Optimal bounds for complete elliptic integral from the quasi-arithmetic mean and Seiffert-like mean

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Research Article

Keywords: Quasi-arithmetic mean, Seiffert-like mean, Complete elliptic integral, inequality

Posted Date: April 12th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2795274/v1

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Additional Declarations: No competing interests reported.
Optimal bounds for complete elliptic integral from the quasi-arithmetic mean and Seiffert-like mean

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Abstract.
We give the new double inequalities for the quasi-arithmetic mean and Seiffert-like mean. As applications, two new optimal bounds are presented for the complete elliptic integral of the second kind. And numerical comparisons illustrate that some bounds of our results are better than before.

AMS(2020) subject Classification: 26E60, 33C75, 26D20.

Keywords: Quasi-arithmetic mean; Seiffert-like mean; Complete elliptic integral; inequality

1 Introduction

For \(a, b > 0\), the harmonic mean \(H(a,b)\), arithmetic mean \(A(a,b)\), geometric mean \(G(a,b)\) and Logarithmic mean \(L(a,b)\) are respectively defined by (see [5])

\[
H(a,b) = \frac{2ab}{a+b}, \quad A(a,b) = \frac{a + b}{2}, \quad G(a,b) = \sqrt{ab}, \quad L(a,b) = \frac{\log a - \log b}{a - b}.
\]

The generalized Heronian mean \(H_w(a,b)\) defined as (see [3])

\[
H_w(a,b) = \left\{ \begin{array}{ll}
\frac{a + w\sqrt{ab} + b}{w + 2}, & 0 \leq w < \infty \\
\sqrt{ab}, & w = \infty.
\end{array} \right.
\]

In particular, \(H_4\) is denoted by \(\tilde{H}(a,b)\), and \(H_1\) is denoted by \(He(a,b)\). One can see that

\[
He(a,b) = 2A(a,b)/3 + G(a,b)/3, \quad \tilde{H}(a,b) = A(a,b)/3 + 2G(a,b)/3.
\]
Zhao et al. [8] defined the quasi-arithmetic mean $E(a, b)$ as

$$E(a, b) = \begin{cases} 
4a \left[ \varepsilon \left( \sqrt{1 - (b/a)} \right) \right]^2 / \pi^2, & a \geq b \\
4b \left[ \varepsilon \left( \sqrt{1 - (a/b)} \right) \right]^2 / \pi^2, & a < b
\end{cases}$$

where $a, b > 0$ and $\varepsilon$ is the complete elliptic integral of the second kind. This mean $E(a, b)$ has attracted the attention of several researchers (e.g., [4, 8]). From $H(a, b) < E(a, b) < A(a, b)$, Wang et al. [7] proved that the double inequalities

$$\lambda_1 A(a, b) + (1 - \lambda_1) H(a, b) < E(a, b) < \mu_1 A(a, b) + (1 - \mu_1) H(a, b)$$

hold for $a, b > 0$ with $a \neq b$ iff $\lambda_1 \leq 8/\pi^2$, $\mu_1 \geq 7/8$. Letting $a = 1, b = 1 - r^2$, they obtained that

$$L_1 < \varepsilon(r) < S_1, r \in (0, 1)$$

where

$$L_1 = \pi / 2 \left[ \lambda_1 \frac{1 + r^2}{2} + (1 - \lambda_1) \frac{2r^2}{1 + r^2} \right], \quad S_1 = \pi / 2 \left[ \mu_1 \frac{1 + r^2}{2} + (1 - \mu_1) \frac{2r^2}{1 + r^2} \right],$$

and $\lambda_1 = 8/\pi^2$, $\mu_1 = 7/8$, $r' = \sqrt{1 - r^2}$.

Zhang et al. (see [6, 9, 10]) introduced the Seiffert-like mean $V(a, b)$ as follows

$$V(a, b) = \frac{\pi H(a, b)}{2\varepsilon(\frac{a-b}{a+b})} = \frac{\pi H(a, b)}{2\varepsilon(\frac{1 - G^2(a,b)}{A^2(a,b)})}.$$ 

