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Spectral theorems for Wigner transform associated with the Laguerre-Bessel localization operators

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Abstract
The main crux of this work is to study the Wigner transform associated with the Laguerre-Bessel operator and to give some results related to this transform. Next motivated by Wong’s point of view we define a class of pseudo-differential operator \( L_{u,v}(\sigma) \) called localization operator which depend on a symbol \( \sigma \) and two functions \( u \) and \( v \), we give a criteria in terms of the symbol \( \sigma \) for its boundedness and compactness, we also show that these operators belongs to the Schatten-Von Neumann class \( S^p \) for all \( p \in [1;+\infty] \) and we give a trace formula.

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1 Introduction

The Wigner transform has a long story which started in 1932 with Eugene Wigner’s in [21] as a probability quasi-distribution to study quantum corrections to classical statistical mechanics, the goal was to link the wave function that appears in Schrödinger’s equation to a probability distribution in phase space see [7, 21]. A mathematical object closely related to the Wigner transform is the windowed Fourier transform used in signal theory and time-frequency analysis, using this connection we will define and study the localization operators for the Fourier-Wigner transform.
associated with the Laguerre-Bessel operator.

The classical Fourier transform in $\mathbb{R}^d$ can be defined by many ways, its most basic formulation it is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\lambda \cdot x} f(x) \, dx.$$ 

Alternatively, one can rewrite this transform as

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} K(\lambda, x) f(x) \, dx,$$

where $K(\lambda, x)$ is the unique solution of the system of partial differential equations

$$\begin{cases}
\partial_{x_j} K(\lambda, x) = -i\lambda_j K(\lambda, x), & \text{for } j = 1, \ldots, d, \\
K(\lambda, 0) = 1, & \lambda \in \mathbb{R}^d.
\end{cases}$$

A lot of attention has been given to various generalization of the classical Fourier transform. This paper focuses on the generalized Fourier transform associated with the Laguerre-Bessel operator called the Laguerre-Bessel transform, more precisely we consider a system of partial differential operator $\Delta_1$ and $\Delta_2$ defined on $\mathbb{R}^d$ by

$$\begin{align*}
\Delta_1 &= \frac{\partial^2}{\partial t^2} + \frac{2\alpha}{t} \frac{\partial}{\partial t}, & \alpha \geq 0, & t > 0, \\
\Delta_2 &= \frac{\partial^2}{\partial x^2} + \frac{2\alpha + 1}{x} \frac{\partial}{\partial x} + x^2 \Delta_1, & x > 0.
\end{align*}$$

The eigenfunctions of this system are related to the Bessel and Laguerre functions and they satisfy a product formula which permits to develop a new harmonic analysis associated to these operators. One of the aims of the Fourier transform is the study of the theory of localization operators called also Gabor multipliers, Toeplitz operators or Anti-Wick operators, this theory was initiated by Daubechies in \cite{5,6}, developed and detailed in the book \cite{22} by Wong. Wong was the first one who defined the localization operators on the Weyl Heisenberg group in \cite{23}, next Boggiatto and Wong have extended this results on $L^p(\mathbb{R}^d)$ in \cite{3}. Then Wong studies the localization operators associated to left regular representation of locally compact and Hausdorff group $G$ on $L^p(G)$ in \cite{24}. Some results for wavelets multipliers which are localization operators associated to modulation on the additive group on $\mathbb{R}^d$ are given by Ma and Wong in \cite{10}.

The theory of localization operators associated with the Fourier-Wigner transform has been studied and known remarkable development in many settings for example in the Riemann-Liouville setting \cite{11}, in the spherical mean setting \cite{12}, in the Laguerre setting \cite{13}, in the Dunkl setting \cite{14}, in the Weinstein setting \cite{17}, in the Heckman-Opdam-Jacobi setting \cite{1}, so its natural to ask whether there exists the equivalent of the theory of localization operators in other setting as the Laguerre Bessel settings. Following Wong’s point of view, our main aim in this paper is to prove the analogues of the
results on the localization operators studies by the authors in [1],[11],[12],[13],[14],[17] in the Laguerre-Bessel frame. The remainder of this paper is arranged as follows, in section 2 we recall the main results concerning the harmonic analysis associated with the Laguerre-Bessel transform and Schatten-Von Neumann classes, in section 3 we will study the boundedness, compactness and the Schatten properties of the two-Wavelet localization operator associated with the Laguerre-Bessel-Wigner transform.

2 Preliminaries

In this section we set some notations and we recall some results in harmonic analysis related to the Laguerre-Bessel transform and the Schatten-Von Neumann classes, for more details we refer the reader to [3,9,15].

In the following we denote by

- $K := [0,+\infty) \times [0,+\infty)$ equipped with the weighted Lebesgue measure $\mu_\alpha$ on $K$ given by
  \[ d\mu_\alpha(x,t) := \frac{x^{2\alpha+1}t^{2\alpha}dxdt}{\pi^\Gamma(\alpha + 1)(\alpha + \frac{1}{2})}, \quad \alpha \geq 0, \]
  where $\Gamma$ is the Gamma function.
- $L^p_{\alpha}(K), 1 \leq p \leq \infty$, the space of measurable functions on $K$, satisfying
  \[ \|f\|_{L^p_{\alpha}(K)} := \left( \int_K |f(x,t)|^p d\mu_\alpha(x,t) \right)^{1/p} < \infty, \quad 1 \leq p < \infty, \]
  \[ \|f\|_{L^\infty_{\alpha}(K)} := \text{ess sup}_{(x,t) \in K} |f(x,t)| < \infty, \quad p = \infty. \]
- $S_r(K)$ the space of function $f : \mathbb{R}^2 \to \mathbb{C}$, even with respect to the first variable, $C^\infty$ on $\mathbb{R}^2$ and rapidly decreasing together with their derivations, i.e. for all $k,p,q \in \mathbb{N}$ we have:
  \[ N_{k,p,q}(f) = \sup_{(x,t) \in K} (1 + x^2 + t^2)^k \left| \frac{\partial^{p+q}}{\partial x^p \partial t^q} f(x,t) \right| < \infty. \]
- $L_m^{(\alpha)}(x)$ is the Laguerre function defined on $[0, +\infty)$ by
  \[ L_m^{(\alpha)}(x) := \frac{c_\alpha^2}{L_m^{(\alpha)}(0)}, \]
  where $L_m^{(\alpha)}$ is the laguerre polynomial of degree $m$ and order $\alpha$ given by
  \[ L_m^{(\alpha)}(x) = \sum_{k=0}^{m} (-1)^k \Gamma(m + \alpha + 1) \frac{x^k}{\Gamma(k + \alpha + 1) k!(m-k)!}. \]
• $\hat{\mathbb{K}} := [0, +\infty) \times \mathbb{N}$ equipped with weighted Lebesgue measure $\gamma_\alpha$ given by

$$d\gamma_\alpha(\lambda, m) = \frac{\lambda^{2\alpha+1}}{2^{2\alpha-1}\Gamma(\alpha + \frac{1}{2})} L_m^{(\alpha)}(0) d\lambda \otimes \delta_m,$$

where $\delta_m$ is the Dirac measure at $m$ and $d\lambda$ is the classical Lebesgue measure in $[0, +\infty)$.

