Generalized Hukuhara-Clarke Derivative of Interval-valued Functions and its Properties

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Abstract This paper is devoted to the study of *gH-Clarke derivative* for interval-valued functions. To develop the properties of gH-Clarke derivative, the concepts of limit superior, limit inferior, and sublinear interval-valued functions are studied in the sequel. It is proved that the upper gH-Clarke derivative of a gH-Lipschitz continuous *interval- valued function* (IVF) always exists. Further, it is found that for a convex and gH-Lipschitz IVF, the upper gH-Clarke derivative coincides with the gH-directional derivative. It is observed that the upper gH-Clarke derivative is a sublinear IVF. Several numerical examples are provided to support the study.

Keywords Interval-valued functions · Upper gH-Clarke derivative · Sublinear IVF · gH-Lipschitz function.

1 Introduction

Clarke derivative [9] is applied in nonsmooth analysis where a function do not have a unique linear approximation at a point. Advances of nonsmooth analysis [6,29] show the essential need of this derivative to handle nondifferentiable functions, especially in the absence of convexity. The topics of optimization [21], control theory [21], variational

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method [1], generalized convexity [28], etc. are wide application areas of Clarke derivative.

The analysis of interval-valued functions (IVFs) enables one to effectively deal with the errors/uncertainties that appear while modeling practical problems. Thus, in this paper, we aim to develop Clarke derivative for IVFs.

As the topic of this study is Clarke derivatives for IVFs, we describe below the literature on calculus of Interval-Valued Functions (IVFs). It is to be noted that there is a relatively large joint intersection of the literature survey of this paper with the recent paper by Ghosh et al. [14]. However, the proposed work in this paper is completely different than that in [14]. In [14], the properties of IVFs that are differentiable in the sense of gH-Gâteaux and gH-Fréchet derivatives have been studied. On the other hand, we attempt to derive the properties upper gH-Clarke differentiable IVFs in this article.

1.1 Literature Survey

In the study of interval analysis, in addition to interval arithmetic, ordering of intervals and the calculus of IVFs play key roles. As a linear ordering of intervals is not found so far, the development of optimization theory with interval-valued function is not a trivial extension of the conventional optimization theory. Most often [4,14,32,37], IOPs have been analyzed with respect to a partial ordering [20]. Some researchers [3,11] used ordering relations of intervals based on the parametric comparison of intervals. In [7], an ordering relation of intervals is defined by a bijective correspondence between intervals and the points in $\mathbb{R}^2$. However, these ordering relations [3,11,7] of intervals can be derived from the relations described in [20]. Sengupta et al. [30] proposed an acceptability function for intervals, just like a fuzzy membership function. Recently, Ghosh et al. [15] investigated variable ordering relations for intervals and used them in IOPs.

To observe the properties of an IVF, calculus plays an essential role. Initially, in order to develop the calculus of IVFs, Hukuhara [19] introduced the concept of differentiability (H-differentiability) of IVFs with the help of the H-difference. However, H-differentiability is restrictive [4]. For instance, for a given interval $\mathbf{I}$ and a differentiable function $h : \mathbb{R} \to \mathbb{R}$ with with $h'(x) < 0$, the IVF $\mathbf{H}(x) = \mathbf{I} \odot h(x)$ is not H-differentiable [2]. To remove the deficiencies of H-differentiability, Bede and Gal [2] defined strongly generalized derivative (G-derivative) for IVFs and derived a Newton-Leibnitz-type formula. In order to formulate the mean-value theorem for IVFs, Markov [25] introduced a new concept of difference of intervals and defined differentiability of IVFs by this difference. In [32], Stefanini and Bede defined the generalized Hukuhara differentiability (gH-differentiability) of IVF by using the concept of generalized Hukuhara difference. Some fracional calculus for IVFs is reported in [24]. In defining the calculus of IVFs, the concepts of gH-partial derivative, gH-gradient, and gH-differentiability for IVFs have been developed in [4,10,25,32,34].

To derive a Karush-Kuhan-Tucker (KKT) condition for interval optimization problems (IOPs), Guo et al. [17] defined gH-symmetric derivative for IVFs. Ghosh [12] analyzed the notion of gH-differentiability of multi-variable IVFs to propose a Newton-type method for IOPs. The concept of second order differentiablity of IVFs is used in [35] to find existence-uniqueness results on solving interval differential equations. Lupulescu [23] defined delta generalized Hukuhara differentiability by using gH-difference and Riemann delta integrability for IVFss. Chalco et al. [5] introduced the concept of π-derivative for IVFs that generalizes Hukuhara derivative and G-derivative, and it is proved that this derivative is equivalent to gH-derivative. In [33], Stefanini and Bede defined level-wise generalized Hukuhara differentiability for IVFs and generalized fuzzy differentiability with LU-parametric representation for fuzzy-valued functions. Kalani et al. [22] analyzed the concept of interval-valued fuzzy derivative for perfect and semi-perfect interval-valued fuzzy mappings to derive a method for solving interval-valued fuzzy differential equations. Recently, Ghosh et al. [14] have provided the idea of gH-directional derivative, gH-Gâteaux derivative, and gH-Fréchet derivative of IVFs to derive the optimality conditions for IOPs.

1.2 Motivation and Contribution

From the literature on the analysis of IVFs, one can notice that the study of traditional generalized derivative (Clarke derivative) for IVFs have not been developed so far. However, the basic properties of generalized derivatives might be beneficial for characterizing and capturing the optimal solutions of IOPs with nonsmooth IVFs. To define and find properties of Clarke derivative of IVFs, we need to establish the notions of limit superior and sublinearity for IVFs. In this article, after illustrating the concept of limit superior, limit inferior, and sublinearity of IVFs, we define upper and lower gH-Clarke derivatives of IVFs. Although both of the upper and lower gH-Clarke derivatives of IVFs are defined in this article, only the properties of the upper gH-Clarke derivative are studied since the results for the lower derivative can be used analogously. It is shown that if an IVF is upper gH-Clarke differentiable at a point, then its derivative is a sublinear IVF. We further prove that the upper gH-Clarke derivative exists at any point if the IVF is convex gH-Lipschitz and the derivative is equal to gH-directional derivative.

1.3 Delineation

The rest of the article is demonstrated as this sequence. The next section covers some basic terminology and notions of convex analysis and interval analysis, followed by the convexity and calculus of IVFs that are required in this paper. Also, a few properties of intervals, the gH-directional of an IVF is discussed in Section 2. The concepts of limit superior, sublinear IVF and their properties are given in Section 3. In the same section, we define upper gH-Clarke derivative, lower gH-Clarke derivative of IVFs, and prove that a gH-Lipschitz IVF is always upper gH-Clarke differentiable. Also, it is found that upper gH-Clarke derivative coincides with gH-directional derivative for a convex gH-Lipschitz continuous IVF. Further, the sublinearity of the upper gH-Clarke derivative is shown in the same section.

2 Preliminaries and Terminology

This section is devoted to some basic notions on intervals and the convexity and calculus of IVFs. We use the following notations in the entire paper.

