2}}) = \frac{z_2 \left(\sqrt{1 + \sqrt{2}} + \sqrt{z_1^2 + (1 + \sqrt{2})}\right)}{z_1}, \tag{3.11}
\]
respectively, where we take
\[
m_1 = 6(4 + 3\sqrt{2})^{1/4} + 2^{7/8} \cdot 3^{3/4} \sqrt{6 + 6\sqrt{2} + \sqrt{6}},
\]
\[
m_2 = 6(4 + 3\sqrt{2})^{1/4} - 2^{7/8} \cdot 3^{3/4} \sqrt{6 + 6\sqrt{2} + \sqrt{6}},
\]
\[
z_1 = 6 + 2^{5/8} \cdot 3^{3/4} (-4 + 3\sqrt{2})^{1/4} \sqrt{6 + 6\sqrt{2} + \sqrt{6}}
\]
and
\[
z_2 = 6 - 2^{5/8} \cdot 3^{3/4} (-4 + 3\sqrt{2})^{1/4} \sqrt{6 + 6\sqrt{2} + \sqrt{6}}.
\]

Similarly, employing the values \( s_{4,32} = 2^{-1/8}(1 + \sqrt{2})^{3/8}(4 + \sqrt{2} + 10\sqrt{2})^{1/4} \) from [3] and \( h_{3,2/3} = 2^{5/8}(3\sqrt{2} - 4)^{-3/4}(6 + 6\sqrt{2} + \sqrt{6})^{1/2} \) from [11] in Theorem 3.1(i) and (ii), we evaluate
\[
U(e^{-\pi\sqrt{2}/3}) = \frac{2a}{2^{3/4}b(1 + \sqrt{2})^{3/4}\sqrt{1 + 4a^2 + 2^{3/2}(1 + \sqrt{2})^{1/2}b^2l}}, \tag{3.12}
\]
On Ramanujan’s continued fractions of order twenty-four

and

\[ V(e^{-\pi \sqrt{3}/3}) = \frac{2^{3/4}a(1 + \sqrt{2})^{3/4}\sqrt{t} - \sqrt{4b^2 + 2^{3/2}(1 + \sqrt{2})^{3/2}a^2t}}{-2b}, \]  

(3.13)

respectively, where

\[ a = 2^{5/8} \cdot 3^{1/4} - (-4 + 3\sqrt{2})^{3/4}\sqrt{6 + 6\sqrt{2} + \sqrt{6}}, \]
\[ b = 2^{5/8} \cdot 3^{1/4} + (-4 + 3\sqrt{2})^{3/4}\sqrt{6 + 6\sqrt{2} + \sqrt{6}} \]

and

\[ t = 4 + \sqrt{2(1 + 5\sqrt{2})}. \]

**Theorem 3.3.** We have

(i) \( U^2(-q) = \frac{U^2(q^2)}{U^2(q)} \),

(ii) \( V^2(-q) = \frac{V^2(q^2)}{V^2(q)} \).

**Proof.** Setting \( n = 2 \) in Theorem 2.3 (i) and (ii), we arrived at (i) and (ii), respectively. \( \square \)

**Remark 3.4.** From Theorem 3.3, it is clear that if we know the values of \( U^2(q) \) and \( U^2(q^2) \), values of \( U^2(-q) \) can be easily evaluated. For example, employing (3.10) and (3.12) in Theorem 3.3(i), we obtain

\[ U^2(-e^{-\pi \sqrt{3}/3}) = \frac{-4a^2m_2^2(\sqrt{1 + \sqrt{2}} - \sqrt{m_3^2 + (1 + \sqrt{2})})^2}{m_1^2[(2^{3/4}(1 + \sqrt{2})^{3/4}b\sqrt{7} + \sqrt{4a^2 + 2^{3/2}(1 + \sqrt{2})^{3/2}b^2})^2].} \]

(3.14)

Again, if we know the values of \( V^2(q) \) and \( V^2(q^2) \), we can find the values of \( V^2(-q) \). For example, employing (3.11) and (3.13) in Theorem 3.3(ii), we evaluate

\[ V^2(-e^{-\pi \sqrt{3}/3}) = \frac{-z_2^2(2^{3/4}(1 + \sqrt{2})^{3/4}a\sqrt{1} - \sqrt{4b^2 + 2^{3/2}(1 + \sqrt{2})^{3/2}a^2})^2}{4b^2z_2^2(\sqrt{1 + \sqrt{2}} + \sqrt{z_1^2 + (1 + \sqrt{2})})^2}. \]

(3.15)

One can determine other values of \( U(q), \ V(q), \ U^2(-q) \) and \( V^2(-q) \) by routine calculation.

### 4. Colour partition identities from theta-function identities of \( U(q) \) and \( V(q) \)

A partition of a positive integer \( n \) is a non-increasing sequence of positive integers, called parts, whose sum equals \( n \). For example, \( n = 3 \) has three partitions, namely,

\[ 3, \ 2 + 1, \ 1 + 1 + 1. \]

If \( p(n) \) denotes the number of partitions of \( n \), then \( p(3) = 3 \). The generating function for \( p(n) \) due to Euler is given by

\[ \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty}. \]

(4.1)

A part in a partition of \( n \) is said to have \( r \) colours if each part has \( r \) copies and all of them are viewed as distinct objects. For any positive integer \( n \) and \( r \), let \( C_r(n) \) denote the number of partition of \( n \) with each part have \( r \) distinct colours. For example, if each part of partition of \( 3 \) has 2 colours, say white (indicated by the suffix \( w \)) and brown (indicated by the suffix \( b \)), then the number of 2 colour
Proof. Partitions of 3 are 10, namely $3_1, 3_2, 2_1 + 1_1, 2_2 + 1_2, 2_1 + 1_2, 2_1 + 1_1, 1_1 + 1_1 + 1_1, 1_1 + 1_1 + 1_2, 1_1 + 1_2 + 1_2, 1_1 + 1_2 + 1_1$. The generating function of $C_r(n)$ is given by

$$\sum_{n=0}^{\infty} C_r(n)q^n = \frac{1}{(q; q)_{\infty}}. \quad (4.2)$$

For positive integers $s, m$ and $r$, the quotient

$$\frac{1}{(q^s; q^m)_{\infty}}$$

is the generating function of the number of partitions of $n$ with parts congruent to $s$ modulo $m$ and each part has $r$ colours. For example,

$$\frac{1}{(q^s; q^m)_{\infty}} = \frac{1}{(q^{s_1}; q^{s_2}; q^m)_{\infty}} \quad (4.4)$$

is the generating function of the number of partitions of positive integer with parts congruent to $s_1$ or $s_2$ modulo $m$ and each part has four colours. Here we use the notation

$$\langle q^r; q^s; q^t \rangle := \langle q^r; q^{t-r}; q^s \rangle_{\infty}, \quad (4.5)$$

where $r$ and $t$ are positive integers and $r < t$.