• $L_p^p(\hat{\mathbb{K}})$ with $p \in [1, +\infty]$ the space of measurable functions on $\hat{\mathbb{K}}$ satisfying

$$\|g\|_{p, \gamma_\alpha} := \left( \int_{\hat{\mathbb{K}}} |g(\lambda, m)|^p d\gamma_\alpha(\lambda, m) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$\|g\|_{\infty, \gamma_\alpha} := \operatorname{ess\ sup}_{(\lambda, m) \in \hat{\mathbb{K}}} |g(\lambda, m)| < \infty, \quad p = \infty.$$

### 2.1 The eigenfunctions of the partial Differential operators $\Delta_1$ and $\Delta_2$

For $(\lambda, m) \in \hat{\mathbb{K}}$ we consider the following Cauchy problem

$$(S) : \begin{cases} \Delta_1(u) = -\lambda^2 u, \\ \Delta_2(u) = -2\lambda(2m + \alpha + 1) u, \\ u(0,0) = 1, \quad \frac{\partial u}{\partial x}(0,0) = \frac{\partial u}{\partial t}(0,0) = 0. \end{cases}$$

From [9], we put $\psi(x, t) = j_{\alpha - \frac{1}{2}}(\lambda t) f \left( x^2 \right)$ where $j_{\alpha}$ is the spherical Bessel function given by

$$j_{\alpha}(x) = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k (x)^{2k}}{2^{\alpha}k!\Gamma(\alpha + k + 1)}.$$

see [16, 20] for more informations about this function. We have $\psi$ is the solution of $(S)$ if and only if the function $f$ satisfies the following differential equation:

$$\begin{cases} \frac{d^2}{dx^2} f + (\alpha + 1) \frac{d}{dx} f - \lambda^2 f = -2\lambda(2m + \alpha + 1) f, \\ f(0) = 1; \quad \frac{d}{dx} f(0) = 0. \end{cases}$$

But from [19], this differential equation has a unique solution given by

$$f \left( x^2 \right) = L_m^{(\alpha)} \left( \lambda x^2 \right),$$

Hence the Cauchy problem $(S)$ admits a unique solution given by

$$\varphi_{\lambda, m}(x, t) = j_{\alpha - \frac{1}{2}}(\lambda t) L_m^{(\alpha)} \left( \lambda x^2 \right) \quad \text{for } (x, t) \in \mathbb{K} \quad \text{and } (\lambda, m) \in \hat{\mathbb{K}}.$$
This function is infinitely differentiable on $\mathbb{R}^2$, even with respect to each variable and we have the following important result

$$
\sup_{(x,t) \in K} |\varphi_{\lambda,m}(x,t)| = 1.
$$

(2.1)

We suppose that $\alpha > 0$, and from [2] we have the Laguerre function $L_\alpha^m$ satisfies the following product formula

$$
L_\alpha^m(\lambda x^2) L_\alpha^m(\lambda y^2) = b_\alpha \int_0^\pi \varphi_{\lambda,m}(\lambda A_\theta^2(x,y)) j_{\alpha-\frac{1}{2}}(\lambda xy \sin(\theta))(\sin(\theta))^{2\alpha} d\theta,
$$

with $b_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+\frac{1}{2})}$ and $A_\theta(x,y) = \sqrt{x^2 + y^2 - 2xy \cos(\theta)}$ also from the product formula for the Bessel function see [16] and by a change of variables and by a simple computation we find that:

$$
\varphi_{\lambda,m}(x,t) \varphi_{\lambda,m}(y,s) = \int_K \varphi_{\lambda,m}(z,v) K_\alpha[(x,t), (y,s), (z,v)] d\mu_\alpha,
$$

(2.2)

where $K_\alpha$ is a positive kernel given explicitly in [9].

The product formula (2.2) permits to define a translation operator, a convolution product and to develop a new harmonic analysis associated to the operators $\Delta_1$ and $\Delta_2$.

2.2 The Laguerre-Bessel transform

**Definition 2.1.** The Laguerre-Bessel transform $F_\alpha$ defined on $L_1^\alpha(K)$ by

$$
F_\alpha(f)(\lambda,m) = \int_K \varphi_{\lambda,m}(x,t)f(x,t) d\mu_\alpha(x,t) \quad \text{for} \ (\lambda,m) \in \hat{K}.
$$

Some basic properties of this transform are as follows, for the proofs, we refer the reader to [9].

**Proposition 2.1.**

(1) For every $f \in L_1^\alpha(K), m \in \mathbb{N}$ the function

$$
\lambda \mapsto F_\alpha(f)(\lambda,m),
$$

is continuous on $\mathbb{R}$ and we have

$$
\|F_\alpha\|_{\infty,\gamma_\alpha} \leq \|f\|_{1,\mu_\alpha}.
$$

(2.3)

(2)(Inversion formula) For $f \in (L_1^\alpha \cap L_2^\alpha)(K)$ such that $F_\alpha \in L_1^\alpha(\hat{K})$ we have

$$
f(x,t) = \int_K \varphi_{\lambda,m}(x,t) F_\alpha(f)(\lambda,m) d\gamma_\alpha(\lambda,m), \quad \text{a.e} \quad (x,t) \in K
$$

(2.4)
(3) (Parseval formula) For all $f, g \in L^2_\alpha(\mathbb{K})$ we have
\[
\langle f, g \rangle_{\mu_\alpha} = \langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g) \rangle_{\gamma_\alpha},
\]
where
\[
\langle f, g \rangle_{\mu_\alpha} = \int_{\mathbb{K}} f(x,t)g(x,t) d\mu_\alpha(x,t), \quad \text{and}
\]
\[
\langle \mathcal{F}_\alpha(f), \mathcal{F}_\alpha(g) \rangle_{\gamma_\alpha} = \int_{\mathbb{K}} \mathcal{F}_\alpha(f)(\lambda, m)\mathcal{F}_\alpha(g)(\lambda, m) d\gamma_\alpha(\lambda, m).
\]
In particular we have
\[
\|f\|_{2,\mu_\alpha} = \|\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}.
\]
(2.6)

(4) (Plancherel theorem) The Laguerre-Bessel transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2_\alpha(\mathbb{K})$ into $L^2_\alpha(\hat{\mathbb{K}})$.