- $\mathcal{X}$ denotes a real normed linear space with the norm $\|\cdot\|$
- $\mathcal{S}$ represents a nonempty subset of $\mathcal{X}$
- $\mathcal{B}(\bar{x}, \delta)$ denotes the open ball centered at $\bar{x} \in \mathcal{X}$ with radius δ
- $\overline{\mathcal{B}}(\bar{x}, \delta)$ denotes $\mathcal{B}(\bar{x}, \delta) \setminus \{\bar{x}\}$
- $\mathbb{R}$ represents the set of real numbers
- $\mathbb{R}_+$ is the set of nonnegative real numbers
- $I(\mathbb{R})$ represents the set of compact intervals

2.1 Arithmetic of Intervals and their Dominance Relation

Throughout the article, bold letters $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \ldots$, are used for denoting the elements of $I(\mathbb{R})$. An element $\mathbf{X}$ of $I(\mathbb{R})$ is presented by the corresponding small letter: $\mathbf{X} = [\underline{x}, \overline{x}]$.

Let $\mathbf{X}, \mathbf{Y} \in I(\mathbb{R})$ and $\delta \in \mathbb{R}$. Moore's [26,27] interval addition, subtraction, product, division and scalar multiplication are denoted by $\mathbf{X} \oplus \mathbf{Y}$, $\mathbf{X} \ominus \mathbf{Y}$, $\mathbf{X} \odot \mathbf{Y}$, $\mathbf{X} \oslash \mathbf{Y}$, and $\delta \odot \mathbf{X}$, respectively. In defining $\mathbf{X} \oslash \mathbf{Y}$, it is assumed that $0 \notin \mathbf{Y}$. Since $\mathbf{X} \ominus \mathbf{X} \neq \mathbf{0}$ for any nondegenerate interval $\mathbf{X}$, we use the following concept of difference of intervals in this article.

Definition 1 (*gH-difference* [31]). The gH-difference of $\mathbf{X}, \mathbf{Y} \in I(\mathbb{R})$ is an interval $\mathbf{Z}$ such that

$$\mathbf{X} = \mathbf{Y} \oplus \mathbf{Z} \quad \text{or} \quad \mathbf{Y} = \mathbf{X} \ominus \mathbf{Z}.$$

We denote then $\mathbf{Z} = \mathbf{X} \ominus_{gH} \mathbf{Y}$. For $\mathbf{X} = [\underline{x},\ \overline{x}]$ and $\mathbf{Y} = \left[\underline{y},\ \overline{y}\right]$,

$$\mathbf{X} \ominus_{gH} \mathbf{Y} = \left[\min\{\underline{x} - \underline{y}, \overline{x} - \overline{y}\},\ \max\{\underline{x} - \underline{y}, \overline{x} - \overline{y}\}\right] \text{ and } \mathbf{X} \ominus_{gH} \mathbf{X} = \mathbf{0}.$$

In the following, we provide a domination relation on intervals that is used throughout the paper. We remark that *domination* in the following definition is based on a *minimization* type optimization problems: the *smaller value* the *better*.

Definition 2 (*Dominance of intervals* [36]). Consider two intervals $\mathbf{X} = [\underline{x}, \overline{x}]$ and $\mathbf{Y} = [\underline{y}, \overline{y}]$ in $I(\mathbb{R})$.

(i) $\mathbf{Y}$ is said to be *dominated by* $\mathbf{X}$ if $\underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$, and then we write $\mathbf{X} \preceq \mathbf{Y}$;
(ii) $\mathbf{Y}$ is called *strictly dominated by* $\mathbf{X}$ if either '$\underline{x} \leq \underline{y}$ and $\overline{x} < \overline{y}$' or '$\underline{x} < \underline{y}$ and $\overline{x} \leq \overline{y}$', and then we write $\mathbf{X} \prec \mathbf{Y}$;
(iii) if $\mathbf{Y}$ is not dominated by $\mathbf{X}$, then we write $\mathbf{X} \npreceq \mathbf{Y}$; if $\mathbf{Y}$ is not strictly dominated by $\mathbf{X}$, then we write $\mathbf{X} \nprec \mathbf{Y}$;
(iv) if either $\mathbf{X}$ is dominated by $\mathbf{Y}$ or $\mathbf{Y}$ is dominated by $\mathbf{X}$, then it will be said that $\mathbf{X}$ and $\mathbf{Y}$ are *comparable*,
(v) if $\mathbf{X} \npreceq \mathbf{Y}$ and $\mathbf{Y} \npreceq \mathbf{X}$, then it will be said that *none of* $\mathbf{X}$ *and* $\mathbf{Y}$ *dominates the other*, or $\mathbf{X}$ and $\mathbf{Y}$ are *not comparable*.

Notice that if $\mathbf{Y}$ is strictly dominated by $\mathbf{X}$, then $\mathbf{Y}$ is dominated by $\mathbf{X}$. Moreover, if $\mathbf{Y}$ is not dominated by $\mathbf{X}$, then $\mathbf{Y}$ is not strictly dominated by $\mathbf{X}$.

2.2 Few Results of Intervals

This subsection covered by some results of compact intervals that are used later in the paper. In the rest of the paper, by the norm of an interval, we refer to the following definition.

Definition 3 (*Norm on* $I(\mathbb{R})$ [26]). Let $\mathbf{Y} = [\underline{y}, \overline{y}]$. A function $\|\cdot\|_{I(\mathbb{R})} : I(\mathbb{R}) \to \mathbb{R}_+$, defined by

$$\|\mathbf{Y}\|_{I(\mathbb{R})} = \left\| [\underline{y}, \overline{y}] \right\|_{I(\mathbb{R})} = \max\{|\underline{y}|, |\overline{y}|\},$$

is a *norm* $I(\mathbb{R})$.

Lemma 1 *Let* $\boldsymbol{Z} \in I(\mathbb{R})$. *Then, for all* $a,\ b \in \mathbb{R}$

(i) $\boldsymbol{Z} \succeq \boldsymbol{0} \implies |a+b| \odot \boldsymbol{Z} \preceq |a| \odot \boldsymbol{Z} \oplus |b| \odot \boldsymbol{Z}$ *and*
(ii) $\boldsymbol{Z} \preceq \boldsymbol{0} \implies |a+b| \odot \boldsymbol{Z} \succeq |a| \odot \boldsymbol{Z} \oplus |b| \odot \boldsymbol{Z}$.

Proof See Appendix A.

Lemma 2 *For all* $\boldsymbol{A},\ \boldsymbol{B},\ \boldsymbol{C},\ \boldsymbol{D} \in I(\mathbb{R})$,

(i) $(\boldsymbol{A} \ominus_{gH} \boldsymbol{C}) \oplus (\boldsymbol{C} \ominus_{gH} \boldsymbol{B}) \not\prec \boldsymbol{A} \ominus_{gH} \boldsymbol{B}$,
(ii) $\boldsymbol{B} \preceq \boldsymbol{A} \oplus [L, L]$, *where* $L = \|\boldsymbol{B} \ominus_{gH} \boldsymbol{A}\|_{I(\mathbb{R})}$, *and*
(iii) $\|(\boldsymbol{A} \ominus_{gH} \boldsymbol{B}) \ominus_{gH} (\boldsymbol{C} \ominus_{gH} \boldsymbol{D})\|_{I(\mathbb{R})} \leq \|\boldsymbol{A} \ominus_{gH} \boldsymbol{C}\|_{I(\mathbb{R})} \oplus \|\boldsymbol{B} \ominus_{gH} \boldsymbol{D}\|_{I(\mathbb{R})}$.

Proof See Appendix B.

Remark 1 The following two points are noticeable.