**Theorem 4.1.** Let $C_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1, \pm 5, \pm 7$ or $\pm 12 \,(mod\, 24)$ such that the parts congruent to $\pm 1$ and $\pm 12 \,(mod\, 24)$ have 2 colours. Let $C_2(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 5, \pm 7, \pm 11$ or $\pm 12 \,(mod\, 24)$ such that parts congruent to $\pm 11$ and $\pm 12 \,(mod\, 24)$ have 2 colours. Let $C_3(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1, \pm 6$ or $\pm 11 \,(mod\, 24)$ with 2 colours. Then for any positive integer $n \geq 5$,

$$C_1(n) - C_2(n - 5) - C_3(n) = 0.\quad \Box$$

Proof. Employing (1.13), (1.5) and (1.6) in first part of Theorem 2.1(i), we obtain

$$\frac{(q^{11\pm}; q^{24})_{\infty}}{(q^{11\pm}; q^{24})_{\infty}} - q^{5/2} \frac{(q^{1\pm}; q^{24})_{\infty}}{(q^{1\pm}; q^{24})_{\infty}} - \frac{(q^{5\pm}; q^{7\pm}; q^{24})_{\infty}(q^{12\pm}; q^{24})_{\infty}}{(q^{5\pm}; q^{7\pm}q^{24})_{\infty} (q^{12\pm}; q^{24})_{\infty}} = 0. \quad (4.6)$$

Dividing (4.6) by $(q^{1\pm, 5\pm, 7\pm, 11\pm}; q^{24})_{\infty}(q^{12\pm}, q^{24})_{2\infty}$, we obtain

$$\frac{1}{(q^{1\pm, 5\pm, 7\pm, 11\pm}; q^{24})_{\infty}(q^{1\pm, 12\pm}; q^{24})_{2\infty}} - \frac{q^5}{(q^{5\pm, 7\pm}; q^{24})_{\infty}(q^{11\pm, 12\pm}; q^{24})_{2\infty}} - \frac{1}{(q^{1\pm, 6\pm, 11\pm}; q^{24})_{2\infty}} = 0. \quad (4.7)$$

The above quotients of (4.7) represent the generating functions for $C_1(n), C_2(n)$, and $C_3(n)$, respectively. Hence, (4.7) is equivalent to

$$\sum_{n=0}^{\infty} C_1(n)q^n - q^5 \sum_{n=0}^{\infty} C_2(n)q^n - \sum_{n=0}^{\infty} C_3(n)q^n = 0, \quad (4.8)$$

where we set $C_1(0) = C_2(0) = C_3(0) = 1$. Equating coefficients of $q^n$ on both sides, we arrive at the desired result.

**Example:** The following table illustrates the case $n = 5$ in Theorem 4.1:

**Theorem 4.2.** Let $C_1(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 1, \pm 5, \pm 11$ or $\pm 12 \,(mod\, 24)$ such that the parts congruent to $\pm 1$ and $\pm 12 \,(mod\, 24)$ have 2 colours. Let $C_2(n)$ denote the number of partitions of $n$ into parts congruent to $\pm 5, \pm 7, \pm 11$ or $\pm 12 \,(mod\, 24)$ such that parts...
From the first part of Theorem 2.1 (ii), employing the same procedure as in Theorem 4.1, we obtain the desired result.

Proof. From the first part of Theorem 2.1 (ii), employing the same procedure as in Theorem 4.1, we obtain

\[
\left(\frac{1}{(q^{1+11\pm}; q^{24})_{\infty}(q^{7\pm, 12\pm}; q^{24})_{\infty}^2} - \frac{q}{(q^{1+11\pm}; q^{24})_{\infty}(q^{7\pm, 12\pm}; q^{24})_{\infty}^2} - \frac{1}{(q^{5\pm, 6\pm, 7\pm}; q^{24})_{\infty}^2}\right) = 0. \tag{4.10}
\]

The above quotients of (4.10) represent the generating functions for \(C_1(n), C_2(n)\) and \(C_3(n)\), respectively. Hence, (4.10) is equivalent to

\[
\sum_{n=0}^{\infty} C_1(n)q^n - \sum_{n=0}^{\infty} C_2(n)q^n - \sum_{n=0}^{\infty} C_3(n)q^n = 0, \tag{4.11}
\]

where we set \(C_1(0) = C_2(0) = C_3(0) = 1\). Equating coefficients of \(q^n\) on both sides of (4.11), we obtain the desired result. \(\square\)

Example: The following table illustrates the case \(n = 5\) in Theorem 4.2:

<table>
<thead>
<tr>
<th>(n)</th>
<th>(C_1(n))</th>
<th>(C_2(n))</th>
<th>(C_3(n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5g</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Theorem 4.3. Let \(C_1(n)\) denote the number of partitions of \(n\) into parts congruent to \(\pm 1, \pm 5\) or \(\pm 12\) (mod 24) such that the parts congruent to \(\pm 12\) (mod 24) has 4 colours. Let \(C_2(n)\) denote the number of partitions of \(n\) into parts congruent to \(\pm 1, \pm 7\) or \(\pm 12\) (mod 24) such that parts congruent to \(\pm 1\) and \(\pm 7\) (mod 24) have 2 colours and parts congruent to \(\pm 12\) (mod 24) have 4 colours. Let \(C_3(n)\) denote the number of partitions of \(n\) into parts congruent to \(\pm 5, \pm 11\) or \(\pm 12\) (mod 24) such that parts congruent to \(\pm 5\) and \(\pm 11\) (mod 24) have 2 colours and parts congruent to \(\pm 12\) (mod 24) have 4 colours. Let \(C_4(n)\) denote the number of partitions of \(n\) into parts congruent to \(\pm 7, \pm 11\) or \(\pm 12\) (mod 24) such that parts congruent to \(\pm 7\) and \(\pm 11\) (mod 24) have 2 colours and parts congruent to \(\pm 12\) (mod 24) have 4 colours. Let \(C_5(n)\) denote the number of partitions of \(n\) into parts congruent to \(\pm 1, \pm 5, \pm 6, \pm 7, \) or \(\pm 11\) (mod 24) such that parts congruent to \(\pm 6\) (mod 24) have 4 colours. Then for any positive integer \(n \geq 6\),

\[
C_1(n) - C_2(n - 1) - C_3(n - 5) + C_4(n - 6) - C_5(n) = 0. \tag{4.12}
\]
Proof. From Theorem 2.1 (iii), employing the same procedure, we obtain

\[
\frac{1}{(q^{1,\pm, 5}; q^{24})_{\infty}^2 (q^{12,\pm}; q^{24})_{\infty}} - \frac{q}{(q^{1,\pm, 7}; q^{24})_{\infty}^2 (q^{12,\pm}; q^{24})_{\infty}} - \frac{q^5}{(q^{3,\pm, 11}; q^{4})_{\infty}^2 (q^{12,\pm}; q^{24})_{\infty}}
\]

\[
+ \frac{q^6}{(q^{3,\pm, 11}; q^{24})_{\infty}^2 (q^{12,\pm}; q^{24})_{\infty}} + \frac{1}{(q^{3,\pm, 5}; q^{24})_{\infty}^2 (q^{12,\pm}; q^{24})_{\infty}} = 0.
\]