2.3 The translation operator associated with the Laguerre-Bessel transform

The product formula (2.2) permits to define the translation operator as follows:

**Definition 2.2.** Let $(x,t), (y,s) \in \mathbb{K}$ and $f \in \mathcal{S}_*(\mathbb{K})$ the translation operator is defined by:
\[
\mathcal{T}_\alpha^{(x,t)}(f)(y,s) = \int_{\mathbb{K}} f(z,v)K_\alpha((x,t), (y,s), (z,v)) d\mu_\alpha(z,v).
\]

The following proposition summarizes some properties of the Laguerre-Bessel translation operator see [9]:

**Proposition 2.2.** For all $(x,t), (y,s) \in \mathbb{K}$, $f \in C_*(\mathbb{K})$ we have

(1) \[
\mathcal{T}_\alpha^{(x,t)}(f)(y,s) = \mathcal{T}_\alpha^{(x,t)}(f)(x,t).
\]

(2) \[
\mathcal{T}_\alpha^{(x,t)}(\varphi_{\lambda,m})(y,s) = \varphi_{\lambda,m}(x,t)\varphi_{\lambda,m}(y,s).
\]

(3) \[
\int_{\mathbb{K}} \mathcal{T}_\alpha^{(x,t)}(f)(y,s) d\mu_\alpha(y,s) = \int_{\mathbb{K}} f(y,s) d\mu_\alpha(y,s).
\]

(2.8)

(4) for $f \in L^p_\alpha(\mathbb{K})$ with $p \in [1; +\infty]$ \[
\mathcal{T}_\alpha^{(x,t)}(f) \in L^p_\alpha(\hat{\mathbb{K}}). \quad \text{and we have}
\]
\[
\left\| \mathcal{T}_\alpha^{(x,t)}(f) \right\|_{p,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha}.
\]

(2.9)

By using the generalized translation, we define the generalized convolution product of $f, g \in \mathcal{S}_*(\mathbb{K})$ by
\[
(f \ast_\alpha g)(x,t) = \int_{\mathbb{K}} \mathcal{T}_\alpha^{(x,t)}(f)(y,s)g(y,s) d\mu_\alpha(y,s).
\]

This convolution is commutative, associative and its satisfies the following properties Proposition 2.3.
(1) (Young’s inequality) for all \( p, q, r \in [1; +\infty] \) such that: \( \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \) and for all \( f \in L^p_\alpha(\mathbb{K}), g \in L^q_\alpha(\mathbb{K}) \) the function \( f *_\alpha g \) belongs to the space \( L^r_\alpha(\mathbb{K}) \) and we have

\[
\|f *_\alpha g\|_{r,\mu_\alpha} \leq \|f\|_{p,\mu_\alpha} \|g\|_{q,\mu_\alpha}.
\]  

(2) For \( f, g \in L^2_\alpha(\mathbb{K}) \) the function \( f *_\alpha g \) belongs to \( L^2_\alpha(\mathbb{K}) \) if and only if the function \( \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g) \) belongs to \( L^2_\alpha(\mathfrak{H}) \) and in this case we have

\[
\mathcal{F}_\alpha(f *_\alpha g) = \mathcal{F}_\alpha(f)\mathcal{F}_\alpha(g).
\]  

(3) For \( f, g \in L^2_\alpha(\mathbb{K}) \) then we have

\[
\int_{\mathbb{K}} |f *_\alpha g(x,t)|^2 d\mu_\alpha(x,t) = \int_{\mathbb{K}} |\mathcal{F}_\alpha(f)(\lambda,m)|^2 |\mathcal{F}_\alpha(g)(\lambda,m)|^2 d\gamma_\alpha(\lambda,m),
\]  

(2.11) where both integrals are simultaneously finite or infinite.

### 2.4 The Schatten-Von Neumann classes

**Notation:** we denote by

- \( l^p(\mathbb{N}), 1 \leq p \leq \infty \), the set of all infinite sequences of real (or complex) numbers \( u := (u_j)_{j \in \mathbb{N}} \), such that

\[
\|u\|_p := \left( \sum_{j=1}^{\infty} |u_j|^p \right)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty,
\]

\[
\|u\|_\infty := \sup_{j \in \mathbb{N}} |u_j| < \infty, \quad \text{if} \quad p = +\infty.
\]

For \( p = 2 \), we provide this space \( l^2(\mathbb{N}) \) with the scalar product

\[
(u, v)_2 := \sum_{j=1}^{\infty} u_j \overline{v_j}.
\]

- \( B(\mathcal{L}^p_\alpha(\mathbb{K})), 1 \leq p \leq \infty \), the space of bounded operators from \( \mathbb{R}^d_+ \) into itself.

For \( p = 2 \), we define the space \( S_\infty := B(\mathcal{L}^2_\alpha(\mathbb{K})) \), equipped with the norm,

\[
\|A\|_{S_\infty} := \sup_{\|v\|_{\mathcal{L}^2_\alpha(\mathbb{K})} = 1} \|Av\|_{2,\mu_\alpha}.
\]  

(2.12)

**Definition 2.3.**

(1) The singular values \( (s_n(A))_{n \in \mathbb{N}} \) of a compact operator \( A \) in \( B(\mathcal{L}^p_\alpha(\mathbb{R}^d_+)) \) are the eigenvalues of the positive self-adjoint operator \( |A| = \sqrt{A^*A} \).
(2) For \( 1 \leq p < \infty \), the Schatten class \( S_p \) is the space of all compact operators whose singular values lie in \( l^p(\mathbb{N}) \). The space \( S_p \) is equipped with the norm

\[
\|A\|_{S_p} := \left( \sum_{n=1}^{\infty} (s_n(A))^p \right)^{\frac{1}{p}}.
\]

**Remark 2.1.** We note that the space \( S_2 \) is the space of Hilbert-Schmidt operators, and \( S_1 \) is the space of trace class operators.

**Definition 2.4.** The trace of an operator \( A \) in \( S_1 \) is defined by

\[
\text{tr}(A) = \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_{\mu_n},
\]

where \((e_n)_n \) is any orthonormal basis of \( L^2_\alpha(\mathbb{R}^d_+) \).

**Remark 2.2.** If \( A \) is positive, then

\[
\text{tr}(A) = \|A\|_{S_1}.
\]

Moreover, a compact operator \( A \) on the Hilbert space \( L^2_\alpha(\mathbb{R}^d_+) \) is Hilbert-Schmidt, if the positive operator \( A^* A \) is in the space of trace class \( S_1 \). Then

\[
\|A\|_{HS}^2 := \|A\|_{S_2}^2 = \|A^* A\|_{S_1} = \text{tr}(A^* A) = \sum_{n=1}^{\infty} \|Ae_n\|_{\mu_n}^2,
\]

for any orthonormal basis \((v_n)_n \) of \( L^2_\alpha(\mathbb{K}) \).

For more informations about the Schatten-Von Neumann classes one can see [3,15].

### 2.5 Fourier-Wigner transform associated with the Laguerre-Bessel transform

In this section we define and give some results for the Wigner transform associated with the Laguerre-Bessel transform.

**Notation :** we denote by

- \( S(\mathbb{K} \times \hat{\mathbb{K}}) \) the Schwartz space defined in \( \mathbb{K} \times \hat{\mathbb{K}} \) equipped with its usual topology.
- \( L^p_p(\mathbb{K} \times \hat{\mathbb{K}}), 1 \leq p \leq +\infty \) the space of measurable functions on \( \mathbb{K} \times \hat{\mathbb{K}} \)

satisfying

\[
\|f\|_{p,\theta_\alpha} := \begin{cases} \left( \int_{\mathbb{K} \times \hat{\mathbb{K}}} |f((x,t), (\lambda, m))|^p d\theta_\alpha((x,t), (\lambda, m)) \right)^{\frac{1}{p}}, & \text{if } p \in [1, +\infty], \\ \text{ess sup} |f((x,t), (\lambda, m))|, & \text{if } p = +\infty. \end{cases}
\]
where \( \theta_{\alpha} \) is the measure defined on \( \mathbb{K} \times \hat{\mathbb{K}} \) by

\[
d\theta_{\alpha}((x,t), (\lambda,m)) := d\gamma_{\alpha}(\lambda, m) d\mu_{\alpha}(x,t)
\]

for all \((x,t) \in \mathbb{K}, (\lambda, m) \in \hat{\mathbb{K}}\).