(i) For two elements $\mathbf{A}$ and $\mathbf{B}$ of $I(\mathbb{R})$, if $\mathbf{B} = \mathbf{A} \oplus (\mathbf{B} \ominus_{gH} \mathbf{A})$, then (ii) of Lemma 2 is an obvious property since $\mathbf{B} \ominus_{gH} \mathbf{A} \preceq [L, L]$. However, $(\mathbf{A} \oplus (\mathbf{B} \ominus_{gH} \mathbf{A}))$ is not always equal to $\mathbf{B}$. For instance, for $\mathbf{A} = [4, 10]$ and $\mathbf{B} = [-3, 2]$,

$$\mathbf{A} \oplus (\mathbf{B} \ominus_{gH} \mathbf{A}) = [4, 10] \oplus [-8, -7] = [-4, 3] \neq \mathbf{B}.$$

Therefore, (ii) of Lemma 2 is not a trivial property.

(ii) For any $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ in $I(\mathbb{R})$, if

$$\mathbf{B} \ominus_{gH} \mathbf{A} \preceq \mathbf{C} \implies \mathbf{B} \preceq \mathbf{A} \oplus \mathbf{C}, \tag{2.1}$$

then replacing $\mathbf{C}$ by $[L, L]$, we see that (ii) of Lemma 2 is an obvious property. However, (2.1) is not always true. For instance, if $\mathbf{B} = [-3, 2]$, $\mathbf{A} = [4, 10]$ and $\mathbf{C} = [-7.5, -6]$, then

$$\mathbf{B} \ominus_{gH} \mathbf{A} = [-8, -7] \text{ and } \mathbf{A} \oplus \mathbf{C} = [-3.5, 4].$$

Hence, $\mathbf{B} \ominus_{gH} \mathbf{A} \preceq \mathbf{C}$, but $\mathbf{B}$ and $\mathbf{A} \oplus \mathbf{C}$ are not comparable. Thus, (ii) of Lemma 2 is not an obvious property.

2.3 Convexity and Calculus of IVFs

A function $\mathbf{G}$ from $\mathcal{S}$ to $I(\mathbb{R})$ is known as an IVF. For each $x \in \mathcal{S}$, $\mathbf{G}$ can be presented by intervals

$$\mathbf{G}(x) = \left[\underline{g}(x),\ \overline{g}(x)\right],$$

where $\underline{g}$ and $\overline{g}$ are real-valued functions on $\mathcal{S}$ such that $\underline{g}(x) \leq \overline{g}(x)$ for all $x \in \mathcal{S}$.
If $\mathcal{S}$ is convex, then the IVF $\mathbf{G}$ is said to be convex [36] on $\mathcal{S}$ if for any $x,\ y \in \mathcal{S}$,

$$\mathbf{G}(\delta_1 x + \delta_2 y) \preceq \delta_1 \odot \mathbf{G}(x) \oplus \delta_2 \odot \mathbf{G}(y) \text{ for all } \delta_1, \delta_2 \in [0,1] \text{ with } \delta_1 + \delta_2 = 1.$$

The IVF $\mathbf{G}$ is *gH-continuous* [10] at an interior point $\bar{y} \in \mathcal{S}$ if

$$\lim_{\|v\| \to 0} (\mathbf{G}(\bar{y} + v) \ominus_{gH} \mathbf{G}(\bar{y})) = \mathbf{0}.$$

If $\mathbf{G}$ is gH-continuous at each $\bar{y}$ in $\mathcal{S}$, then $\mathbf{G}$ is said to be gH-continuous on $\mathcal{S}$.
The IVF $\mathbf{G}$ is *gH-Lipschitz continuous* [14] at $\bar{y} \in \mathcal{S}$ if there exist constants $K' > 0$ and $\delta > 0$ such that

$$\|\mathbf{G}(\bar{y}) \ominus_{gH} \mathbf{G}(y)\|_{I(\mathbb{R})} \leq K' \|\bar{y} - y\| \text{ for all } y \in \mathcal{S} \cap \mathcal{B}(\bar{y}, \delta).$$

The constant K' is called a Lipschitz constant of $\mathbf{F}$ at $\bar{x}$. If there exists a $K > 0$ such that

$$\|\mathbf{G}(x) \ominus_{gH} \mathbf{G}(y)\|_{I(\mathbb{R})} \leq K \|x - y\| \text{ for all } x, y \in \mathcal{S},$$

then the IVF $\mathbf{G}$ is gH-Lipschitz continuous on $\mathcal{S}$ with Lipschitz constant K.

Lemma 3 *(See [36]). If $\mathbf{G}$ is a convex IVF on a convex set $\mathcal{S} \subseteq \mathcal{X}$, then $\underline{g}$ and $\overline{g}$ are convex on $\mathcal{S}$ and vise-versa.*

Lemma 4 *Let $\mathbf{G}$ be an IVF defined on $\mathcal{S}$. Then,*

(i) if $\mathbf{G}$ is gH-continuous on $\mathcal{S}$, then $\underline{g}$ and $\overline{g}$ are continuous on $\mathcal{S}$ and vise-versa.
(ii) if $\mathbf{G}$ is gH-Lipschitz continuous on $\mathcal{S}$, then $\underline{g}$ and $\overline{g}$ are Lipschitz continuous on $\mathcal{S}$ and vise-versa.
(iii) if $\mathbf{G}$ is a gH-Lipschitz continuous on $\mathcal{S}$, then $\mathbf{G}$ is gH-continuous on $\mathcal{S}$.

Proof (i) See [16].
(ii) Let $\mathbf{G}$ be gH-Lipschitz continuous on $\mathcal{X}$. Thus, there exists $K > 0$ such that for any $x,\ y \in \mathcal{X}$ we have

$$\begin{aligned} &\|\mathbf{G}(x) \ominus_{gH} \mathbf{G}(y)\|_{I(\mathbb{R})} \leq K\|x - y\| \\ \implies &\max\left\{\left|\underline{g}(x) - \underline{g}(y)\right|,\ |\overline{g}(x) - \overline{g}(y)|\right\} \leq K\|x - y\| \\ \implies &\left|\underline{g}(x) - \underline{g}(y)\right| \leq K\|x - y\| \text{ and } |\overline{g}(x) - \overline{g}(y)| \leq K\|x - y\|. \end{aligned}$$

Hence, $\underline{g}$ and $\overline{g}$ are Lipschitz continuous on $\mathcal{X}$.
Conversely, let the functions $\underline{g}$ and $\overline{g}$ be Lipschitz continuous on $\mathcal{X}$. Thus, there exist $K_1,\ K_2 > 0$ such that for all $x,\ y \in \mathcal{X}$,

$$\begin{aligned} &\left|\underline{g}(x) - \underline{g}(y)\right| \leq K_1\|x - y\| \text{ and } |\overline{g}(x) - \overline{g}(y)| \leq K_2\|x - y\| \\ \implies &\max\left\{\left|\underline{g}(x) - \underline{g}(y)\right|,\ |\overline{g}(x) - \overline{g}(y)|\right\} \leq \bar{K}\|x - y\|, \text{ where } \bar{K} = \max\{K_1,\ K_2\} \\ \implies &\|\mathbf{G}(x) \ominus_{gH} \mathbf{G}(y)\|_{I(\mathbb{R})} \leq \bar{K}\|x - y\|. \end{aligned}$$

Hence, $\mathbf{G}$ is gH-Lipschitz continuous IVF on $\mathcal{X}$.