The above quotients of (4.13) represent the generating functions for \(C_1(n), C_2(n), C_3(n), C_4(n)\) and \(C_5(n)\), respectively. Hence, (4.13) is equivalent to

\[
\sum_{n=0}^{\infty} C_1(n)q^n - q \sum_{n=0}^{\infty} C_2(n)q^n - q^5 \sum_{n=0}^{\infty} C_3(n)q^n + q^6 \sum_{n=0}^{\infty} C_4(n)q^n - \sum_{n=0}^{\infty} C_5(n)q^n = 0,
\]

where we set \(C_1(0) = C_2(0) = C_3(0) = C_4(0) = C_5(0) = 1\). Equating coefficients of \(q^n\) on both sides of (4.14), we obtain the desired result.

Example: The following table illustrates the case \(n = 6\) in Theorem 4.3:

<table>
<thead>
<tr>
<th>(C_1(6) = 11)</th>
<th>(C_2(5) = 6)</th>
<th>(C_3(1) = 0)</th>
<th>(C_4(0) = 1)</th>
<th>(C_5(6) = 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5r + 1r)</td>
<td>(1r + 1r + 1r + 1r + 1r)</td>
<td>(6r)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5r + 1g)</td>
<td>(1r + 1r + 1r + 1r + 1g)</td>
<td>(6g)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5g + 1r)</td>
<td>(1r + 1r + 1r + 1g + 1g)</td>
<td>(6g)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(5g + 1g)</td>
<td>(1r + 1r + 1g + 1g + 1g)</td>
<td>(6g)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1r + 1r + 1r + 1r + 1r + 1r)</td>
<td>(1r + 1g + 1g + 1g + 1g)</td>
<td>(5 + 1)</td>
<td>(1 + 1 + 1 + 1 + 1)</td>
<td></td>
</tr>
<tr>
<td>(1r + 1r + 1r + 1r + 1r + 1g + 1g)</td>
<td>(1g + 1g + 1g + 1g + 1g)</td>
<td></td>
<td>(1 + 1 + 1 + 1 + 1)</td>
<td></td>
</tr>
<tr>
<td>(1r + 1r + 1r + 1g + 1g + 1g)</td>
<td>(1r + 1r + 1g + 1g + 1g)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1r + 1g + 1g + 1g + 1g + 1g)</td>
<td>(1r + 1g + 1g + 1g + 1g)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1g + 1g + 1g + 1g + 1g + 1g)</td>
<td>(1g + 1g + 1g + 1g + 1g)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Dissection formulas for \(U^*(q)\) and \(1/U^*(q)\)

In this section, we present \(2-, 4-\) and \(8-\) dissections of the continued fractions \(U^*(q) := q^{-5/2}U(q)\) and \(1/U^*(q)\). We will use the fact that,

\[
\sum_{n=0}^{\infty} a_n q^n + \sum_{n=0}^{\infty} a_n (-q)^n = 2 \sum_{n=0}^{\infty} a_{2n} q^{2n},
\]

and

\[
\sum_{n=0}^{\infty} a_n q^n - \sum_{n=0}^{\infty} a_n (-q)^n = 2 \sum_{n=0}^{\infty} a_{2n+1} q^{2n+1}.
\]

We will use the following two identities from [4, p. 45, Entry 29] and [4, p. 51, Entry 31, Example (v)], respectively:

\[
f(a, b)f(c, d) = f(ac, bd)f(ad, bc) + af(b/c, ac^2d)f(b/d, acd^2); \quad \text{for} \quad ab = cd
\]

and

\[
f(q, q^5) = \psi(-q^3)\chi(q).
\]
Theorem 5.1. If

\[ U^*(q) := q^{-5/2} U(q) = \frac{f(-q, -q^{23})}{f(-q^{11}, -q^{13})} = \sum_{n=0}^{\infty} a_n q^n, \]

then 2-dissections of \( U^*(q) \) are

\[ \sum_{n=0}^{\infty} a_{2n} q^n = \frac{\psi(q^6 f(-q^7, -q^{17})}{\phi(-q^{12}) f(-q^{11}, -q^{13})}, \] (5.5)

and

\[ \sum_{n=0}^{\infty} a_{2n+1} q^n = -\frac{\psi(-q^6 f(-q^9, -q^{19})}{\phi(-q^{12}) f(-q^{11}, -q^{13})}. \] (5.6)

Proof. We have

\[ \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q, -q^{23})}{f(-q^{11}, -q^{13})} = \frac{f(-q, -q^{23}) f(q^{11}, q^{13})}{f(-q^{11}, -q^{13}) f(q^{11}, q^{13})}. \] (5.7)

Setting \( a = -q, b = -q^{23}, c = q^{11}, d = q^{13} \) in (5.3), we obtain

\[ f(-q, -q^{23}) f(q^{11}, q^{13}) = f(-q^{12}, -q^{36}) f(-q^{14}, -q^{34}) - qf(-q^{12}, -q^{36}) f(-q^{10}, -q^{38}). \] (5.8)

Setting \( a = q^{11} \) and \( b = q^{13} \) in (2.9), we obtain

\[ f(q^{11}, q^{13}) f(-q^{11}, -q^{13}) = f(-q^{22}, -q^{26}) \phi(-q^{24}). \] (5.9)

Employing (5.8) and (5.9) in (5.7), we obtain

\[ \sum_{n=0}^{\infty} a_n q^n = \frac{f(-q^{12}, -q^{36}) f(-q^{14}, -q^{34}) - qf(-q^{12}, -q^{36}) f(-q^{10}, -q^{38})}{f(-q^{22}, -q^{26}) \phi(-q^{24})}. \] (5.10)

Replacing \( q \) by \(-q\) in (5.10), we obtain

\[ \sum_{n=0}^{\infty} a_n (-1)^n q^n = \frac{f(-q^{12}, -q^{36}) f(-q^{14}, -q^{34}) + qf(-q^{12}, -q^{36}) f(-q^{10}, -q^{38})}{f(-q^{22}, -q^{26}) \phi(-q^{24})}. \] (5.11)

Adding (5.10) and (5.11) and using (5.1), we obtain

\[ \sum_{n=0}^{\infty} a_{2n} q^{2n} = \frac{f(-q^{12}, -q^{36}) f(-q^{14}, -q^{34})}{f(-q^{22}, -q^{26}) \phi(-q^{24})}. \] (5.12)