**Definition 2.5.** The Wigner transform associated with the Laguerre-Bessel transform is defined on \( S_* \mathbf{K} \times S_* \mathbf{K} \) by

\[
W(f,g)((x,t), (\lambda,m)) := \int_{\mathbb{K}} f(y,s) T_{\alpha}^{(x,t)}(g)(y,s) \varphi_{\lambda,m}(y,s) d\mu_{\alpha}(y,s).
\]

**Remark 2.3.** the transform \( W \) is a bilinear mapping from \( S_* \mathbf{K} \times S_* \mathbf{K} \) into \( S_* \left( \mathbb{K} \times \hat{\mathbb{K}} \right) \) and can be written in the following form

\[
W(f,g)((x,t), (\lambda,m)) = F_{\alpha}(f T_{\alpha}^g)(\lambda)
\]

\[
= (g * f \psi_{\alpha}(\lambda))(x).
\]

We have the following result.

**Proposition 2.4.** Let \( f,g \in L^2_{\alpha}(\mathbb{K}) \) then \( W(f,g) \) is well defined and belongs to \( L^2_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}) \cap L^\infty_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}) \) and we have

\[
\|W(f,g)\|_{2,\theta_{\alpha}} \leq \|f\|_{2,\mu_{\alpha}} \|g\|_{2,\mu_{\alpha}},
\]

and

\[
\|W(f,g)\|_{\infty,\theta_{\alpha}} \leq \|f\|_{2,\mu_{\alpha}} \|g\|_{2,\mu_{\alpha}}.
\]

**Proof.** see [4]

**Remark 2.4.** For \( p, q, r \in [1, +\infty] \) such that \( \frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{r} \) and for \( f \in L^p_{\alpha}(\mathbb{K}), g \in L^q_{\alpha}(\mathbb{K}) \) we define the Wigner transform \( W(f,g) \) by the relation \( (2.18) \), then we have the following result.

**Proposition 2.5.** Let \( p, q, r \in [1, +\infty] \) such that \( \frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r} \) and for \( f \in L^p_{\alpha}(\mathbb{K}), g \in L^q_{\alpha}(\mathbb{K}) \) and \((x,t) \in \mathbb{K}\) then the function

\[
x \mapsto W(f,g)(x,\lambda)
\]

belongs to \( L^r_{\alpha}(\mathbb{K}) \) and we have

\[
\|W(f,g)(\cdot,\lambda)\|_{r,\mu_{\alpha}} \leq \|f\|_{p,\mu_{\alpha}} \|g\|_{q,\mu_{\alpha}}.
\]

**Proof.** By the relation \( (2.18) \) we have

\[
W(f,g)(\cdot,\lambda) = g * f \psi_{\alpha}(\lambda),
\]
by Young’s inequality (2.14) we find that
\[ \|W(f,g)((\cdot , \lambda))\|_{r,\mu_\alpha} \leq \|g\|_{q,\mu_\alpha} \|f\psi_\alpha(\lambda)\|_{p,\mu_\alpha}, \]
the relation (2.5) give the desired result.

3 Localization operators associated with the Laguerre-Bessel-Wigner transform

3.1 Introduction

In this section we will define and give sufficient conditions for the boundedness, compactness and Schatten class properties of localization operators \( \mathcal{L}_{u,v}(\sigma) \) associated with the Laguerre-Bessel-Wigner transform in terms of properties of the symbol \( \sigma \) and the functions \( u \) and \( v \).

Definition 3.1. Let \( u \) and \( v \) be measurable functions on \( \mathbb{K} \), \( \sigma \) be a measurable function on the set \( \mathbb{K} \times \hat{\mathbb{K}} \), we define the localization operator \( \mathcal{L}_{u,v} \) associated with the Laguerre-Bessel-Wigner transform by

\[ \mathcal{L}_{u,v}(f)(y,s) := \int_{\mathbb{K} \times \hat{\mathbb{K}}} \sigma((x,t),(\lambda,m))W(f,u)((x,t),(\lambda,m))\varphi_{\lambda,m}(y,s)T_{\alpha}^{\sigma}(v)(y,s)d\theta_\alpha((x,t),(\lambda,m)). \]

(3.1)

Remark 3.1. In accordance with the different choices of the symbol \( \sigma \) and the different continuities required, we need to impose different conditions on \( u,v \), and then we obtain an operator on \( L^p_\alpha(\mathbb{K}) \) for all \( 1 \leq p \leq +\infty \).

we have the following result.

Proposition 3.1. Let \( f,g \in L^2_\alpha(\mathbb{K}) \) then we have

\[ \langle \mathcal{L}_{u,v}(\sigma)(f) \mid g \rangle_{\mu_\alpha} = \int_{\mathbb{K} \times \hat{\mathbb{K}}} \sigma((x,t),(\lambda,m))W(f,u)((x,t),(\lambda,m))W(g,v)((x,t),(\lambda,m))d\theta_\alpha((x,t),(\lambda,m)). \]

(3.2)

Proof. Let \( f,g \in L^2_\alpha(\mathbb{K}) \) we have

\[ \langle \mathcal{L}_{u,v}(\sigma)(f) \mid g \rangle_{\mu_\alpha} = \int_{\mathbb{K}} \mathcal{L}_{u,v}(f)(y,s)\overline{g(y,s)}d\mu_\alpha(y) \]

by (3.1), Fubini’s theorem we find the result. \( \Box \)

Proposition 3.2. The adjoint of the linear operator

\[ \mathcal{L}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \longrightarrow L^2_\alpha(\mathbb{K}) \]

is the operator

\[ \mathcal{L}^*_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \longrightarrow L^2_\alpha(\mathbb{K}) \]
where
\[ \mathcal{L}_{u,v}^*(\sigma) = \mathcal{L}_{v,u}(\sigma). \] (3.3)

**Proof.** Let \( f, g \in L^2_\alpha(K) \) by the relation (3.2) we have

\[
\langle \mathcal{L}_{u,v}(\sigma)(f) \mid g \rangle_{\mu_\alpha} = \int_{K \times \hat{K}} \sigma((x,t), (\lambda, m)) W(f, u)((x,t), (\lambda, m)) W(g, v)((x,t), (\lambda, m)) d\theta_\alpha((x,t), (\lambda, m)) = (\mathcal{L}_{v,u}(\sigma)(g) \mid f)_{\alpha} = (f \mid \mathcal{L}_{v,u}(\sigma)(g))_{\alpha},
\]

this show that
\[ \mathcal{L}_{u,v}^*(\sigma) = \mathcal{L}_{v,u}(\sigma). \]

\[ \square \]

In the sequel of this section, \( u \) and \( v \) will be any functions in \( L^2_\alpha(K) \) such that
\[ \|u\|_{2,\mu_\alpha} = \|v\|_{2,\mu_\alpha} = 1. \]
We note that this hypothesis is not essential and the result still true up some constant depending on \( \|u\|_{2,\mu_\alpha} \) and \( \|v\|_{2,\mu_\alpha} \).