(iii) Let $\mathbf{G}$ be gH-Lipschitz continuous on $\mathcal{X}$. Then, there exists an $K > 0$ such that for all $x,\ y \in \mathcal{X}$, we have

$$\|\mathbf{G}(y) \ominus_{gH} \mathbf{G}(x)\|_{I(\mathbb{R})} \leq K\|y - x\|.$$

For $h = y - x \in \mathcal{X}$,

$$\begin{aligned} & \|\mathbf{G}(x+h) \ominus_{gH} \mathbf{G}(x)\|_{I(\mathbb{R})} \leq K\|h\| \\ \implies & \lim_{\|h\|\to 0} \|\mathbf{G}(x+h) \ominus_{gH} \mathbf{G}(x)\|_{I(\mathbb{R})} = 0 \\ \implies & \lim_{\|h\|\to 0} (\mathbf{G}(x+h) \ominus_{gH} \mathbf{G}(x)) = \mathbf{0}. \end{aligned}$$

Hence, $\mathbf{G}$ is gH-continuous at $x \in \mathcal{X}$.

A consequence of Lemma 4 is that gH-continuity and gH-Lipschitz continuity of IVFs can be defined classically, i.e., without the concept of gH-difference. Then, the prefix gH- in continuity and Lipschitz continuity could be omitted.

Remark 2 Converse of (iii) of Lemma 4 is not true. For example, consider $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = [0, 10]$, and the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$, which is defined by

$$\mathbf{G}(y) = \sqrt{y} \odot [2, 5].$$

Since $\underline{g}(y) = 2\sqrt{y}$ and $\overline{g}(y) = 5\sqrt{y}$ are continuous on $\mathcal{S}$, $\mathbf{G}$ is gH-continuous on $\mathcal{S}$ by (i) of Lemma 4. If $\mathbf{G}$ is gH-Lipschitz continuous on $\mathcal{S}$, then by (ii) of Lemma 4, $\underline{g}$ and $\overline{g}$ are Lipschitz continuous on $\mathcal{S}$, which is not true. Consequently, $\mathbf{G}$ is not gH-Lipschitz continuous on $\mathcal{S}$.

Definition 4 (*gH-directional derivative* [14,34]). Let $\mathbf{T}$ be an IVF on $\mathcal{S}$. Then, $\mathbf{T}$ is gH-directional differentiable at $\bar{y} \in \mathcal{S}$ in direction $v \in \mathcal{X}$ if the limit

$$\lim_{\delta\to 0+} \frac{1}{\delta} \odot \big(\mathbf{T}(\bar{y} + \delta v) \ominus_{gH} \mathbf{T}(\bar{y})\big)$$

exists finitely. The limit is denoted by $\mathbf{T}_{\mathscr{D}}(\bar{x})(h)$, is known as *$gH$-directional derivative* of $\mathbf{T}$ at $\bar{y}$ in the direction v.

3 gH-Clarke Derivative of IVF

In this section, extended concepts of the gH-directional derivative, namely upper and lower gH-Clarke derivatives, for IVFs are given. A short discussion on the required notions of limit superior and sublinearity for IVFs is provided.

Definition 5 (*Supremum and limit superior of an IVF*). Let $\mathbf{G}$ be an IVF defined on $\mathcal{S}$. Then, the *supremum* of $\mathbf{G}$ over $\mathcal{S}$ is defined by

$$\sup_{\mathcal{S}} \mathbf{G} = \left[\sup_{\mathcal{S}} \underline{g},\ \ \sup_{\mathcal{S}} \overline{g}\right],$$

where $\sup_{\mathcal{S}} \underline{g} = \sup\left\{\underline{g}(y) : y \in \mathcal{S}\right\}$ and $\sup_{\mathcal{S}} \overline{g} = \sup\left\{\overline{g}(y) : y \in \mathcal{S}\right\}$.
The *limit superior* of $\mathbf{G}$ at a limit point $\bar{y}$ of $\mathcal{S}$ is defined by

$$\limsup_{y \to \bar{y}} \mathbf{G}(y) = \left[\limsup_{y \to \bar{y}} \underline{g}(y),\ \ \limsup_{y \to \bar{y}} \overline{g}(y)\right],$$

where $\limsup_{y \to \bar{y}} \underline{g}(y) = \lim_{\delta \to 0}\left(\sup_{y \in \overline{\mathcal{B}}(\bar{y},\delta) \cap \mathcal{S}} \underline{g}(y)\right)$ and $\limsup_{y \to \bar{y}} \overline{g}(y) = \lim_{\delta \to 0}\left(\sup_{y \in \overline{\mathcal{B}}(\bar{y},\delta) \cap \mathcal{S}} \overline{g}(y)\right)$.

Definition 6 (*Infimum and limit inferior of an IVF*). Let $\mathbf{G}$ be an IVF defined on $\mathcal{S}$. Then, the *infimum* of $\mathbf{G}$ over $\mathcal{S}$ is defined by

$$\inf_{\mathcal{S}} \mathbf{G} = \left[\inf_{\mathcal{S}} \underline{g},\ \ \inf_{\mathcal{S}} \overline{g}\right],$$

where $\inf_{\mathcal{S}} \underline{g} = \inf\left\{\underline{g}(y) : y \in \mathcal{S}\right\}$ and $\inf_{\mathcal{S}} \overline{g} = \inf\left\{\overline{g}(y) : y \in \mathcal{S}\right\}$.
The *limit inferior* of $\mathbf{G}$ at a limit point $\bar{y}$ of $\mathcal{S}$ is defined by

$$\liminf_{y \to \bar{y}} \mathbf{G}(y) = \left[\liminf_{y \to \bar{y}} \underline{g}(y),\ \ \liminf_{y \to \bar{y}} \overline{g}(y)\right],$$

where $\liminf_{y \to \bar{y}} \underline{g}(y) = \lim_{\delta \to 0}\left(\inf_{y \in \overline{\mathcal{B}}(\bar{y},\delta) \cap \mathcal{S}} \underline{g}(y)\right)$ and $\liminf_{y \to \bar{y}} \overline{g}(y) = \lim_{\delta \to 0}\left(\inf_{y \in \overline{\mathcal{B}}(\bar{y},\delta) \cap \mathcal{S}} \overline{g}(y)\right)$.

Lemma 5 *Let $\boldsymbol{H}$ and $\boldsymbol{G}$ be two IVFs defined on $\mathcal{S}$. Then, at any $\bar{y} \in \mathcal{S}$, the following properties are true:*

(i) $\limsup_{y \to \bar{y}} (\boldsymbol{H}(y) \oplus \boldsymbol{G}(y)) \preceq \limsup_{y \to \bar{y}} \boldsymbol{H}(y) \oplus \limsup_{y \to \bar{y}} \boldsymbol{G}(y)$,
(ii) $\limsup_{y \to \bar{y}} (\delta \odot \boldsymbol{G}(y)) = \delta \odot \limsup_{y \to \bar{y}} \boldsymbol{G}(y)$ *for all* $\delta \geq 0$, *and*
(iii) $\left\| \limsup_{y \to \bar{y}} \boldsymbol{G}(y) \right\|_{I(\mathbb{R})} \leq \limsup_{y \to \bar{y}} \| \boldsymbol{G}(y) \|_{I(\mathbb{R})}$.