Subtracting (5.11) from (5.10) and using (5.2), we obtain

\[ \sum_{n=0}^{\infty} a_{2n+1} q^{2n+1} = \frac{-q f(-q^{12}, -q^{36}) f(-q^{10}, -q^{38})}{f(-q^{22}, -q^{26}) \phi(-q^{24})}. \] (5.13)

Dividing both sides of (5.13) by \( q \), we obtain

\[ \sum_{n=0}^{\infty} a_{2n+1} q^{2n} = \frac{-q f(-q^{12}, -q^{36}) f(-q^{10}, -q^{38})}{f(-q^{22}, -q^{26}) \phi(-q^{24})}. \] (5.14)

Now, replacing \( q^2 \) by \( q \) in (5.12) and (5.14), we arrive at the desired result. □

In the next theorem, we give 4-dissections of \( U^*(q) \).
Theorem 5.2. We have

\[
\sum_{n=0}^{\infty} a_{4n} q^n = \frac{\psi(-q^3)f(-q^9, -q^{15}) f(-q^{10}, -q^{14})}{\phi(-q^6)\phi(-q^{12}) f(-q^{11}, -q^{13})}, \tag{5.15}
\]

\[
\sum_{n=0}^{\infty} a_{4n+1} q^n = -\frac{\psi(-q^3)f(-q^9, -q^{16}) f(-q^9, -q^{15})}{\phi(-q^6)\phi(-q^{12}) f(-q^{11}, -q^{13})}, \tag{5.16}
\]

\[
\sum_{n=0}^{\infty} a_{4n+2} q^n = -q^3 \frac{\psi(-q^3)f(-q^3, -q^{21}) f(-q^2, -q^{22})}{\phi(-q^6)\phi(-q^{12}) f(-q^{11}, -q^{13})}, \tag{5.17}
\]

and

\[
\sum_{n=0}^{\infty} a_{4n+3} q^n = \frac{q^2\psi(-q^3)\psi(q^{12})\chi(-q^4)f(-q^3, -q^{21})}{\phi(-q^6)\phi(-q^{12}) f(-q^{11}, -q^{13})}. \tag{5.18}
\]

Proof. From Theorem 5.1, we have

\[
\sum_{n=0}^{\infty} a_{2n} q^n = \frac{\psi(-q^6)f(-q^7, -q^{17}) f(q^{11}, q^{13})}{\phi(-q^{12}) f(-q^{11}, -q^{13}) f(q^{11}, q^{13})}, \tag{5.19}
\]

Setting \(a = -q^7\), \(b = -q^{17}\), \(c = q^{11}\), \(d = q^{13}\) in (5.3), we obtain

\[
f(-q^7, -q^{17}) f(q^{11}, q^{13}) = f(-q^{18}, -q^{30}) f(-q^{20}, -q^{28}) - q^7 f(-q^6, -q^{12}) f(-q^4, -q^{14}). \tag{5.20}
\]

Again, setting \(a = q^{11}\) and \(b = q^{13}\) in (2.9), we obtain

\[
f(q^{11}, q^{13}) f(-q^{11}, -q^{13}) = f(-q^{22}, -q^{26}) \phi(-q^{24}). \tag{5.21}
\]

Employing (5.20) and (5.21) in (5.19), we obtain

\[
\sum_{n=0}^{\infty} a_{2n} q^n = \frac{\psi(-q^6)(f(-q^{18}, -q^{30}) f(-q^{20}, -q^{28}) - q^7 f(-q^6, -q^{12}) f(-q^4, -q^{14}))}{\phi(-q^{12}) f(-q^{22}, -q^{26}) \phi(-q^{24})}. \tag{5.22}
\]

Replacing \(q\) by \(-q\) in (5.22), we obtain

\[
\sum_{n=0}^{\infty} a_{2n}(-1)^n q^n = \frac{\psi(-q^6)(f(-q^{18}, -q^{30}) f(-q^{20}, -q^{28}) - q^7 f(-q^6, -q^{12}) f(-q^4, -q^{14}))}{\phi(-q^{12}) f(-q^{22}, -q^{26}) \phi(-q^{24})}. \tag{5.23}
\]

Combining (5.22) and (5.23), we obtain

\[
\sum_{n=0}^{\infty} a_{4n} q^{2n} = \frac{\psi(-q^6)f(-q^{18}, -q^{30}) f(-q^{20}, -q^{28})}{\phi(-q^{12})\phi(-q^{24}) f(-q^{22}, -q^{26})}. \tag{5.24}
\]

and

\[
\sum_{n=0}^{\infty} a_{4n+2} q^{2n} = -q^6 \frac{\psi(-q^6)f(-q^6, -q^{42}) f(-q^4, -q^{44})}{\phi(-q^{12})\phi(-q^{24}) f(-q^{22}, -q^{26})}. \tag{5.25}
\]

Now, replacing \(q^2\) by \(q\) in (5.24) and (5.25) and using (5.4), we arrive at (5.15) and (5.17), respectively. Proofs of (5.16) and (5.18) are identical to the proof of (5.15) and (5.17), and follow similarly from (5.6). So we omit details. \(\square\)

Next, we give 8-dissections of \(U^*(q)\).
Theorem 5.3. We have

\[
\sum_{n=0}^{\infty} a_{8n}q^n = \frac{f(-q^5, -q^7)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^{10}, -q^{14}) f(-q^{11}, -q^{13}) + q^6 f(q^5, q^{21}) f(-q^2, -q^{22}) f(-q, -q^{23}) \right],
\]

(5.26)

\[
\sum_{n=0}^{\infty} a_{8n+1}q^n = -\frac{f(-q^4, -q^6)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^{10}, -q^{14}) f(-q^{11}, -q^{13}) + q^6 f(q^5, q^{21}) f(-q^2, -q^{22}) f(-q, -q^{23}) \right],
\]

(5.27)

\[
\sum_{n=0}^{\infty} a_{8n+2}q^n = \frac{q^2 f(-q, -q^{11})}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) + f(q^3, q^{21}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) \right],
\]

(5.28)

\[
\sum_{n=0}^{\infty} a_{8n+3}q^n = -\frac{q^3 \psi(q^4) \chi(-q^2)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^{10}, -q^{14}) f(-q^{11}, -q^{13}) + q^3 f(q^3, q^{21}) f(-q^2, -q^{22}) f(-q, -q^{23}) \right],
\]

(5.29)

\[
\sum_{n=0}^{\infty} a_{8n+4}q^n = \frac{q^4 f(q^4, q^8)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^3, q^{21}) f(-q^{10}, -q^{14}) f(-q^{11}, -q^{13}) + q^3 f(q^3, q^{15}) f(-q^2, -q^{22}) f(-q, -q^{23}) \right],
\]

