Abstract

The main relations of tunnel mathematics (theory of functions of spatial complex variable) can be obtained with the aid of tensor analysis. It is commonly known, for instance, that a second-rank tensor being multiplied by vector on a plane can change a direction of this vector, i.e. second-rank tensor can take the vector out of the plane. Besides, tunnel mathematics can be applied for solving problems of fluid dynamics and theory of elasticity where tensor analysis is used very broadly. So, it is naturally to use tensor analysis for building the tunnel mathematics.

Introduction

This article is an addition to article [4] where main relations of tunnel mathematics were obtained by algebraic way. We do a comparison of corresponding results.

Theory

3.1 Definition of a spatial complex number

Similar to how a vector is defined in three-dimensional space, we define a spatial complex number as follows:

\[ L = ReL + iImL + fF\text{ant}L = x + iy + fz. \]

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Where \( ReL \) – is a real part of a spatial complex number \( L \), \( ImL \) - its imaginary part, \( F\text{ant}L \) - its spatial part.

The spatial complex number \( L \) is shown in Fig. 1.

The components of the space vector expressed in terms of its magnitude \( R \) as well as the angles \( \varphi \) and \( \theta \) are defined as follows:

\[ x = R \cos \theta \cos \varphi, \]
\[ y = R \cos \theta \sin \varphi, \]
\[ z = R \sin \theta. \]

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On the other hand, the following obvious equality holds:

\[ e^{i(\phi-\theta)} = e^{i\phi}e^{i\theta} = \cos \theta \cos \phi + i \cos \theta \sin \phi + ie^{i\phi} \sin \theta. \]

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Now, if we introduce the operator

\[ f = ie^{i\phi} = e^{i(\phi+\frac{\pi}{2})} \]

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then on the right-hand in (3) we obtain the expansion of the space vector with unit modulus along the \( x, y, z \) axes. The components of this expansion coincide with the values of the projections in formula (2) at \( R = 1 \).

So the spatial complex number is given by formula (1) where \( x, y, z \) are the real numbers.

The exponential form of the spatial complex number is:

\[ L = Re^{i(\phi-\theta)} = \frac{r}{\cos \theta}e^{i\phi}e^{i\theta} = re^{i\phi} \left( \frac{\cos \theta + isin \theta}{\cos \theta} \right) = (1 + itg \theta)m \]
where \( \phi = \tan^{-1} \frac{y}{x}; \theta = \sin^{-1} \frac{z}{R}; m = re^{i\phi}. \) (6)

The trigonometric form is as follows:

\[
L = R(\cos \theta \cos \phi + i \cos \theta \sin \phi + f \sin \theta)
\]

We define the conjugate spatial complex number as follows:

\[
L^* = xiyf^* = Re^{-i(\phi-\theta)},
\]

where \( f^* = ie^{-i\phi}, \) hence

\[
f \times f^* = -1.
\]

Identical to plane theory, the following relation holds:

\[
L \times L^* = R^2 = x^2 + y^2 + z^2.
\]

Meanwhile the following relationships hold:

\[i^2 = -1\] (as usually);

\[
i \times f = f \times i = -e^{i\phi} = \cos \phi \sin \phi = -\frac{x + iy}{\sqrt{x^2 + y^2}}.
\]

\[
f^* = ie^{-i\phi} = \sin \phi + i \cos \phi = \frac{y + ix}{\sqrt{x^2 + y^2}} = f + 2 \sin \phi = f + \frac{2y}{\sqrt{x^2 + y^2}}.
\]

\[
f^2 = e^{2i\phi} = (\cos 2\phi + i \sin 2\phi) = -\frac{x^2 - y^2 + 2ixy}{x^2 + y^2}.
\]

Considering that

\[
f = ie^{i\phi} = \sin \phi + i \cos \phi = -\frac{-y + ix}{\sqrt{x^2 + y^2}},
\]

we can define spatial complex number on a plane as follows:
3.2 Calculation

3.2.1 The space Cauchy-Riemann conditions

Let us derive conditions similar to the plane theory Cauchy-Riemann conditions which are necessary for the spatial function to be analytic. These conditions (let's call them the space Cauchy-Riemann conditions) as in the plane theory are derived from the requirement that the derivatives of the function of spatial complex variable $P$ in the directions $x, iy, fz$ are equal to each other:

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial iy} = \frac{\partial P}{\partial fz}.$$

Consider the following spatial complex variable function:

$$P = u(x, y, z) + iv(x, y, z) + fw(x, y, z);$$

where functions $u, v$ and $w$ are complex functions of real variables $x, y, z$.