Proof (i) Since

$$\limsup_{y \to \bar{y}} \left(\underline{h}(y) + \underline{g}(y)\right) \leq \limsup_{y \to \bar{y}} \underline{h}(y) + \limsup_{y \to \bar{y}} \underline{g}(y) \text{ and}$$

$$\limsup_{y \to \bar{y}} \left(\overline{h}(y) + \overline{g}(y)\right) \leq \limsup_{y \to \bar{y}} \overline{h}(y) + \limsup_{y \to \bar{y}} \overline{g}(y),$$

therefore

$$\begin{aligned} &\left[\limsup_{y \to \bar{y}} (\underline{h}(y) + \underline{g}(y)),\ \ \limsup_{y \to \bar{y}} \left(\overline{h}(y) + \overline{g}(y)\right)\right] \\ \preceq &\left[\limsup_{y \to \bar{y}} \underline{h}(y), \limsup_{y \to \bar{y}} \overline{h}(y)\right] \oplus \left[\limsup_{y \to \bar{y}} \underline{g}(y), \limsup_{y \to \bar{y}} \overline{g}(y)\right], \end{aligned}$$

which implies $\limsup_{y \to \bar{y}} (\mathbf{H}(y) \oplus \mathbf{G}(y)) \preceq \limsup_{y \to \bar{y}} \mathbf{H}(y) \oplus \limsup_{y \to \bar{y}} \mathbf{G}(y)$.

(ii) Since $\underline{g}$ and $\overline{g}$ are define on $\mathbb{R}$, for any $\delta \geq 0$, we have

$$\limsup_{y\to\bar{y}} \left(\delta \underline{g}(y)\right) = \delta \limsup_{y\to\bar{y}} \underline{g}(y) \text{ and } \limsup_{y\to\bar{y}} \left(\delta \overline{g}(y)\right) = \delta \limsup_{y\to\bar{y}} \overline{g}(y). \tag{3.1}$$

Hence, for any $\delta \geq 0$,

$$\begin{aligned}\limsup_{y\to\bar{y}} \left(\delta \odot \mathbf{G}(y)\right) &= \left[\limsup_{y\to\bar{y}} \left(\delta \underline{g}(y)\right), \ \limsup_{y\to\bar{y}} \left(\delta \overline{g}(y)\right)\right] \\ &= \delta \odot \limsup_{y\to\bar{y}} \mathbf{G}(y) \text{ by (3.1)}.\end{aligned}$$

(iii) Let g be define on $\mathbb{R}$. Then, $\left|\limsup_{y\to\bar{y}} g(y)\right| \leq \limsup_{y\to\bar{y}} |g(y)|$. By the definition of norm on $I(\mathbb{R})$,

$$\begin{aligned}\left\|\limsup_{y\to\bar{y}} \mathbf{G}(y)\right\|_{I(\mathbb{R})} &= \max\left\{\left|\limsup_{y\to\bar{y}} \underline{g}(y)\right|, \ \left|\limsup_{y\to\bar{y}} \overline{g}(y)\right|\right\} \\ &\leq \limsup_{y\to\bar{y}} \ \|\mathbf{G}(y)\|_{I(\mathbb{R})}.\end{aligned}$$

Definition 7 (*Upper gH-Clarke derivative*). Let $\mathbf{G}$ be an IVF defined on $\mathcal{S}$. For $\bar{y} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit superior

$$\limsup_{\substack{y\to\bar{y} \\ t\to 0+}} \frac{1}{t} \odot \left(\mathbf{G}(y+tv) \ominus_{gH} \mathbf{G}(y)\right) = \lim_{\delta\to 0} \left(\sup_{y\in\overline{\mathcal{B}}(\bar{y},\delta)\cap\mathcal{S}, \ t\in(0,\delta)} \frac{1}{t} \odot \left(\mathbf{G}(y+tv) \ominus_{gH} \mathbf{G}(y)\right)\right)$$

exists finitely, then the limit superior value is called *upper gH-Clarke derivative* of $\mathbf{G}$ at $\bar{y}$ in the direction v, and it is denoted by $\mathbf{G}_{\mathscr{C}}(\bar{y})(v)$. If this limit superior exists for all $v \in \mathcal{X}$, then $\mathbf{G}$ is said to be *upper gH-Clarke differentiable* at $\bar{y}$.

Definition 8 (*Lower gH-Clarke derivative*). Let $\mathbf{G}$ be an IVF defined on $\mathcal{S}$. For $\bar{y} \in \mathcal{S}$ and $v \in \mathcal{X}$, if the limit inferior

$$\liminf_{\substack{y\to\bar{y} \\ t\to 0+}} \frac{1}{t} \odot \left(\mathbf{G}(y+tv) \ominus_{gH} \mathbf{G}(y)\right) = \lim_{\delta\to 0} \left(\inf_{y\in\overline{\mathcal{B}}(\bar{y},\delta)\cap\mathcal{S}, \ t\in(0,\delta)} \frac{1}{t} \odot \left(\mathbf{G}(y+tv) \ominus_{gH} \mathbf{G}(y)\right)\right)$$

exists finitely, then the limit inferior value is called *lower gH-Clarke derivative* of $\mathbf{G}$ at $\bar{y}$ in the direction v. If this limit inferior exists for all $v \in \mathcal{X}$, then $\mathbf{G}$ is said to be *lower gH-Clarke differentiable* at $\bar{y}$.

If $\mathbf{G}$ has both upper and lower gH-Clark derivatives at $\bar{y}$ and they are equal, then $\mathbf{G}$ is called gH-Clark differentiable at $\bar{y}$.

Remark 3 Conventionally, for real valued-functions, the terminologies Clarke derivative [6,21] and upper Clarke derivative [8] are interchangeably used. In fact, the upper Clarke derivative is usually referred to Clarke derivative. However, in order to avoid any confusion, we prefix upper and lower with the Clarke derivative corresponding to the values given by limit superior and limit inferior, respectively. In addition, throughout the article, we use the notation $\mathbf{G}_{\mathscr{C}}$ to refer the upper gH-Clarke derivative of an IVF $\mathbf{G}$.

Remark 4 It is to note that $\mathbf{G}$ is lower gH-Clark differentiable at $\bar{y}$ if and only if $[-1,-1]\odot\mathbf{G}$ is upper gH-Clark differentiable at $\bar{y}$. That is why we deal only with the upper gH-Clark differentiability in this study.

Example 1 In this example, we calculate the upper gH-Clarke derivative at $\bar{y}=0$ for the IVF $\mathbf{G}(y)=|y|\odot\mathbf{C}$, where $\mathbf{0}\preceq\mathbf{C}\in I(\mathbb{R})$, $\mathcal{X}$ is the Euclidean space $\mathbb{R}$, and $\mathcal{S}=\mathcal{X}$. For any $v\in\mathcal{X}$, we see that

$$\begin{aligned}&\limsup_{\substack{y\to 0\\ t\to 0+}}\frac{1}{t}\odot(\mathbf{G}(y+tv)\ominus_{gH}\mathbf{G}(y))\\ &\preceq\limsup_{\substack{y\to 0\\ t\to 0+}}\frac{1}{t}\odot(|y|\odot\mathbf{C}\oplus t|v|\odot\mathbf{C}\ominus_{gH}|y|\odot\mathbf{C})\ \text{ by Lemma 1}\\ &=|v|\odot\mathbf{C}.\end{aligned}\tag{3.2}$$

Further,

$$\begin{aligned}&\limsup_{\substack{y\to 0\\ t\to 0+}}\frac{1}{t}\odot(|y+tv|\odot\mathbf{C}\ominus_{gH}|y|\odot\mathbf{C})\\ &\succeq\limsup_{\substack{y\to 0\\ t\to 0+}}\frac{1}{t}\odot(2t|v|\odot\mathbf{C}\ominus_{gH}t|v|\odot\mathbf{C})\ \ (\text{taking } y=tv, v>0)\\ &=|v|\odot\mathbf{C}.\end{aligned}\tag{3.3}$$

Then, from the inequalities (3.2) and (3.3), we get $\mathbf{G}_{\mathscr{C}}(\bar{y})(v)=|v|\odot\mathbf{C}$.