(5.30)

\[
\sum_{n=0}^{\infty} a_{8n+5}q^n = -\frac{q f(-q^9, -q^{11})}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) + q^3 f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^4, -q^{20}) \right],
\]

(5.31)

\[
\sum_{n=0}^{\infty} a_{8n+6}q^n = \frac{q^3 \psi(q^6) \chi(-q^2)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) + q^3 f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^4, -q^{20}) \right],
\]

(5.32)

\[
\sum_{n=0}^{\infty} a_{8n+7}q^n = -\frac{q^2 \psi(q^6) \chi(-q^2)}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) + f(q^3, q^{21}) f(-q^7, -q^{17}) f(-q^8, -q^{16}) \right].
\]

(5.33)

Proof. From (5.15), we have

\[
\sum_{n=0}^{\infty} a_{4n}q^n = \frac{\psi(-q^3) f(-q^9, -q^{15}) f(-q^{10}, -q^{14}) f(q^{11}, q^{13})}{\phi(-q^6)\phi(-q^{12})f(-q^{11}, -q^{13}) f(q^{11}, q^{13})}.
\]

(5.34)

Setting \( a = -q^9 \), \( b = -q^{15} \), \( c = q^{11} \), \( d = q^{13} \) in (5.3), we obtain

\[
f(-q^9, -q^{15}) f(q^{11}, q^{13}) = f(-q^{20}, -q^{28}) f(-q^{22}, -q^{26}) - q^9 f(-q^4, -q^{14}) f(-q^2, -q^{46}).
\]

(5.35)

Again from [4, p.46, Entry 30], we note that

\[ f(a, b) = f(a^3b, ab^3) + af(b/a, a^5b^3). \]

(5.36)

Setting \( a = -q^3 \) and \( b = -q^9 \) in (5.36), we have

\[ \psi(-q^3) = f(-q^3, -q^9) = f(q^{18}, q^{30}) - q^3 f(q^6, q^{12}). \]

(5.37)
Employing (5.35) and (5.37) in (5.34), we have
\[
\sum_{n=0}^{\infty} a_{4n}q^n = \frac{1}{\phi(-q^6)\phi(-q^{12})\phi(-q^{24})} \left[ f(-q^{10}, -q^{14}) \left( f(q^{18}, q^{30}) - q^3 f(-q^6, -q^{42}) \right) \right. \\
\left. + q^9 f(q^4, q^{44}) f(-q^2, -q^{46}) f(-q^4, -q^{20}) \right],
\]
(5.38)
Replacing \( q \) by \(-q\) in (5.38), we obtain
\[
\sum_{n=0}^{\infty} a_{4n}q^n = \frac{1}{\phi(-q^6)\phi(-q^{12})\phi(-q^{24})} \left[ f(-q^{10}, -q^{14}) \left( f(q^{18}, q^{30}) + q^3 f(-q^6, -q^{42}) \right) \right. \\
\left. + q^9 f(q^4, q^{44}) f(-q^2, -q^{46}) f(-q^4, -q^{20}) \right],
\]
(5.39)
Combining (5.38) and (5.39), we obtain
\[
\sum_{n=0}^{\infty} a_{8n}q^{2n} = \frac{f(-q^{10}, -q^{14})}{\phi(-q^6)\phi(-q^{12})\phi(-q^{24})} \left[ f(q^{18}, q^{30}) f(-q^2, -q^{26}) f(-q^4, -q^{46}) \right. \\
\left. + q^{12} f(q^6, q^{42}) f(-q^2, -q^{46}) \right],
\]
(5.40)
and
\[
\sum_{n=0}^{\infty} a_{8n+4}q^{2n+1} = -\frac{f(-q^{10}, -q^{14})}{\phi(-q^6)\phi(-q^{12})\phi(-q^{24})} \left[ q^3 f(q^{18}, q^{30}) f(-q^2, -q^{46}) f(-q^4, -q^{26}) \right. \\
\left. + q^9 f(q^6, q^{42}) f(-q^2, -q^{26}) \right].
\]
(5.41)
Replacing \( q^2 \) by \( q \) in (5.40) and (5.41), we arrived at (5.26) and (5.30), respectively. Proofs (5.27), (5.28), (5.29), (5.31), (5.32) and (5.33) are identical to the proofs of (5.26) and (5.30), so omitted. □

Next, we prove 2-, 4- and 8-dissection formulas of \( 1/U^*(q) \). Since the dissection formulas and their proofs are analogous to the dissection formulas of \( U^*(q) \), we simply state them and omit proofs.

**Theorem 5.4.** If
\[
\frac{1}{U^*(q)} := \frac{f(-q^{11}, -q^{13})}{f(-q, -q^{23})} = \sum_{n=0}^{\infty} c_n q^n,
\]
then 2-dissection formulas are
\[
\sum_{n=0}^{\infty} c_{2n}q^n = \frac{\psi(-q^6) f(-q^{17}, -q^7)}{\phi(-q^{12}) f(-q, -q^{23})},
\]
(5.42)
and
\[
\sum_{n=0}^{\infty} c_{2n+1}q^n = -\frac{\psi(-q^6) f(-q^5, -q^{19})}{\phi(-q^{12}) f(-q, -q^{23})}.
\]
(5.43)

**Theorem 5.5.** We have
\[
\sum_{n=0}^{\infty} c_{4n}q^n = \frac{\psi(-q^3) f(-q^9, -q^{15}) f(-q^{20}, -q^4)}{\phi(-q^6) \phi(-q^{12}) f(-q, -q^{23})},
\]
(5.44)
\[
\sum_{n=0}^{\infty} c_{4n+1}q^n = -\frac{\psi(-q^3) f(-q^3, -q^{21}) f(-q^{14}, -q^{10})}{\phi(-q^6) \phi(-q^{12}) f(-q, -q^{23})},
\]
(5.45)
\[
\sum_{n=0}^{\infty} c_{4n+2}q^n = -\frac{\psi(-q^3) f(-q^3, -q^{21}) f(-q^8, -q^{16})}{\phi(-q^6) \phi(-q^{12}) f(-q, -q^{23})},
\]
(5.46)
\[ \sum_{n=0}^{\infty} c_{4n+3} q^n = \frac{\psi(-q^3 \phi(-q^9 \phi(-q^{15} \phi(-q^2 \phi(-q^{22})}}{\phi(-q^4 \phi(-q^{12}) \phi(-q^{23})}} \cdot (5.47) \]