### 3.2 Boundedness for \( \mathcal{L}_{u,v}(\sigma) \) in \( S_\infty \)

The main purpose of this subsection is to prove that the linear operator
\[ \mathcal{L}_{u,v}(\sigma) : L^2_\alpha(K) \rightarrow L^2_\alpha(K) \]
is bounded for all symbol \( \sigma \in L^p_\alpha(K \times \hat{K}) \) with \( 1 \leq p + \infty \). We consider first the problem for \( \sigma \in L^1_\alpha(K \times \hat{K}) \), next in \( \sigma \in L^\infty_\alpha(K \times \hat{K}) \) and we conclude by using interpolation theory.

**Proposition 3.3.** Let \( \sigma \in L^1_\alpha(K \times \hat{K}) \) then the localization operator \( \mathcal{L}_{u,v}(\sigma) \) is in \( S_\infty \) and we have
\[ \| \mathcal{L}_{u,v}(\sigma) \|_{S_\infty} \leq \| \sigma \|_{1,\theta_\alpha}. \] (3.4)

**Proof.** Let \( f, g \in L^2_\alpha(K) \) by the relation (3.2) we have

\[
| \langle \mathcal{L}_{u,v}(\sigma)(f) \mid g \rangle_{\mu_\alpha} | \leq \int_{K \times \hat{K}} |\sigma((x,t), (\lambda, m))||W(f, u)((x,t), (\lambda, m))||W(g, v)((x,t), (\lambda, m))| d\theta_\alpha((x,t), (\lambda, m)) \\
\leq \|W(f, u)\|_{\infty,\theta_\alpha} \|W(g, v)\|_{\infty,\theta_\alpha} \|\sigma\|_{1,\theta_\alpha},
\]

from the relation (2.24) we get

\[
| \langle \mathcal{L}_{u,v}(\sigma)(f) \mid g \rangle_{\mu_\alpha} | \leq \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha} \|\sigma\|_{1,\theta_\alpha},
\]

by (2.19) we find that
\[ \| \mathcal{L}_{u,v}(\sigma) \|_{S_\infty} \leq \| \sigma \|_{1,\theta_\alpha}. \]

\[ \square \]
Proposition 3.4. Let $\sigma \in L^\infty_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}})$, the localization operators $L_{u,v}(\sigma)$ is in $S_\infty$ and we have

$$\|L_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_\alpha}. \quad (3.5)$$

Proof. Let $f, g \in L^2_{\alpha}(\mathbb{K})$ by the relation (3.2) we have

$$|\langle L_{u,v}(\sigma)(f) | g \rangle_{\mu_\alpha}| \leq \int_{\mathbb{K} \times \hat{\mathbb{K}}} |\sigma((x,t), (\lambda, m))| |W(f, u)((x,t), (\lambda, m))| |W(g, v)((x,t), (\lambda, m))| d\theta_\alpha((x,t), (\lambda, m)),$$

by Hölder’s inequality we find that

$$|\langle L_{u,v}(\sigma)(f) | g \rangle_{\mu_\alpha}| \leq \|\sigma\|_{\infty,\theta_\alpha} \|W(f, u)\|_{2,\theta_\alpha} \|W(g, v)\|_{2,\theta_\alpha},$$

by the relation (2.23) we get

$$|\langle L_{u,v}(\sigma)(f) | g \rangle_{\mu_\alpha}| \leq \|\sigma\|_{\infty,\theta_\alpha} \|f\|_{2,\mu_\alpha} \|g\|_{2,\mu_\alpha},$$

thus

$$\|L_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{\infty,\theta_\alpha}. \quad \square$$

We can now associate a localization operator $L_{u,v}(\sigma)$ to every symbol $\sigma$ in $L^p_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}})$ for all $1 \leq p \leq +\infty$, and prove that $L_{u,v}(\sigma)$ belongs to $S_\infty$.

Theorem 3.1. Let $\sigma \in L^p_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}), 1 \leq p \leq +\infty$ then the localization operator

$$L_{u,v}(\sigma) : L^2_{\alpha}(\mathbb{K}) \longrightarrow L^2_{\alpha}(\mathbb{K})$$

extend to a unique bounded linear operator for all $\sigma \in L^p_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}})$ and we have

$$\|L_{u,v}(\sigma)\|_{S_\infty} \leq \|\sigma\|_{p,\theta_\alpha}. \quad (3.6)$$

Proof. Let $\sigma \in L^p_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}), 1 \leq p \leq +\infty$ and $f \in L^2_{\alpha}(\mathbb{K})$ we consider the following operator

$$T : L^1_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}) \cap L^\infty_{\alpha}(\mathbb{K} \times \hat{\mathbb{K}}) \longrightarrow L^2_{\alpha}(\mathbb{K}),$$

given by

$$T(\sigma) = L_{u,v}(\sigma)(f),$$

then by (3.4) and (3.5) we have

$$\|T(\sigma)\|_{2,\mu_\alpha} \leq \|f\|_{2,\mu_\alpha} \|\sigma\|_{1,\theta_\alpha},$$

and

$$\|T(\sigma)\|_{2,\mu_\alpha} \leq \|f\|_{2,\mu_\alpha} \|\sigma\|_{\infty,\theta_\alpha},$$
by the Riesz–Thorin interpolation Theorem see [18, 22], the operator $T$ may be
uniquely extended to a linear operator on $L^p_α(K \times \hat{K})$ for all $1 \leq p \leq +\infty$ and we have
\[
\|T(\sigma)\|_{2,\mu_α} = \|L_{u,v}(\sigma)(f)\|_{2,\mu_α} \leq \|f\|_{2,\mu_α} \|\sigma\|_{p,θ_α},
\]
for all $f \in L^2_α(K)$ which give the desired result.

### 3.3 $L^p_α$-Boundedness of localization operator $L_{u,v}(\sigma)$

Using Schur’s technique [8] our main purpose of this subsection is to prove that the
linear operator
\[
L_{u,v}(\sigma) : L^p_α(K) \rightarrow L^p_α(\hat{K}),
\]
is bounded for all $1 \leq p \leq +\infty$, we have the following theorem.