Lemma 6 *If $\underline{g}$ and $\overline{g}$ are upper Clarke differentiable at $\bar{y}\in\mathcal{S}\subseteq\mathcal{X}$, then the IVF $\boldsymbol{G}$ is upper gH-Clarke differentiable at $\bar{y}\in\mathcal{S}$.*

Proof Since $\underline{g}$ and $\overline{g}$ are upper Clarke differentiable at $\bar{y}$. Therefore, both of the following limits

$$\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}l_1(t)\ \text{ and }\ \limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}l_2(t)$$

exist, where $l_1(t)=\underline{g}(y+tv)-\underline{g}(y)$ and $l_2(t)=\overline{g}(y+tv)-\overline{g}(y)$. Thus,

$$\begin{aligned}&\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}\left(l_1(t)+l_2(t)\right)\ \text{ and }\ \limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}|l_1(t)-l_2(t)|\ \text{exist}\\ \implies&\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{2t}\Big(l_1(t)+l_2(t)-|l_1(t)-l_2(t)|\Big)\ \text{and}\\ &\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{2t}\Big(l_1(t)+l_2(t)+|l_1(t)-l_2(t)|\Big)\ \text{exist}\\ \implies&\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}\left(\min\{l_1(t),l_2(t)\}\right)\ \text{ and }\ \limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}\left(\max\{l_1(t),l_2(t)\}\right)\ \text{ exist}\\ \implies&\limsup_{\substack{y\to\bar{y}\\ t\to 0+}}\frac{1}{t}\odot(\mathbf{G}(y+tv)\ominus_{gH}\mathbf{G}(y))\ \text{ exists.}\end{aligned}$$

Hence, $\mathbf{G}$ is upper gH-Clarke differentiable IVF at $\bar{y}\in\mathcal{S}$.

Remark 5 Let $\mathbf{G}$ be upper gH-Clarke diffrentiable IVF at a point $\bar{z}$ in $\mathcal{S}$. Then, $\mathbf{G}$ may not be gH-directional differentiable at $\bar{z} \in \mathcal{S}$. For example, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathcal{X}$ and the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$, which is defined by

$$\mathbf{G}(z) = \begin{cases} \frac{\sin^2 z}{z} \odot \mathbf{C} & \text{if } z \neq 0 \\ 5 \odot \mathbf{C} & \text{if } z = 0, \end{cases}$$

where $\mathbf{C} \in I(\mathbb{R})$ with $\mathbf{C} \succeq \mathbf{0}$. For all nonzero v in $\mathcal{X}$, we obtain

$$\begin{aligned} &\lim_{t \to 0+} \frac{1}{t} \odot (\mathbf{G}(z + tv) \ominus_{gH} \mathbf{G}(z)) \\ &= \lim_{t \to 0+} \frac{1}{t} \odot \left(\left(\frac{\sin^2(z + tv)}{z + tv} \right) \odot \mathbf{C} \ominus_{gH} \left(\frac{\sin^2 z}{z} \right) \odot \mathbf{C} \right) \\ &= \lim_{t \to 0+} \frac{1}{t} \odot \left(\left(\frac{\sin^2(z + tv)}{z + tv} - \frac{\sin^2 z}{z} \right) \odot \mathbf{C} \right) \\ &= \left(\frac{v \sin 2z}{z} - \frac{v \sin^2 z}{z^2} \right) \odot \mathbf{C}. \end{aligned}$$

Thus,

$$\lim_{\substack{z \to 0 \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(z + tv) \ominus_{gH} \mathbf{G}(z)) = v \odot \mathbf{C},$$

which implies

$$\limsup_{\substack{z \to 0 \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(z + tv) \ominus_{gH} \mathbf{G}(z)) = v \odot \mathbf{C}.$$

Hence, $\mathbf{G}$ is upper gH-Clarke differentiable at $\bar{z} = 0$. However, the limit

$$\begin{aligned} &\lim_{t \to 0+} \frac{1}{t} \odot (\mathbf{G}(\bar{z} + tv) \ominus_{gH} \mathbf{G}(\bar{z})) \\ &= \lim_{t \to 0+} \frac{1}{t} \odot \left(\left(\frac{\sin^2(tv)}{tv} \right) \odot \mathbf{C} \ominus_{gH} 5 \odot \mathbf{C} \right) \end{aligned}$$

does not exist at $\bar{z} = 0$. Consequently, $\mathbf{G}$ is not gH-directional differentiable at $\bar{z}$.

Remark 6 Let $\mathbf{G}$ be gH-directional diffrentiable IVF at a point $\bar{y}$ in $\mathcal{S}$. Then, $\mathbf{G}$ is not necessarily upper gH-Clarke differentiable at $\bar{y} \in \mathcal{S}$. For instance, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}^2$, $\mathcal{S} = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 \geq 0, y_2 \geq 0\}$ and the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$, which is defined by

$$\mathbf{G}(y_1, y_2) = \begin{cases} y_1^2 \left(1 + \frac{1}{y_2} \right) \odot [3,\ 8] & \text{if } y = (y_1, y_2) \neq (0, 0) \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then, at $\bar{y} = (0, 0)$ and $v = (v_1, v_2) \in \mathcal{X}$ such that for sufficiently small $t > 0$ so that $\bar{y} + tv \in \mathcal{S}$, we have

$$\lim_{t \to 0+} \frac{1}{t} \odot (\mathbf{G}(\bar{y} + tv) \ominus_{gH} \mathbf{G}(\bar{y})) = \begin{cases} \frac{v_1^2}{v_2} \odot [3,\ 8] & \text{if } v_2 \neq 0 \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Hence, $\mathbf{F}$ is a gH-directional differentiable at $\bar{y}$ in every direction $v \in \mathcal{X}$.