From $H(a, b) < V(a, b) < A(a, b)$, they proved a double inequality for the reciprocal of $V(a, b)$

$$\frac{\lambda_2}{H(a, b)} + \frac{1 - \lambda_2}{A(a, b)} < \frac{1}{V(a, b)} < \frac{\mu_2}{H(a, b)} + \frac{1 - \mu_2}{A(a, b)}$$

holds for all $a, b > 0$ with $a \neq b$ iff $\lambda_2 \leq 2/\pi$, $\mu_2 \geq 3/4$. Putting $a = 1 + r, b = 1 - r$, they obtained

$$L_3 < \varepsilon(r) < S_3, r \in (0, 1)$$

where

$$L_3 = 1 + \left( \frac{\pi}{2} - 1 \right)r^2, \quad S_3 = \frac{\pi}{2} \cdot \frac{3 + r^2}{4}.$$ 

In fact, one can see that $L_3 < L_1 < \varepsilon(r) < S_1 < S_3$, see the last section.
A natural question is whether some better bounds of \( \varepsilon \) can be found? Recently, Wang et al. [7] proved that \( G(a, b) < L(a, b) < He(a, b) \) for all \( a, b > 0 \) with \( a \neq b \). Then it is clear that

\[
H(a, b) < V(a, b) < He(a, b) < A(a, b), \tag{5}
\]

\[
H(a, b) < G(a, b) < \tilde{H}(a, b) < E(a, b) < A(a, b), \tag{6}
\]

for all \( a, b > 0 \) with \( a \neq b \).

In this paper, we will present the new and better double inequalities of \( E(a, b) \) and \( V(a, b) \) with the help of (5) and (6). As applications, we obtain two new optimal bounds of \( \varepsilon(r) \). And numerical comparisons illustrate that some bounds of our results are better than (2) and (4).

2 Preliminaries

Let \( r \in (0, 1) \), the complete elliptic integral of the first kind \( \kappa(r) \) and the second kind \( \varepsilon(r) \) in the Lergedre’s expression form are respectively defined by (see [2])

\[
\kappa(r) = \int_{0}^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 \theta}} \, d\theta, \quad \kappa(0^+) = \frac{\pi}{2}, \quad \kappa(1^-) = \infty,
\]

\[
\varepsilon(r) = \int_{0}^{\pi/2} \sqrt{1 - r^2 \sin^2 \theta} \, d\theta, \quad \varepsilon(0^+) = \frac{\pi}{2}, \quad \varepsilon(1^-) = 1.
\]

For convenience, \( \varepsilon \) and \( \kappa \) are sometimes used instead of \( \varepsilon(r) \) and \( \kappa(r) \).

It follows from [8] that the function \( r \mapsto \varepsilon(r) \) is strictly decreasing from \( (0, 1) \) onto \( (1, \pi/2) \), and it satisfies the formulas

\[
\varepsilon \left( \frac{2\sqrt{r}}{1 + r} \right) = \frac{2\varepsilon - r^2 \kappa}{1 + r}, \quad \frac{d\varepsilon}{dr} = \frac{\varepsilon - \kappa}{r},
\]

where \( r' = \sqrt{1 - r^2} \).

**Lemma 1** [7, Lemma 3.1] The function \( r \mapsto \frac{4 (2\varepsilon - r^2 \kappa)^2 / \pi^2 - (1 - r^2)}{r^2} \) is strictly increasing from \( (0, 1) \) onto \( (3/2, 16/\pi^2) \).

**Lemma 2** [9, Lemma 2.2] The function \( r \mapsto (\kappa - \varepsilon)/r^2 \) is strictly increasing from \( (0, 1) \) onto \( (\pi/4, \infty) \).
Lemma 3 [1, Theorem 1.25] For $-\infty < a < b < \infty$, let $f, g : [a, b] \to R$ be continuous on $[a, b]$, and differentiable on $(a, b)$, let $g'(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$ 

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

3 The new double inequalities of $E(a, b)$ and $V(a, b)$

The following result improves the double inequalities (1).

Proposition 1 Let $\alpha_1, \beta_1 \in (0, 1)$, then the double inequality

$$\alpha_1 A(a, b) + (1 - \alpha_1)\tilde{H}(a, b) < E(a, b) < \beta_1 A(a, b) + (1 - \beta_1)\tilde{H}(a, b),$$

holds for all $a, b > 0$, with $a \neq b$ iff $\alpha_1 \leq 5/8$ and $\beta_1 \geq 12/\pi^2 - 1/2$.