**Theorem 5.6.** We have

\[ \sum_{n=0}^{\infty} c_{8n} q^n = \frac{\phi(-q^{10}, -q^2)}{\phi(-q^6 \phi(-q^9 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^5, -q^{19}) \ f (-q^{16}, -q^{8}) \right. \\
+ q^2 \ f (q^5, q^{19}) \ f (-q^7, -q^{17}) \ f (-q^4, -q^{20}) \right], (5.48) \]

\[ \sum_{n=0}^{\infty} c_{8n+1} q^n = -\frac{\phi(-q^7, -q^2)}{\phi(-q^6 \phi(-q^9 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^2, -q^{22}) \ f (-q^{13}, -q^{11}) \right. \\
+ q^2 \ f (q^3, q^{21}) \ f (-q^{10}, -q^{14}) \ f (-q, -q^{23}) \right], (5.49) \]

\[ \sum_{n=0}^{\infty} c_{8n+2} q^n = -\frac{\phi(-q^4, -q^8)}{\phi(-q^6 \phi(-q^9 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^2, -q^{22}) \ f (-q^{13}, -q^{11}) \right. \\
+ q^2 \ f (q^3, q^{21}) \ f (-q^{10}, -q^{14}) \ f (-q, -q^{23}) \right], (5.50) \]

\[ \sum_{n=0}^{\infty} c_{8n+3} q^n = \frac{\phi(-q, -q^{11})}{\phi(-q^9 \phi(-q^6 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^5, -q^{19}) \ f (-q^{16}, -q^{8}) \right. \\
+ q^2 \ f (q^5, q^{21}) \ f (-q^7, -q^{17}) \ f (-q^4, -q^{20}) \right], (5.51) \]

\[ \sum_{n=0}^{\infty} c_{8n+4} q^n = -\frac{\phi(-q^{10}, -q^2)}{\phi(-q^9 \phi(-q^6 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^7, -q^{17}) \ f (-q^4, -q^{20}) \right. \\
+ q \ f (q^3, q^{21}) \ f (-q^5, -q^{19}) \ f (-q^{16}, -q^{8}) \right], (5.52) \]

\[ \sum_{n=0}^{\infty} c_{8n+5} q^n = \frac{\phi(-q^7, -q^5)}{\phi(-q^6 \phi(-q^9 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^{10}, -q^{14}) \ f (-q, -q^{23}) \right. \\
+ q \ f (q^3, q^{21}) \ f (-q^2, -q^{22}) \ f (-q^{13}, -q^{11}) \right], (5.53) \]

\[ \sum_{n=0}^{\infty} c_{8n+6} q^n = -\frac{\phi(-q^4, -q^8)}{\phi(-q^9 \phi(-q^6 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^{10}, -q^{14}) \ f (-q, -q^{23}) \right. \\
+ q \ f (q^3, q^{21}) \ f (-q^2, -q^{22}) \ f (-q^{13}, -q^{11}) \right], (5.54) \]

\[ \sum_{n=0}^{\infty} c_{8n+7} q^n = \frac{\phi(-q, -q^{11})}{\phi(-q^9 \phi(-q^6 \phi(-q^{12}) \phi(-q^{23}))} \left[ \phi(q^9, q^{15}) \ f (-q^7, -q^{17}) \ f (-q^4, -q^{20}) \right. \\
+ q \ f (q^3, q^{21}) \ f (-q^5, -q^{19}) \ f (-q^{16}, -q^{8}) \right] \cdot (5.55) \]
6. Dissection formulas for $V^*(q)$ and $1/V^*(q)$

In this section, we establish $2-, 4-$ and $8-$ dissection formulas of $V^*(q) := q^{-1/2}V(q)$ and $1/V^*(q)$.

**Theorem 6.1.** If

$$V^*(q) := q^{-1/2}V(q) = \frac{\mathcal{f}(q, -q^9)}{\mathcal{f}(q^9, -q^{19})} = \sum_{n=0}^{\infty} b_n q^n,$$

then $2$-dissection formulas are

$$\sum_{n=0}^{\infty} b_{2n} q^n = \frac{\psi(-q^6)\mathcal{f}(-q^{11}, -q^{13})}{\phi(-q^{12})\mathcal{f}(-q^9, -q^{17})}, \quad (6.1)$$

and

$$\sum_{n=0}^{\infty} b_{2n+1} q^n = -q^2 \frac{\psi(-q^6)\mathcal{f}(-q, -q^{23})}{\phi(-q^{12})\mathcal{f}(-q^7, -q^{17})}. \quad (6.2)$$

**Proof.** We have

$$\sum_{n=0}^{\infty} b_n q^n = \frac{\mathcal{f}(q^9, -q^{19})\mathcal{f}(q^7, q^{17})}{\mathcal{f}(q^9, -q_{17})\mathcal{f}(q^7, q^{17})}. \quad (6.3)$$

Setting $a = -q^9, b = -q^9, c = q^7, d = q^{17}$ in (5.3), we obtain

$$\mathcal{f}(-q^5, -q^{19})\mathcal{f}(q^7, q^{17}) = \mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^{22}, -q^{26}) - q^5\mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^{2}, -q^{46}). \quad (6.4)$$

Setting $a = q^7$ and $b = q^{17}$ in (2.9), we obtain

$$\mathcal{f}(q^7, q^{17})\mathcal{f}(-q^7, -q^{17}) = \mathcal{f}(-q^{14}, -q^{34})\phi(-q^{24}). \quad (6.5)$$

Employing (6.4) and (6.5) in (6.3), we have

$$\sum_{n=0}^{\infty} b_n q^n = \frac{\mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^{22}, -q^{26}) - q^5\mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^2, -q^{46})}{\mathcal{f}(-q^{14}, -q^{34})\phi(-q^{24})}. \quad (6.6)$$

Replacing $q$ by $-q$ in (6.6) we have

$$\sum_{n=0}^{\infty} b_n (-1)^n q^n = \frac{\mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^{22}, -q^{26}) + q^5\mathcal{f}(-q^{12}, -q^{36})\mathcal{f}(-q^2, -q^{46})}{\mathcal{f}(-q^{14}, -q^{34})\phi(-q^{24})}. \quad (6.7)$$

Combining (6.6) and (6.7) and using (1.5) and (1.6), we have

$$\sum_{n=0}^{\infty} b_{2n} q^{2n} = \frac{\psi(-q^{12})\mathcal{f}(-q^{22}, -q^{26})}{\phi(-q^{24})\mathcal{f}(-q^{14}, -q^{34})}, \quad (6.8)$$

and

$$\sum_{n=0}^{\infty} b_{2n+1} q^{2n+1} = -q^2 \frac{\psi(-q^{12})\mathcal{f}(-q^2, -q^{46})}{\phi(-q^{24})\mathcal{f}(-q^{14}, -q^{34})}. \quad (6.9)$$

Replacing $q^2$ by $q$ in (6.8) and (6.9), we arrive at desired result. \hfill \square