**Theorem 3.2.** Let $\sigma \in L^1_1(K \times \hat{K})$ and $u, v \in L^1_1(K) \cap L^\infty_1(K)$ then the localization
operator $L_{u,v}(\sigma)$ extend to a unique bounded linear operator from $L^p_α(K)$ into itself for
all $1 \leq p \leq +\infty$, furthermore we have
\[
\|L_{u,v}(\sigma)\|_{B(L^p_α(K))} \leq \max(\|u\|_{1,\mu_α} \|v\|_{\infty,\mu_α}, \|u\|_{\infty,\mu_α} \|v\|_{1,\mu_α}) \|\sigma\|_{1,θ_α}.
\]

**Proof.** Let $F$ be the function defined on $\mathbb{R}^d_+$ by
\[
F((y, s), (w, z)) = \int_{K \times \hat{K}} \sigma((x, t), (\lambda, m)) \varphi_{\lambda, m}(y, s) \overline{T^{(y,s)}(v)(x, t)} \varphi_{\lambda, m}(w, z) T^{(x,t)}(u)(w, z) d\theta((x, t), (\lambda, m)),
\]
by Fubini’s theorem we find that
\[
L_{u,v}(\sigma)(f)(y, s) = \int_K F((y, s), (w, z)) f(w, z) d\mu_α(w, z),
\]
Furthermore by the relation (2.9) and Fubini’s theorem we find that
\[
\int_K |F((y, s), (w, z))| d\mu_α(y, s) \leq \|u\|_{\infty,\mu_α} \|v\|_{1,\mu_α} \|\sigma\|_{1,θ_α}
\]
(3.7) and
\[
\int_K |F((y, s), (w, z))| d\mu_α(z, w) \leq \|u\|_{1,\mu_α} \|v\|_{\infty,\mu_α} \|\sigma\|_{1,θ_α}
\]
(3.8)
by (3.7), (3.8) and Schur’s lemma [8] we can conclude that the linear operator
\[
L_{u,v}(\sigma) : L^p_α(K) \rightarrow L^p_α(\hat{K}),
\]
is bounded for all $1 \leq p \leq +\infty$ and we have
\[
\|L_{u,v}(\sigma)\|_{B(L^p_α(K))} \leq \max(\|u\|_{1,\mu_α} \|v\|_{\infty,\mu_α}, \|u\|_{\infty,\mu_α} \|v\|_{1,\mu_α}) \|\sigma\|_{1,θ_α}.
\]
3.4 Trace of the localization operators

The main result of this subsection is to prove that the localization operator

\[ L_{u,v}(\sigma) : L^2(\mathbb{K}) \rightarrow L^2(\mathbb{K}) \]

is in the Schatten-Von Neumann class \( S^p \) for all \( 1 \leq p \leq +\infty \), firstly we have the following result.

**Theorem 3.3.** Let \( \sigma \in L^1(\mathbb{K} \times \hat{\mathbb{K}}) \) the localization operator

\[ L_{u,v}(\sigma) : L^2(\mathbb{K}) \rightarrow L^2(\mathbb{K}) \]

is an Hilbert-Schmidt operator in particular it is compact and we have

\[ \| L_{u,v}(\sigma) \|_{HS} \leq 1 + \| \sigma \|_{1,0_\alpha}^2. \]

**Proof.** Let \( (e_k)_k \) be an orthonormal basis of \( L^2(\mathbb{K}) \), by Fubini’s theorem and the relation (3.2) we get

\[ \| L_{u,v}(\sigma) (e_k) \|_{2,\mu_\alpha}^2 = \langle L_{u,v}(\sigma) (e_k) | L_{u,v}(\sigma) (e_k) \rangle_{\mu_\alpha} \]

\[ = \int_{\mathbb{K} \times \hat{\mathbb{K}}} \sigma((x,t),(\lambda,m))W(e_k,u)((x,t),(\lambda,m))W(L_{u,v}(\sigma)(e_k),v)((x,t),(\lambda,m))d\theta_\alpha((x,t),(\lambda,m)), \]

by (2.17) we find that

\[ \| L_{u,v}(\sigma) (e_k) \|_{2,\mu_\alpha}^2 \leq \int_{\mathbb{K} \times \hat{\mathbb{K}}} \| \sigma((x,t),(\lambda,m)) \| \| \varphi_{\lambda,m} T^{(x,t)}(\varphi_{\lambda,m}) \| d\theta_\alpha((x,t),(\lambda,m)), \]

by the relation (3.3) we get

\[ \| L_{u,v}(\sigma)(e_k)T^{(x,t)}(v) \| (\lambda) = \langle e_k | L_{v,u}(\sigma) (T^{(x,t)}(v)\varphi_{\lambda,m}) \rangle_{\mu_\alpha}, \]

So we find that

\[ \| L_{u,v}(\sigma) (e_k) \|_{2,\mu_\alpha}^2 \leq \frac{1}{2} \int_{\mathbb{K} \times \hat{\mathbb{K}}} \| \sigma((x,t),(\lambda,m)) \| \| \varphi_{\lambda,m} T^{(x,t)}(\varphi_{\lambda,m}) \| d\theta_\alpha((x,t),(\lambda,m)) \]

\[ + \frac{1}{2} \int_{\mathbb{K} \times \hat{\mathbb{K}}} \| \sigma((x,t),(\lambda,m)) \| \| \varphi_{\lambda,m} T^{(x,t)}(\varphi_{\lambda,m}) \| ^2 d\theta_\alpha((x,t),(\lambda,m)). \]
by Fubini’s theorem we find that
\[ \|L_{u,v}(\sigma)\|_{HS}^2 \leq \frac{1}{2} \int_{\mathbb{K} \times \hat{\mathbb{K}}} |\sigma((x,t),(\lambda,m))| \left[ \sum_{k=1}^{+\infty} \left| \left( \varphi_{\lambda,m} \mathcal{T}_{\alpha}^{(x,t)}(u) \right) e_k \right|_{\mu_\alpha}^2 \right] + \sum_{k=1}^{+\infty} \left| \left( L_{u,v}(\sigma)(\mathcal{T}_{\alpha}^{(x,t)}(v)\varphi_{\lambda,m}) \right) e_k \right|_{\mu_\alpha}^2 d\mu_\alpha((x,t),(\lambda,m)) \].

By Parseval’s identity, the relations (2.1),(2.9),(3.4) and the fact that \( \|u\|_{2,\mu_\alpha} = \|v\|_{2,\mu_\alpha} = 1 \) we find that
\[ \|L_{u,v}(\sigma)\|_{HS}^2 \leq \frac{1}{2} \|\sigma\|_{1,\mu_\alpha} \left( 1 + \|\sigma\|_{1,\mu_\alpha}^2 \right) \leq \left( 1 + \|\sigma\|_{1,\mu_\alpha}^2 \right)^2 < \infty \]

which proves that \( L_{u,v}(\sigma) \) is an Hilbert-Schmidt operator so compact and we have
\[ \|L_{u,v}(\sigma)\|_{HS} \leq 1 + \|\sigma\|_{1,\mu_\alpha}^2. \]

**In the following we prove that the localization operator**
\[ L_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K}) \]
**is compact for all \( \sigma \in L^p_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \).**

**Proposition 3.5.** Let \( \sigma \in L^p_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \), \( 1 \leq p < +\infty \) then the localization operator
\[ L_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \to L^2_\alpha(\mathbb{K}) \]
**is compact.**

**Proof.** Let \( \sigma \in L^p_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \) with \( 1 \leq p < +\infty \) and let \( (\sigma_n)_n \) be a sequence of functions in \( L^1_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \cap L^\infty_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \) such that \( \sigma_n \to \sigma \) in \( L^p_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \) as \( n \to \infty \) then by the relation (3.6) we have
\[ \|L_{u,v}(\sigma_n) - L_{u,v}(\sigma)\|_{S_{\infty}} \leq \|\sigma_n - \sigma\|_{p,\mu_\alpha}, \]

hence \( L_{u,v}(\sigma_n) \to L_{u,v}(\sigma) \) in \( S_{\infty} \) as \( n \to \infty \) on the other hand by theorem (3.3) we have \( L_{u,v}(\sigma_n) \) is in \( S_2 \) hence compact, it follows that \( L_{u,v}(\sigma) \) is compact.