Again, for $y = (y_1, y_2) \in \mathcal{S}$ and $v = (v_1, v_2) \in \mathcal{X}$, we have

$$\begin{aligned}
&\lim_{t \to 0+} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \\
&= \lim_{t \to 0+} \frac{1}{t} \odot \left((y_1 + tv_1)^2 \left(1 + \frac{1}{y_2 + tv_2} \right) \odot [3, 8] \ominus_{gH} y_1^2 \left(1 + \frac{1}{y_2} \right) \odot [3,\ 8] \right) \\
&= \Bigg[\min \left\{ 3 \left(2y_1 v_1 + \frac{2y_1 v_1}{y_2} - \frac{y_1^2 v_2}{y_2^2} \right),\ 8 \left(2y_1 v_1 + \frac{2y_1 v_1}{y_2} - \frac{y_1^2 v_2}{y_2^2} \right) \right\}, \\
&\quad \max \left\{ 3 \left(2y_1 v_1 + \frac{2y_1 v_1}{y_2} - \frac{y_1^2 v_2}{y_2^2} \right),\ 8 \left(2y_1 v_1 + \frac{2y_1 v_1}{y_2} - \frac{y_1^2 v_2}{y_2^2} \right) \right\} \Bigg].
\end{aligned}$$

Along $y_2 = my_1$, where m is any real number,

$$\begin{aligned}
&\lim_{\substack{y \to 0 \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \\
&= \left[\min \left\{ 3 \left(\frac{2v_1}{m} - \frac{v_2}{m^2} \right),\ 8 \left(\frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\}, \max \left\{ 3 \left(\frac{2v_1}{m} - \frac{v_2}{m^2} \right),\ 8 \left(\frac{2v_1}{m} - \frac{v_2}{m^2} \right) \right\} \right].
\end{aligned}$$

Hence, for $v_2 > 0$, $\left(\frac{2v_1}{m} - \frac{v_2}{m^2}\right) \to -\infty$ as $m \to 0$. Consequently,

$$\limsup_{\substack{y \to 0 \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \quad \text{does not exist.}$$

This implies that $\mathbf{G}$ has no upper gH-Clarke derivative at $\bar{y} \in \mathcal{S}$.

The following theorem extends the well-known result from [21] for Lipschitz continuous functions to gH-Lipschitz continuous IVFs with the help of Lemma 6.

Theorem 1 *Let $\mathcal{S} \subset \mathcal{X}$ and $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$ be a gH-Lipschitz continuous IVF at an interior point $\bar{y}$ with a Lipschitz constant K'. Then, $\mathbf{G}$ is upper gH-Clarke differentiable at $\bar{y}$ and*

$$\|\mathbf{G}_{\mathscr{C}}(\bar{y})(v)\|_{I(\mathbb{R})} \leq K' \|v\| \text{ for all } v \in \mathcal{X}.$$

Proof Since $\mathbf{G}$ is gH-Lipschitz continuous at $\bar{y} \in \mathcal{S}$, for any $v \in \mathcal{X}$, we get for $t > 0$ that

$$\left\| \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \right\|_{I(\mathbb{R})} \leq \frac{1}{t} K' \|y + tv - y\| = K' \|v\|, \tag{3.4}$$

if y and t are sufficiently close to $\bar{y}$ and 0, respectively. From inequality (3.4) we have

$$\left| \frac{1}{t} \left(\underline{g}(y + tv) - \underline{g}(y) \right) \right| \leq K' \|v\| \text{ and } \left| \frac{1}{t} \left(\overline{g}(y + tv) - \overline{g}(y) \right) \right| \leq K' \|v\|.$$

Hence, the limit superior $\underline{g}_{\mathscr{C}}(\bar{y})(v)$ and $\overline{g}_{\mathscr{C}}(\bar{y})(v)$ exist at $\bar{y}$ (cf. p. 69 of [21]). By Lemma 6, the limit superior $\mathbf{G}_{\mathscr{C}}(\bar{y})(v)$ exists.
Furthermore, by gH-Lipschitz continuity of $\mathbf{G}$ on $\mathcal{S}$, we have the following for all $v \in \mathcal{X}$:

$$\|\mathbf{G}_{\mathscr{C}}(\bar{y})(v)\|_{I(\mathbb{R})} = \left\| \limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \right\|_{I(\mathbb{R})}$$

$$\leq \limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \left\| \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)) \right\|_{I(\mathbb{R})} \text{ by Lemma 5}$$
$$\leq K' \|v\| \text{ by (3.4).}$$

For convex and gH-Lipschitz continuous IVFs, gH-directional derivative and upper gH-Clarke derivative coincide as the next theorem states.

Theorem 2 *Let $\boldsymbol{G} : \mathcal{X} \to I(\mathbb{R})$ be convex IVF on a convex set $\mathcal{X}$ and gH-Lipschitz continuous at some $\bar{y} \in \mathcal{X}$. Then, the upper gH-Clarke derivative and the gH-directional of $\boldsymbol{G}$ at $\bar{y}$ in the direction $v \in \mathcal{X}$ are equal.*

Proof Since $\mathbf{G}$ is a convex IVF on $\mathcal{X}$, we get by Theorem 3.1 of [14] that $\mathbf{G}$ is gH-directionally differentiable at $\bar{y}$ in every direction $v \in \mathcal{X}$. Also, as $\mathbf{G}$ is gH-Lipschitz continuous at $\bar{y}$, from Theorem 1, we get that $\mathbf{G}$ is upper gH-Clarke differentiable at any $\bar{y}$ in every direction $v \in \mathcal{X}$. Thus, by Definitions 4 and 7, we observe that

$$\mathbf{G}_{\mathscr{D}}(\bar{y})(v) \preceq \mathbf{G}_{\mathscr{C}}(\bar{y})(v) \text{ for all } v. \tag{3.5}$$

For the reverse of inequality (3.5), we write

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) = \limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y))$$
$$= \lim_{\substack{\delta \to 0+ \\ \epsilon \to 0+}} \sup_{\|y - \bar{y}\| < \delta} \sup_{0 < t < \epsilon} \frac{1}{t} \odot (\mathbf{G}(y + tv) \ominus_{gH} \mathbf{G}(y)).$$

Due to convexity of $\mathbf{G}$ on $\mathcal{X}$ and Lemma 3.1 of [14], we have the following equality

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) = \lim_{\substack{\delta \to 0+ \\ \epsilon \to 0+}} \sup_{\|y - \bar{y}\| < \delta} \frac{1}{\epsilon} \odot (\mathbf{G}(y + \epsilon v) \ominus_{gH} \mathbf{G}(y)). \tag{3.6}$$

For an arbitrary $\alpha > 0$ and from (3.6), we obtain

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) = \lim_{\epsilon \to 0+} \sup_{\|y - \bar{y}\| < \epsilon\alpha} \frac{1}{\epsilon} \odot (\mathbf{G}(y + \epsilon v) \ominus_{gH} \mathbf{G}(y)).$$

Since $\mathbf{G}$ is gH-Lipschitz continuous at $\bar{y}$ and $\|y - \bar{y}\| < \epsilon\alpha$ for sufficiently small $\epsilon > 0$,

$$\left\| \tfrac{1}{\epsilon} \odot (\mathbf{G}(y + \epsilon v) \ominus_{gH} \mathbf{G}(y)) \ominus_{gH} \tfrac{1}{\epsilon} \odot (\mathbf{G}(\bar{y} + \epsilon v) \ominus_{gH} \mathbf{G}(\bar{y})) \right\|_{I(\mathbb{R})}$$
$$\leq \left\| \tfrac{1}{\epsilon} \odot (\mathbf{G}(y + \epsilon v) \ominus_{gH} \mathbf{G}(\bar{y} + \epsilon v)) \right\|_{I(\mathbb{R})} + \left\| \tfrac{1}{\epsilon} \odot (\mathbf{G}(y) \ominus_{gH} \mathbf{G}(\bar{y})) \right\|_{I(\mathbb{R})}$$
by (iii) of Lemma 2
$$\leq \tfrac{1}{\epsilon} K' \|y - \bar{y}\| + \tfrac{1}{\epsilon} K' \|y - \bar{y}\|, \text{ where } K' \text{ is the Lipschitz constant of } \mathbf{G} \text{ at } \bar{y} \in \mathcal{X}$$
$$\leq 2K'\alpha.$$