**Theorem 6.2.** We have

$$\sum_{n=0}^{\infty} b_{4n} q^n = \frac{\psi(-q^3)\mathcal{f}(-q^9, -q^{15}) \mathcal{f}(-q^{14}, -q^{10})}{\phi(-q^9)\phi(-q^{12})\mathcal{f}(-q^7, -q^{17})}, \quad (6.10)$$

$$\sum_{n=0}^{\infty} b_{4n+1} q^n = -q^3 \frac{\psi(-q^3)\mathcal{f}(-q^4, -q^{20}) \mathcal{f}(-q^9, -q^{15})}{\phi(-q^9)\phi(-q^{12})\mathcal{f}(-q^7, -q^{17})}, \quad (6.11)$$

$$\sum_{n=0}^{\infty} b_{4n+2} q^n = -q^3 \psi(-q^3)\mathcal{f}(-q^2, -q^{21}) \mathcal{f}(-q^2, -q^{22})}{\phi(-q^9)\phi(-q^{12})\mathcal{f}(-q^7, -q^{17})}, \quad (6.12)$$
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Theorem 6.3. We have

\[ \sum_{n=0}^{\infty} b_{4n+3} q^n = \frac{f(-q^2) f(-q^8, -q^{16}) f(-q^2, -q^{21})}{\phi(-q^2) \phi(-q^{12}) f(-q^7, -q^{17})}. \]  

(6.13)

\[ \sum_{n=0}^{\infty} b_{4n+2} q^n = \frac{q f(-q^2, -q^{10})}{\phi(-q^3) \phi(-q^6) \phi(-q^{12})} \left[ f(q^9, q^{15}) f(-q^8, -q^{16}) f(-q^{13}, -q^{11}) \right] + q^5 f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}). \]  

(6.14)

\[ \sum_{n=0}^{\infty} b_{4n+1} q^n = \frac{q^2 f(-q^2, -q^{10})}{\phi(-q^2) \phi(-q^{10})} \left[ f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}) \right], \]  

(6.15)

\[ \sum_{n=0}^{\infty} b_{4n} q^n = \frac{q^2 f(-q^7, -q^{11})}{\phi(-q^4) \phi(-q^{12}) \phi(-q^{24})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^2, -q^{22}) \right] + q^5 f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^{10}, -q^{14}). \]  

(6.16)

\[ \sum_{n=0}^{\infty} b_{4n+3} q^n = -\frac{q f(-q^7, -q^5)}{\phi(-q^4) \phi(-q^6) \phi(-q^{12}) \phi(-q^{24})} \left[ f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^{10}, -q^{14}) \right] + q^5 f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}). \]  

(6.17)

\[ \sum_{n=0}^{\infty} b_{4n+6} q^n = -\frac{f(-q^2, -q^{10})}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) \phi(-q^{16})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^{13}, -q^{11}) \right] + q^5 f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}). \]  

(6.18)

\[ \sum_{n=0}^{\infty} b_{4n+5} q^n = -\frac{f(-q^2, -q^{10})}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) \phi(-q^{16})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^{13}, -q^{11}) \right] + q^5 f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}). \]  

(6.19)

\[ \sum_{n=0}^{\infty} b_{4n+4} q^n = -\frac{f(-q^7, -q^5)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) \phi(-q^{24})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^{13}, -q^{11}) \right] + q^5 f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q, -q^{23}). \]  

(6.20)

\[ \sum_{n=0}^{\infty} b_{4n+7} q^n = \frac{f(-q^4, -q^8)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) \phi(-q^{16}) \phi(-q^{24})} \left[ f(q^9, q^{15}) f(-q^5, -q^{19}) f(-q^{10}, -q^{14}) \right] + q^5 f(q^3, q^{21}) f(-q^7, -q^{17}) f(-q^2, -q^{22}). \]  

(6.21)

In next theorems, we state the 2-, 4- and 8-dissection formulas of 1/V*(q).
Theorem 6.4. If
\[
\frac{1}{V^*(q)} := f(-q^7, -q^{17}) \implies f(-q^9, -q^{19}) = \sum_{n=0}^{\infty} d_n q^n,
\]
then 2-dissection formulas are
\[
\sum_{n=0}^{\infty} d_{2n} q^n = \psi(-q^6) f(-q^{13}, -q^{11}) \phi(-q^{12}) f(-q^7, -q^{19}),
\]
and
\[
\sum_{n=0}^{\infty} d_{2n+1} q^n = -q^2 \psi(-q^6) f(-q, -q^{22}) \phi(-q^{12}) f(-q^7, -q^{19}).
\]

Theorem 6.5. We have
\[
\sum_{n=0}^{\infty} d_{4n} q^n = \psi(-q^3) f(-q^9, -q^{15}) f(-q^{16}, -q^8),
\]
\[
\sum_{n=0}^{\infty} d_{4n+1} q^n = -q^2 \psi(-q^3) f(-q^9, -q^{21}) f(-q^{10}, -q^{14}),
\]
\[
\sum_{n=0}^{\infty} d_{4n+2} q^n = -q^2 \psi(-q^3) f(-q^9, -q^{21}) f(-q^4, -q^{20}),
\]
\[
\sum_{n=0}^{\infty} d_{4n+3} q^n = -q^3 \psi(-q^3) f(-q^9, -q^{15}) f(-q^2, -q^{22}).
\]

Theorem 6.6. We have
\[
\sum_{n=0}^{\infty} d_{8n} q^n = \frac{f(-q^9, -q^4)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^{14}, -q^{10}) + q^4 f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^2, -q^{22}) \right],
\]
\[
\sum_{n=0}^{\infty} d_{8n+1} q^n = \frac{q^2 f(-q^9, -q^7)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^9, q^{15}) f(-q^8, -q^{16}) f(-q, -q^{23}) + f(q^3, q^{21}) f(-q^6, -q^{20}) f(-q^{11}, -q^{13}) \right],
\]
\[
\sum_{n=0}^{\infty} d_{8n+2} q^n = -\frac{q f(-q^9, -q^{10})}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^9, q^{15}) f(-q^4, -q^{20}) f(-q^{11}, -q^{13}) + q^2 f(q^3, q^{21}) f(-q^5, -q^{16}) f(-q, -q^{23}) \right],
\]
\[
\sum_{n=0}^{\infty} d_{8n+3} q^n = -\frac{q^2 f(-q, -q^{11})}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^3, q^{21}) f(-q^7, -q^{17}) f(-q^{14}, -q^{10}) + q f(q^9, q^{15}) f(-q^5, -q^{19}) f(-q^2, -q^{22}) \right],
\]
\[
\sum_{n=0}^{\infty} d_{8n+4} q^n = -\frac{q f(-q^8, -q^4)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^3, q^{21}) f(-q^7, -q^{17}) f(-q^{14}, -q^{10}) + q f(q^9, q^{15}) f(-q^5, -q^{19}) f(-q^2, -q^{22}) \right],
\]
\[
\sum_{n=0}^{\infty} d_{8n+5} q^n = -\frac{f(-q^5, -q^7)}{\phi(-q^3) \phi(-q^6) \phi(-q^{12}) f(-q^7, -q^{19})} \left[ f(q^9, q^{15}) f(-q^4, -q^{20}) f(-q^{11}, -q^{13}) + q^3 f(q^3, q^{21}) f(-q^8, -q^{16}) f(-q, -q^{23}) \right],
\]
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\[ \sum_{n=0}^{\infty} d_{8n+6}q^n = \frac{q^3(f(-q^2, -q^{10}))}{\phi(-q^3)\phi(-q^6)\phi(-q^{10})f(-q^5, -q^{19})} \left[ f(q^9, q^{15}) f(-q^8, -q^{16}) f(-q, -q^{23}) \right. \\
+ f(q^3, q^{21}) f(-q^4, -q^{20}) f(-q^{11}, -q^{13}) \left. \right], \quad (6.34) \]