In the next theorem we obtain a \( L^1_\alpha \)-compactness result for the localization operator \( L_{u,v}(\sigma) \).

**Theorem 3.4.** Let \( \sigma \in L^1_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \), \( u \) and \( v \) in \( L^1_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \cap L^\infty_\alpha(\mathbb{K} \times \hat{\mathbb{K}}) \) then the localization operator
\[ L_{u,v}(\sigma) : L^1_\alpha(\mathbb{K}) \to L^1_\alpha(\mathbb{K}) \]
**is compact.**
Proof. By theorem (3.1) the linear operator
\[ \mathcal{L}_{u,v}(\sigma) : L^1_\alpha(\mathbb{K}) \longrightarrow L^1_\alpha(\mathbb{K}) \]
is well defined, let \( (f_n) \subset L^1_\alpha(\mathbb{K}) \) such that \( f_n \longrightarrow 0 \) weakly in \( L^1_\alpha(\mathbb{K}) \) as \( n \longrightarrow \infty \), it is enough to prove that \( \lim_{n \to +\infty} \|\mathcal{L}(\sigma)(f_n)\|_{1,\mu_\alpha} = 0 \).

By the relation (3.1) we have
\[
\|\mathcal{L}(\sigma)(f_n)\|_{1,\mu_\alpha} \\
\leq \int_{\mathbb{K}} \left| \int_{\mathbb{K}\times\mathbb{K}} |\sigma((x,t),(\lambda,m))||\mathcal{W}(f_n,u)((x,t),(\lambda,m))||\mathcal{T}_{\alpha}^{(x,t)}(v)(y,s)|d\theta_\alpha((x,t),(\lambda,m))|d\mu_\alpha(y,s) \right|.
\]

Using the fact that \( f_n \longrightarrow 0 \) weakly in \( L^1_\alpha(\mathbb{K}) \) as \( n \longrightarrow \infty \), we deduce that
\[
\lim_{n \to +\infty} |\mathcal{W}(f_n,u)((x,t),(\lambda,m))||\mathcal{T}_{\alpha}^{(x,t)}(v)(y,s)| = 0,
\]
for all \((x,t),(\lambda,m) \in \mathbb{K}\). On the other hand as \( f_n \longrightarrow 0 \) weakly in \( L^1_\alpha(\mathbb{K}) \) as \( n \longrightarrow \infty \), there exists a positive constant \( c \) such that \( \|f_n\|_{1,\mu_\alpha} \leq c \), so we find that
\[
|\mathcal{W}(f_n,u)((x,t),(\lambda,m))||\mathcal{T}_{\alpha}^{(x,t)}(v)(y,s)| \leq c|\sigma((x,t),(\lambda,m))||u||_{\infty,\mu_\alpha}|v(y,s)|,
\]
by Fubuni’s theorem we get
\[
\int_{\mathbb{K}} \left| \int_{\mathbb{K}\times\mathbb{K}} |\sigma((x,t),(\lambda,m))||\mathcal{W}(f_n,u)((x,t),(\lambda,m))||\mathcal{T}_{\alpha}^{(x,t)}(v)(y,s)|d\theta_\alpha((x,t),(\lambda,m))|d\mu_\alpha(y) \right| \\
\leq c||\sigma||_{1,\theta_\alpha}||u||_{\infty,\mu_\alpha}||v||_{1,\mu_\alpha} < \infty.
\]

Thus from the relation (3.9),(3.10),(3.11),(3.12) and the Lebesgue dominated convergence theorem we deduce that \( \lim_{n \to +\infty} \|\mathcal{L}(\sigma)(f_n)\|_{1,\mu_\alpha} = 0 \) and the proof is complete.

In the following we show that the localization operator \( \mathcal{L}_{u,v} \) is in the trace class \( S_1 \).

**Theorem 3.5.** Let \( \sigma \in L^1_\alpha(\mathbb{K}\times\mathbb{K}) \) then the localization operator
\[ \mathcal{L}_{u,v}(\sigma) : L^2_\alpha(\mathbb{K}) \longrightarrow L^2_\alpha(\mathbb{K}) \]
is in the trace class operators \( S_1 \) and we have
\[
||\tilde{\sigma}||_{1,\theta_\alpha} \leq ||\mathcal{L}_{u,v}(\sigma)||_{S_1} \leq ||\sigma||_{1,\theta_\alpha},
\]
where \( \tilde{\sigma} \) is given by
\[
\tilde{\sigma}((x,t),(\lambda,m)) = \left\langle \mathcal{L}_{u,v}(\sigma) \left( \varphi_{\lambda,m} \mathcal{T}_{\alpha}^{(x,t)}(u) \right), \varphi_{\lambda,m} \mathcal{T}_{\alpha}^{(x,t)}(v) \right\rangle_{\mu_\alpha}.
\]
Proof. Let \( \sigma \in \mathcal{L}^1(K \times \tilde{K}) \) by the theorem (4) we have \( \mathcal{L}_{u,v}(\sigma) \) is a compact operator, using [22], there exists an orthonormal basis \( \phi_j \) for \( j = 1, 2, \ldots \) for the orthogonal complement of the kernel of the operator \( \mathcal{L}_{u,v}(\sigma) \) consisting of eigenvectors of \( |\mathcal{L}_{u,v}(\sigma)| \) and \( \gamma_j \), \( j = 1, 2, \ldots d \), an orthonormal set in \( \mathcal{L}^2(K) \) such that the localization operators \( \mathcal{L}_{u,v}(\sigma) \) can be diagonalized as

\[
\mathcal{L}_{u,v}(\sigma)(f) = \sum_{j=1}^{+\infty} s_j (f | \phi_j)_{\alpha} h_j,
\]

where \( s_j \) for \( j = 1, 2, \ldots d \), are the positive singular values of \( \mathcal{L}_{u,v}(\sigma) \) corresponding to \( \phi_j \), then we get :

\[
\|\mathcal{L}_{u,v}(\sigma)\|_{S^1} = \sum_{j=1}^{+\infty} s_j = \sum_{j=1}^{+\infty} \langle \mathcal{L}_{u,v}(\sigma) (\phi_j) | h_j \rangle_{\mu_\alpha},
\]

by (2.16) and (3.2) we have

\[
\langle \mathcal{L}_{u,v}(\sigma) (\phi_j) | h_j \rangle_{\mu_\alpha} = \int_{K \times \tilde{K}} |\sigma((x, t), (\lambda, m))| \left| \langle \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(u) \varphi_j \rangle_{\alpha} \right| \left| \langle \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(v) \rangle_{\alpha} \right| d\theta_{\alpha}((x, t), (\lambda, m),
\]