In order to take the complex vector $m = re^{i\phi} = x + iy$ out of the plane and obtain (1) we will multiply it by the operator $\hat{T}$ which is a second-rank tensor $T_{ik}$:

$$L = \hat{T}m = T_{ik}m_k.$$

Further, burning in mind (1), we rewrite (18) in matrix form:

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} x \\ iy \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ iy \\ fz \end{pmatrix}.$$

Equating the coefficients in (18) we find the second-rank tensor $T_{ik}$:

$$T_{ik} = \begin{pmatrix} 1 & 0 & \frac{yz}{x\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{xz}{y\sqrt{x^2+y^2}} \\ -\frac{yz}{x\sqrt{x^2+y^2}} & \frac{xz}{y\sqrt{x^2+y^2}} & -1 + \frac{iz}{\sqrt{x^2+y^2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{\tan\phi}{r} \\ 0 & 1 & \frac{\cot\phi}{r} \\ -\frac{\tan\phi}{r} & \frac{\cot\phi}{r} & -1 + itan\theta \end{pmatrix}.$$

Contracting $T_{ik}$ we obtain such relation:

$$T_{ii} = 1 + itan\theta,$$

what fully correspond to (5).
In similar way we proceed with spatial complex variable function $P$ from (17):

$$P = \hat{T}'l = T'_{ik}l_k,$$

where $l_k = \begin{pmatrix} u \\ iv \\ 0 \end{pmatrix}$. (23)

The second-rank tensor $T'_{ik}$ have such form:

$$T'_{ik} = \begin{pmatrix} 1 & 0 & -\frac{yw}{x\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{xw}{y\sqrt{x^2+y^2}} \\ -\frac{yw}{x\sqrt{x^2+y^2}} & \frac{xw}{y\sqrt{x^2+y^2}} & -1 + \frac{iw}{\sqrt{x^2+y^2}} \end{pmatrix}.$$

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Now, in order to find the space Cauchy-Riemann conditions we determine corresponding derivatives (16) from (22):

$$\frac{\partial P}{\partial x} = \frac{\partial T'_{ik}l_k}{\partial x} + T'_{ik}\frac{\partial l_k}{\partial x} = \frac{\partial u}{\partial x} + \frac{(uw) y}{(x^2 + y^2)^{3/2}} \left( 2 + \left( \frac{y}{x} \right)^2 \right) + \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (uv)}{\partial x}$$

$$+ i \left[ \frac{\partial v}{\partial x} + \frac{(uv) y}{(x^2 + y^2)^{3/2}} - \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (uw)}{\partial x} \right] + f \left[ \frac{1}{x} \cdot \frac{\partial (uw)}{\partial x} + \frac{1}{y} \cdot \frac{\partial (vw)}{\partial x} \right];$$

$$\frac{\partial P}{\partial y} = -i \frac{\partial P}{\partial y} = -i \left( \frac{\partial T'_{ik}l_k}{\partial y} + T'_{ik}\frac{\partial l_k}{\partial y} \right) = \frac{\partial v}{\partial y} - \frac{(uw) x}{(x^2 + y^2)^{3/2}} \left( 2 + \left( \frac{x}{y} \right)^2 \right) - \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (uv)}{\partial y}$$

$$+ i \left[ -\frac{\partial u}{\partial y} + \frac{(uw) x}{(x^2 + y^2)^{3/2}} - \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (vw)}{\partial y} \right] + f \left[ -\frac{i}{y} \cdot \frac{\partial (vw)}{\partial y} - \frac{i}{x} \cdot \frac{\partial (uw)}{\partial y} \right];$$

$$\frac{\partial P}{\partial fz} = -f^* \cdot \frac{\partial P}{\partial z} = -f^* \left( \frac{\partial T'_{ik}l_k}{\partial z} + T'_{ik}\frac{\partial l_k}{\partial z} \right) = -\frac{2y}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial z} + \frac{1}{x} \cdot \frac{\partial (uw)}{\partial z} + \frac{1}{y} \cdot \frac{\partial (vw)}{\partial z} - \frac{2y}{(x^2 + y^2)} \frac{\partial (vw)}{\partial z}$$

$$+ i \left[ -\frac{2y}{\sqrt{x^2 + y^2}} \frac{\partial v}{\partial z} + \frac{2y}{(x^2 + y^2)} \frac{\partial (uv)}{\partial z} \right] + f \left[ -\frac{\partial u}{\partial z} - i \frac{\partial v}{\partial z} + \frac{i}{\sqrt{x^2 + y^2}} \cdot \frac{\partial (uv)}{\partial z} - \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{\partial (vw)}{\partial z} \right].$$

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Equating the corresponding parts in (25)-(27) we find the space Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} + \frac{(uw) y}{(x^2 + y^2)^{3/2}} \left( 2 + \left( \frac{y}{x} \right)^2 \right) + \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (vw)}{\partial x} = \frac{\partial v}{\partial y} - \frac{(uv) x}{(x^2 + y^2)^{3/2}} \left( 2 + \left( \frac{x}{y} \right)^2 \right) - \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (uv)}{\partial y}$$

$$= -\frac{2y}{\sqrt{x^2 + y^2}} \frac{\partial u}{\partial z} + \frac{1}{x} \cdot \frac{\partial (uw)}{\partial z} + \frac{1}{y} \cdot \frac{\partial (vw)}{\partial z} - \frac{2y}{(x^2 + y^2)} \frac{\partial (vw)}{\partial z}.$$
Relations (28)-(30) are similar with relations (22)-(24) in [4], which was obtained by algebraic way, but not the same exactly.