Proof. According $A(a, b) > \tilde{H}(a, b)$, so it is clear that the above double inequality is equivalent to

$$\alpha_1 < \frac{E(a, b) - \tilde{H}(a, b)}{A(a, b) - \tilde{H}(a, b)} < \beta_1$$

holding for all $a, b > 0$, with $a \neq b$ iff $\alpha_1 \leq 5/8$ and $\beta_1 \geq 12/\pi^2 - 1/2$.

Since $E(a, b)$, $A(a, b)$ and $\tilde{H}(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a = (1 + r^2) > b$, $b = (1 - r^2)$ where $r \in (0, 1)$. Then simple computations lead to

$$E(a, b) = \frac{4(1 + r)^2}{\pi^2} \cdot \left[ \varepsilon \left( \frac{2\sqrt{r}}{1 + r} \right) \right]^2 = \frac{\pi^2}{4} \left( 2\varepsilon - r^2 \kappa \right)^2,$$

$$\tilde{H}(a, b) = 1 - \frac{1}{3} r^2.$$ 

Consequently,

$$\frac{E(a, b) - \tilde{H}(a, b)}{A(a, b) - \tilde{H}(a, b)} = \frac{4 \left( 2\varepsilon - r^2 \kappa \right)^2 / \pi^2 - (1 - r^2/3)}{(1 + r^2) - (1 - r^2/3)}$$

$$= \frac{3}{4} \cdot \frac{4 \left( 2\varepsilon - r^2 \kappa \right)^2 / \pi^2 - (1 - r^2)}{r^2} - \frac{1}{2} = f(r).$$
By Lemma 1, \( f(r) \) is increasing from \((0, 1)\) onto \((5/8, 12/\pi^2 - 1/2)\). Therefore the result directly follows. \(\square\)

The following result improves the double inequalities (3).

**Proposition 2** The double inequality

\[
\frac{\alpha_2}{H(a, b)} + \frac{1 - \alpha_2}{He(a, b)} < \frac{1}{V(a, b)} < \frac{\beta_2}{H(a, b)} + \frac{1 - \beta_2}{He(a, b)}
\]

holds for all \(a, b > 0\) with \(a \neq b\) iff \(\alpha_2 \leq 2/\pi\) and \(\beta_2 \geq 7/10\).

**Proof.** From the fact \(He(a, b) > H(a, b)\) for all \(a, b > 0\) with \(a \neq b\), one can see that Proposition 2 is equivalent to

\[
\alpha_2 < \frac{1}{V(a, b)} - \frac{1}{He(a, b)} < \beta_2
\]

holding for all \(a, b > 0\) with \(a \neq b\) iff \(\alpha_2 \leq 2/\pi\) and \(\beta_2 \geq 7/10\).

Since \(H(a, b)\) and \(He(a, b)\) are symmetric and homogeneous of degree 1, without loss of generality, we assume that \(a = 1 + r, b = 1 - r\) where \(r \in (0, 1)\). Then

\[
\begin{align*}
\frac{1}{V(a, b)} &= \frac{2\varepsilon}{\pi r^2 A(a, b)} = \frac{2\varepsilon}{\pi r^2}, \\
\frac{1}{H(a, b)} &= \frac{2ab}{a + b} = \frac{1}{r^2}, \\
\frac{1}{He(a, b)} &= \frac{3}{a + \sqrt{ab} + b} = \frac{3}{2 + r'}.
\end{align*}
\]

Thus

\[
\frac{1}{V(a, b)} - \frac{1}{He(a, b)} = \frac{2\varepsilon/\pi r^2 - 3/(2 + r')}{1/r^2 - 3/(2 + r')} = \frac{2\varepsilon/\pi - 3r^2/(2 + r')}{1 - 3r^2/(2 + r')} = G(r) = \frac{G_1(r)}{G_2(r)}.
\]

Therefore

\[
\begin{align*}
\frac{G_1'(r)}{G_2'(r)} &= \frac{2(\varepsilon - \kappa)/(\pi r) + 3r(4 + r')/(2 + r')^2}{3r(4 + r')/(2 + r')^2} \\
&= \frac{2(\varepsilon - \kappa)}{\pi r} \cdot \frac{(2 + r')^2}{3r(4 + r')} + 1 \\
&= 1 - \frac{2}{3\pi} \cdot \frac{\kappa - \varepsilon}{r^2} \cdot \frac{(2 + r')^2}{4 + r'}.
\end{align*}
\]
One can see that \( r \mapsto \frac{(2 + r)^2}{4 + r} \) is strictly increasing from \((0, 1)\) onto \((1, 9/5)\). It follows from Lemma 2 that \( G_1'(r)/G_2'(r) \) is strictly decreasing on \((0, 1)\). Thus by Lemma 3 we see that \( G(r) \) is strictly decreasing on \((0, 1)\), and by L’Hôpital’s Rule,

\[
G(0^+) = \lim_{r \to 0^+} \frac{G_1'(r)}{G_2'(r)} = \frac{7}{10}, \quad G(1^-) = \frac{G_1(1)}{G_2(1)} = \frac{2}{\pi}.
\]

\[\square\]