So we find that

\[
\|\mathcal{L}_{u,v}(\sigma)\|_{S^1} \leq \frac{1}{2} \int_{K \times \tilde{K}} |\sigma((x, t), (\lambda, m))| \left[ \sum_{j=1}^{+\infty} \left| \langle \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(u) \varphi_j \rangle_{\alpha} \right|^2 + \right.
\]

\[
\left. \sum_{j=1}^{+\infty} \left| \langle \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(v) \rangle_{\alpha} \right|^2 \right| d\theta_{\alpha}((x, t), (\lambda, m)),
\]

by Parseval’s identity we get

\[
\|\mathcal{L}_{u,v}(\sigma)\|_{S^1} \leq \int_{K \times \tilde{K}} |\sigma(x, \lambda)| \left( \|\varphi_{\lambda,m} T^{(x,t)}_{\alpha}(u)\|_{2,\mu_\alpha}^2 + \|\varphi_{\lambda,m} T^{(x,t)}_{\alpha}(v)\|_{2,\mu_\alpha}^2 \right) d\theta_{\alpha}((x, t), (\lambda, m)),
\]

By the relation (2.1), (2.9) and the fact that \( \|u\|_{2,\mu_\alpha} = \|v\|_{2,\mu_\alpha} = 1 \) we get

\[
\|\mathcal{L}_{u,v}(\sigma)\|_{S^1} \leq \|\sigma\|_{1,\theta_\alpha}.
\]

Now we prove that \( \mathcal{L}_{u,v}(\sigma) \) satisfies the first member of (3.13), it is easy to see that \( \tilde{\sigma} \in \mathcal{L}^1(K \times \tilde{K}) \) and by using (3.14) we find that

\[
|\tilde{\sigma}(x, \lambda)| = |\langle \mathcal{L}_{u,v}(\sigma) \left( \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(u) \right) | \varphi_{\lambda,m} T^{(x,t)}_{\alpha}(v) \rangle_{\mu_\alpha}|.
\]

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\begin{equation*}
\leq \frac{1}{2} \sum_{j=1}^{+\infty} s_j \left( |\langle \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) | \phi_j \rangle_{\alpha} |^2 + | \langle h_j | \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(v) \rangle_{\alpha} |^2 \right),
\end{equation*}

by Fubini’s theorem we get
\begin{equation*}
\int_{K \times \bar{K}} |\tilde{\sigma}(x, \lambda)| d\theta_{\alpha}((x, t), (\lambda, m)) \leq \frac{1}{2} \sum_{j=1}^{+\infty} s_j \int_{K \times \bar{K}} |\langle \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) | \phi_j \rangle_{\alpha} |^2 + | \langle h_j | \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(v) \rangle_{\alpha} |^2 | d\theta_{\alpha}((x, t), (\lambda, m)),
\end{equation*}

by (2.19) and the fact that \( \|u\|_{2, \mu_\alpha} = \|v\|_{2, \mu_\alpha} = 1 \) we get
\begin{equation*}
\int_{K \times \bar{K}} |\tilde{\sigma}(x, t)(\lambda, m))| d\theta_{\alpha}((x, t), (\lambda, m)) \leq \frac{1}{2} \sum_{j=1}^{+\infty} s_j (\|u\|_{2, \mu_\alpha}^2 + \|v\|_{2, \mu_\alpha}^2) = \sum_{j=1}^{+\infty} s_j = \|L_{u, v}(\sigma)\|_{S_1},
\end{equation*}

the proof is complete. \( \square \)

In the following we give a trace formula for the localization operators \( L_{u, v}(\sigma) \).

**Theorem 3.6.** Let \( \sigma \in L_1^X(K \times \bar{K}) \) we have the following trace formula
\begin{equation*}
\text{Tr}(L_{u, v}(\sigma)) = \int_{K \times \bar{K}} \sigma((x, t), (\lambda, m)) \langle \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) | \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(v) \rangle_{\mu_\alpha} d\theta_{\alpha}((x, t), (\lambda, m)).
\end{equation*}

**Proof.** Let \( \{\phi_j, j = 1, 2, \ldots\} \) be an orthonormal basis for \( L_2^X(K) \). From Theorem (3.5), the localization operator \( L_{u, v}(\sigma) \) belongs to \( S_1 \), then by the definition of the trace given by the relation (2.13), Fubini’s theorem and Parseval’s identity, we get
\begin{equation*}
\text{Tr}(L_{u, v}(\sigma)) = \sum_{j=1}^{\infty} \langle L_{u, v}(\sigma) (\phi_j), \phi_j \rangle_{\mu_\alpha}
\end{equation*}

\begin{equation*}
= \sum_{j=1}^{\infty} \int_{K \times \bar{K}} \sigma((x, t), (\lambda, m)) \langle \phi_j, \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) \rangle_{\mu_\alpha} \langle \phi_j, \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(v) \rangle_{\mu_\alpha} d\theta_{\alpha}((x, t), (\lambda, m))
\end{equation*}

\begin{equation*}
= \int_{K \times \bar{K}} \sigma((x, t), (\lambda, m)) \sum_{j=1}^{\infty} \langle \phi_j, \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) \rangle_{\mu_\alpha} \langle \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(v) | \varphi_{\lambda, m} T_{\alpha}^{(x,t)}(u) \rangle_{\mu_\alpha} d\theta_{\alpha}((x, t), (\lambda, m))
\end{equation*}

and the proof is complete. \( \square \)

**Corollary 3.1.** If \( u = v \) and if \( \sigma \) is a real valued, and nonnegative function in \( L_1^X(K \times \bar{K}) \) then the localization operator
\begin{equation*}
L_u(\sigma) : L_2^X(K) \rightarrow L_2^X(K)
\end{equation*}
is a positive operator and by (2.17) and (3.15) we have

\[ \|L_u(\sigma)\|_{S_1} = \int_{\mathbb{K} \times \hat{\mathbb{K}}} \sigma((x,t),(\lambda,m))|\varphi(\lambda,m)T_{\alpha}^{(x,t)}(u)|^2_{L_{\alpha}^2} d\theta_{\alpha}((x,t),(\lambda,m)), \]

here \( L_u \) denote for simply the operator \( L_{u,u} \). In the following we give the main result of this section.

**Corollary 3.2.** Let \( \sigma \) in \( L_p^p(\mathbb{K} \times \hat{\mathbb{K}}), 1 \leq p \leq +\infty \) then, the localization operator

\[ L_u(\sigma) : L^2_\alpha(\mathbb{K}) \rightarrow L^2_\alpha(\mathbb{K}) \]

is in \( S^p \) and we have

\[ \|L_{u,v}(\sigma)\|_{S^p} \leq \|\sigma\|_{L^p,\alpha}. \]

**Proof.** The result follows from (3.5) and (3.13) and by interpolation theory see [22]. \( \square \)

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**References**