\[ \sum_{n=0}^{\infty} d_{8n+7}q^n = \frac{f(-q, -q^{11})}{\phi(-q^3)\phi(-q^6)\phi(-q^{12})f(-q^5, -q^{19})} \left[ f(q^9, q^{15}) f(-q^7, -q^{17}) f(-q^{14}, -q^{10}) \right. \\
+ q^4 f(q^3, q^{21}) f(-q^5, -q^{19}) f(-q^2, -q^{22}) \left. \right]. \quad (6.35) \]

**Remark 6.7.** Andrews and Bressoud [2] proved the following general theorem related to vanishing coefficients:

If \( 1 \leq r < k \) are relatively prime integers of opposite parity and

\[ \frac{(q,r_2 q^{2k-r}; q^{2k})}{(q^{k-r}, r q^{k+r}; q^{2k})} = \sum_{n=0}^{\infty} \phi_n q^n, \quad (6.36) \]

then \( \phi_{kn+r(k-r+1)/2} \) is always zero.

In view of above result of Andrews and Bressoud [2], we have the following corollary.

**Corollary 6.8.** We have

\( (i) \quad a_{12n+6} = 0, \quad (ii) \quad b_{12n+20} = 0, \quad (iii) \quad c_{12n+11} = 0, \quad (iv) \quad d_{12n+21} = 0. \)

**Proof.** Taking \( k = 12 \) and \( r = 1, 5, 11, 7, \) respectively in (6.36), we see that

\[ \frac{(q,q^{21}; q^{24})}{(q^{11}, q^{13}; q^{24})} = \sum_{n=0}^{\infty} a_n q^n, \]

\[ \frac{(q^5,q^{19}; q^{24})}{(q,q^{17}; q^{24})} = \sum_{n=0}^{\infty} b_n q^n, \]

\[ \frac{(q^{11},q^{13}; q^{24})}{(q,q^{23}; q^{24})} = \sum_{n=0}^{\infty} c_n q^n \]

and

\[ \frac{(q^7,q^{17}; q^{24})}{(q^5,q^{23}; q^{24})} = \sum_{n=0}^{\infty} d_n q^n, \]

respectively. Hence, the desired results follow immediately. \( \square \)

Further, following corollary follows from the dissection formulas of \( U^*(q) \) and \( V^*(q) \) and their reciprocals.
Corollary 6.9. We have

\[
\begin{align*}
(i) & \quad \sum_{n=0}^{\infty} a_{2n} q^n = \sum_{n=0}^{\infty} c_{2n} q^n \\
& \quad \sum_{n=0}^{\infty} a_{2n+1} q^n = \sum_{n=0}^{\infty} c_{2n+1} q^n \\
(iii) & \quad \sum_{n=0}^{\infty} a_{4n} q^n = \sum_{n=0}^{\infty} b_{4n} q^{n+3} \\
& \quad \sum_{n=0}^{\infty} a_{4n+2} q^n = \sum_{n=0}^{\infty} b_{4n+2} q^n \\
(v) & \quad \sum_{n=0}^{\infty} a_{8n} q^n = \sum_{n=0}^{\infty} b_{8n+4} q^n \\
& \quad \sum_{n=0}^{\infty} a_{8n+2} q^n = \sum_{n=0}^{\infty} b_{8n+2} q^n \\
(vii) & \quad \sum_{n=0}^{\infty} b_{8n} q^n = \sum_{n=0}^{\infty} b_{8n+4} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} b_{8n+4} q^n = \sum_{n=0}^{\infty} b_{8n+2} q^{n+1} \\
(ix) & \quad \sum_{n=0}^{\infty} c_{8n} q^n = \sum_{n=0}^{\infty} c_{8n+4} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} c_{8n+4} q^n = \sum_{n=0}^{\infty} c_{8n+2} q^{n+1} \\
(xi) & \quad \sum_{n=0}^{\infty} d_{8n} q^n = \sum_{n=0}^{\infty} d_{8n+4} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} d_{8n+4} q^n = \sum_{n=0}^{\infty} d_{8n+2} q^{n+1} \\
(ii) & \quad \sum_{n=0}^{\infty} b_{2n} q^n = \sum_{n=0}^{\infty} d_{2n} q^{n+2} \\
& \quad \sum_{n=0}^{\infty} b_{2n+1} q^n = \sum_{n=0}^{\infty} d_{2n+1} q^{n+2} \\
(iv) & \quad \sum_{n=0}^{\infty} c_{4n} q^n = \sum_{n=0}^{\infty} d_{4n} q^{n+2} \\
& \quad \sum_{n=0}^{\infty} c_{4n+1} q^n = \sum_{n=0}^{\infty} d_{4n+1} q^{n+2} \\
(vi) & \quad \sum_{n=0}^{\infty} d_{8n} q^n = \sum_{n=0}^{\infty} d_{8n+6} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} d_{8n+6} q^n = \sum_{n=0}^{\infty} d_{8n+4} q^{n+1} \\
(viii) & \quad \sum_{n=0}^{\infty} b_{8n} q^n = \sum_{n=0}^{\infty} d_{8n+3} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} b_{8n+3} q^n = \sum_{n=0}^{\infty} d_{8n} q^{n+1} \\
(x) & \quad \sum_{n=0}^{\infty} c_{8n} q^n = \sum_{n=0}^{\infty} d_{8n+5} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} c_{8n+5} q^n = \sum_{n=0}^{\infty} d_{8n+3} q^{n+1} \\
(xii) & \quad \sum_{n=0}^{\infty} d_{8n} q^n = \sum_{n=0}^{\infty} d_{8n+1} q^{n+1} \\
& \quad \sum_{n=0}^{\infty} d_{8n+1} q^n = \sum_{n=0}^{\infty} d_{8n+3} q^{n+1}.
\end{align*}
\]

Declarations

Conflict of Interest. The authors declare that there is no conflict of interest regarding the publication of this article.

Human and animal rights. The authors declare that there is no research involving human participants or animals in the contained of this paper.

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References


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