4 The optimal bounds of \( \varepsilon \)

The following two new bounds of \( \varepsilon \) can be derived by assigning some special values to \( a \) and \( b \) in Proposition 1 and Proposition 2.

**Theorem 1** For all \( r \in (0, 1) \), we have

\[
L_2 < \varepsilon(r) < S_2,
\]

where

\[
L_2 = \frac{\pi}{2} \sqrt{\alpha_1 \cdot \frac{1 + r'^2}{2} + (1 - \alpha_1)(\frac{1 + r'^2}{6} + \frac{2r'}{3})}, \quad S_2 = \frac{\pi}{2} \sqrt{\beta_1 \cdot \frac{1 + r'^2}{2} + (1 - \beta_1)(\frac{1 + r'^2}{6} + \frac{2r'}{3})},
\]

\(\alpha_1 = 5/8\), and \(\beta_1 = 12/\pi^2 - 1/2\).

**Proof.** Letting \( a = 1 \) and \( b = r'^2 \), one has

\[
E(1, r'^2) = \frac{4}{\pi^2} [\varepsilon(r)]^2, \quad A(1, r'^2) = \frac{1 + r'^2}{2}, \quad \tilde{H}(1, r'^2) = \frac{1}{3} A(1, r'^2) + \frac{2}{3} G(1, r'^2) = \frac{1 + r'^2}{6} + \frac{2r'}{3}.
\]

Substituting the above results into Proposition 2, we obtain (7).

\[\square\]

**Theorem 2** For all \( r \in (0, 1) \), we have

\[
L_4 < \varepsilon(r) < S_4,
\]

where

\[
L_4 = \alpha_2 \cdot \frac{\pi}{2} + (1 - \alpha_2) \cdot \frac{3\pi r'^2}{4 + 2r'}, \quad S_4 = \beta_2 \cdot \frac{\pi}{2} + (1 - \beta_2) \cdot \frac{3\pi r'^2}{4 + 2r'},
\]

\(\alpha_2 = 2/\pi\), and \(\beta_2 = 7/10\).
Proof. Putting \( a = 1 + r, b = 1 - r \), we have

\[
\begin{align*}
\frac{1}{V(1+r, 1-r)} &= \frac{2\varepsilon}{\pi r'^2}, \\
\frac{1}{H(1+r, 1-r)} &= \frac{1}{r'^2}, \\
\frac{1}{He(1+r, 1-r)} &= \frac{3}{2 + r'^2}.
\end{align*}
\]

Substituting the above results into Proposition 2, we obtain (8). \qed

5 Numerical comparisons

In this section, we will show that some new bounds (7) and (8) of \( \varepsilon(r) \) are better than the bounds (2) and (4), receptively.

By Figures 1-2 and Tables 1-2, one can see that

\( L_1 < L_2 < \varepsilon(r) < S_1 < S_2 \).

By Figures 3-6 and Tables 3-4, one can see that

\( L_3 < L_4 < \varepsilon(r) < S_4 < S_3 \).

Figure 1: \( L_1 < \varepsilon < S_1 \)

Figure 2: \( L_2 < \varepsilon < S_2 \)
### Table 1: The calculation of $L_1, L_2$ and $\varepsilon$

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<th>$L_2$</th>
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### Table 2: The calculation of $S_1, S_2$ and $\varepsilon$

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Table 3: The calculation of $L_3$, $L_4$ and $\varepsilon$

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Table 4: The calculation of $S_3, S_4$ and $\varepsilon$

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Declarations

Competing interests

The authors declare no competing interests.

Authors’ contributions

Na Li and Yong-Guo Shi wrote the main manuscript, and both authors reviewed the manuscript.

Funding

This work is supported by NSF of Sichuan Province (2023NSFSC0065).

Availability of data and materials

Not applicable.

References