### 3.2.2 The flux of a function of the spatial complex variable $P$ through the sphere $S$

It is of interest to find the flux of a function of the spatial complex variable $P$ through the sphere $S$ (Fig. 2).

The calculations with applying Gauss-Ostrogradsky theorem yield to following result:

\[
\oint_S P \, dS = \oint_S T_{ik} l_k n_i \, dS = \iiint_V \frac{\partial T_{ik}^l}{\partial x_i} \, dV = \iiint_V \left( \frac{\partial T_{ik}^l}{\partial x_i} l_k + T_{ik}^l \frac{\partial l_k}{\partial x_i} \right) \, dV,
\]

\[
= \iiint_V \left[ \frac{\partial u}{\partial x} + \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (vw)}{\partial z} + i \left( \frac{\partial v}{\partial y} - \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial (uw)}{\partial z} \right) \right] dV.
\]

Some parts of integrand in (31) are included in (28) what opens a field for further investigations.

### Results And Discussion

Taking into account the first equality in (30) we can impose such conditions on functions $uw$ and $vw$:

\[
\frac{\partial (uw)}{\partial x} + i \frac{\partial (uw)}{\partial y} = 0;
\]

and

\[
\frac{\partial (vw)}{\partial x} + i \frac{\partial (vw)}{\partial y} = 0.
\]

So, functions $uw$ and $vw$ are harmonic functions on the $xy$-plane. i.e.
\[ \Delta (uw) = 0; \]

and

\[ \Delta (vw) = 0. \]

where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is the Laplace operator.

We know from the theory of harmonic functions that function, which is harmonic in some area, may be found in any point of the area with the aid of values of this function and its normal derivative on the boundary of the area. So, taking into account (30), we can formulate the following theorem, the proof of which is obvious:

**Theorem**

Values of harmonic in xy-plane functions \( uw \) and \( vw \) have sense only if that xy-plane moves in space (i.e. it moves along \( z \)-coordinate).

Figure 3 is given for explanation of theorem.

This theorem opens the way to find spatial values of function \( u, v \) and \( w \).

Further, we do standard procedure with first equalities in (28) and (29) to find the Laplacians of complex functions \( u \) and \( v \). We take derivative by \( x \) from one equality, then take derivative by \( y \) from another one and vice versa. Further, we either subtract or add obtained relations in order to find corresponding Laplacian. Applying (32)-(35) we obtain such result:

\[ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \sqrt{x^2 + y^2} \left( \frac{uw}{x^3} - \frac{vw}{y^3} \right); \]

\[ \Delta u + \frac{x}{(x^2 + y^2)} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) - \frac{(uw) y}{x^3 \sqrt{x^2 + y^2}} = 0; \]

\[ \Delta v - \frac{y}{(x^2 + y^2)} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + \frac{(vw) x}{y^3 \sqrt{x^2 + y^2}} = 0; \]

In order to (34) and (35) be fulfilled we need to impose such conditions on functions \( u \) and \( v \):

\[ \Delta u = 0; \]

\[ \Delta v = 0. \]

Relations (39) and (40) agree with (23).

Applying (39) and (40) to (37) and (38) we obtain following connection between functions \( u \) and \( v \):

\[ u y^5 - v x^5 = 0. \]
We need burn in mind that functions \( u \) and \( v \) are the complex functions of real variables \( x, y \) and \( z \), and only their real or imaginary parts separately have certain physical meaning.

**Conclusions**

In this article tensor foundations of tunnel mathematics were laid. The conditions for the analyticity of the spatial function of a complex variable and the theorem which opens the way to find spatial values of this function components were obtained as well. The relations of tunnel mathematics reflect a physical behavior of vector quantities; that is why they can be used, for example, to solving of the problems in fluid dynamics and theory of elasticity.

**Declarations**

**Data Availability**

The data that supports the findings of this study are available within the article.

**Declaration of Interests.** The authors report no conflict of interest.

**References**


**Figures**
Figure 1

Spatial complex number $L$. 
Figure 2

To derivation of the flux of a function of the spatial complex variable $P$ through the sphere $S$. 
Figure 3

To explanation of the theorem.