Then, by (ii) of Lemma 2, we have

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) \preceq \lim_{\epsilon \to 0+} \frac{1}{\epsilon} \odot (\mathbf{G}(\bar{y} + \epsilon v) \ominus_{gH} \mathbf{G}(\bar{y})) \oplus [2K'\alpha, 2K'\alpha]$$
$$= \mathbf{G}_{\mathscr{D}}(\bar{y})(v) \oplus [2K'\alpha, 2K'\alpha].$$

Due to arbitrariness of $\alpha > 0$, we get

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) \preceq \mathbf{G}_{\mathscr{D}}(\bar{y})(v) \text{ for all } v. \tag{3.7}$$

From (3.5) and (3.7), we obtain

$$\mathbf{G}_{\mathscr{C}}(\bar{y})(v) = \mathbf{G}_{\mathscr{D}}(\bar{y})(v).$$

Definition 9 (*Sublinear IVF*). Let $\mathbf{G}$ be an IVF on a linear subspace $\mathcal{S}$ of $\mathcal{X}$. Then, $\mathbf{G}$ is *sublinear* on $\mathcal{S}$ if

(i) $\mathbf{G}(\delta w) = \delta \odot \mathbf{G}(w)$ for all $w \in \mathcal{S}$ and for all $\delta \geq 0$, and
(ii) $\mathbf{G}(v + w) \nsucceq \mathbf{G}(v) \oplus \mathbf{G}(w)$ for all $v,\ w \in \mathcal{S}$.

Example 2 Let $\mathcal{X}$ be the Euclidean space $\mathbb{R}$ and $\mathcal{S} = \mathcal{X}$. Then, the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$ that is defined by

$$\mathbf{G}(y) = |y| \odot \mathbf{A}, \text{ where } \mathbf{A} \in I(\mathbb{R}) \text{ such that } \mathbf{A} \nprec \mathbf{0},$$

is sublinear on $\mathcal{S}$. The reason is as follows.
For all $y, z \in \mathcal{S}$ and $\delta \geq 0$, we have

(i) $\mathbf{G}(\delta y) = |\delta y| \odot \mathbf{A} = \delta |y| \odot \mathbf{A} = \delta \odot \mathbf{G}(y)$.
(ii)

$$\begin{aligned} & |y+z| \odot \mathbf{A} \nsucceq \big(|y| + |z|\big) \odot \mathbf{A} \text{ by (i) Lemma 1} \\ \implies & |y+z| \odot \mathbf{A} \nsucceq |a| \odot \mathbf{A} \oplus |b| \odot \mathbf{A} \text{ since } |y| \text{ and } |z| \text{ are nonnegative} \\ \implies & \mathbf{G}(y+z) \nsucceq \mathbf{G}(y) \oplus \mathbf{G}(z). \end{aligned}$$

Hence, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$.

Example 3 Let Q be a real positive definite matrix of order $n \times n$ and $\mathcal{S} \subset \mathcal{X}$ be a linear space. Consider the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$, which is defined by

$$\mathbf{G}(y) = \big(\sqrt{y^T Q y}\big) \odot \mathbf{A}, \text{ where } \mathbf{A} \nprec \mathbf{0}.$$

Then, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$. The reason is as follows.

We write $\mathbf{G}(y)$ as $g(y) \odot \mathbf{A}$, where $g(y) = \sqrt{y^T Q y}$. By Example 1.2.3 of [18], g satisfies the following conditions:

(a) for $\delta \geq 0$ and $y \in \mathcal{S}$,

$$g(\delta y) = \delta g(y), \tag{3.8}$$

and

(b) for all $x, y \in \mathcal{S}$

$$g(x + y) \leq g(x) + g(y). \tag{3.9}$$

From (3.8), we have

$$g(\delta y) \odot \mathbf{A} = \delta g(y) \odot \mathbf{A}, \text{ or, } \mathbf{G}(\delta y) = \delta \odot \mathbf{G}(y).$$

Since $\mathbf{A} \nprec \mathbf{0}$, from (3.9) and Lemma 1, we obtain

$$\begin{aligned} & g(x+y) \odot \mathbf{A} \nsucceq (g(x) + g(y)) \odot \mathbf{A} \\ \implies & (g(x+y)) \odot \mathbf{A} \nsucceq g(x) \odot \mathbf{A} \oplus g(y) \odot \mathbf{A} \text{ since } g(x) \text{ and } g(y) \text{ are nonnegative} \\ \implies & \mathbf{G}(x+y) \nsucceq \mathbf{G}(x) \oplus \mathbf{G}(y). \end{aligned}$$

Hence, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$.

Example 4 Let $\mathbf{G}$ be a convex IVF on a linear subspace $\mathcal{S}$ of $\mathcal{X}$ such that for all $v \in \mathcal{S}$,

$$\mathbf{G}(\alpha v) = \alpha \odot \mathbf{G}(v) \quad \text{for every } \alpha \geq 0. \tag{3.10}$$

Then, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$. The reason is as follows.
For $v, w \in \mathcal{S}$ and $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned}\mathbf{G}(\delta_1 v + \delta_2 w) &= \mathbf{G}\left(\delta\left(\tfrac{\delta_1}{\delta} v + \tfrac{\delta_2}{\delta} w\right)\right), \text{ where } \delta = \delta_1 + \delta_2 \\ &= \delta \odot \mathbf{G}\left(\tfrac{\delta_1}{\delta} v + \tfrac{\delta_2}{\delta} w\right) \text{ by (3.10)} \\ &\preceq \delta_1 \odot \mathbf{G}(v) \oplus \delta_2 \odot \mathbf{G}(w) \text{ by the convexity of } \mathbf{G}.\end{aligned}$$

Taking $\delta_1 = \delta_2 = 1$, we obtain

$$\mathbf{G}(v + w) \preceq \mathbf{G}(v) \oplus \mathbf{G}(w) \text{ for all } v, w \in \mathcal{S}.$$

Hence, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$.

Remark 7 A sublinear IVF may not be convex. For instance, take $\mathcal{X}$ as the Euclidean space $\mathbb{R}$, $\mathcal{S} = \mathcal{X}$ and the IVF $\mathbf{G} : \mathcal{S} \to I(\mathbb{R})$ that is given by

$$\mathbf{G}(y) = |y| \odot [-3, 2].$$

Clearly, by Example 2, $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$. However, $\underline{g}(y) = -3|y|$ is not convex on $\mathcal{S}$. Therefore, by Lemma 3, $\mathbf{G}$ is not a convex IVF on $\mathcal{S}$.

Theorem 3 *Let $\mathbf{G}$ be an upper gH-Clarke differentiable IVF at an interior point $\bar{y}$. Then, the upper gH-Clarke derivative $\mathbf{G}_{\mathscr{C}}(\bar{y})$ of $\mathbf{G}$ is a sublinear IVF on $\mathcal{S}$.*

Proof For an arbitrary $v \in \mathcal{S}$ and $\delta \geq 0$, we have

$$\begin{aligned}\limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + t\delta v) \ominus_{gH} \mathbf{F}(y)) &= \delta \odot (\limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \tfrac{1}{t\delta} \odot (\mathbf{G}(y + t\delta v) \ominus_{gH} \mathbf{F}(y))) \\ &= \delta \odot \mathbf{G}_{\mathscr{C}}(\bar{y})(v).\end{aligned}$$

Thus, $\mathbf{G}_{\mathscr{C}}(\bar{y})(\delta v) = \delta \odot \mathbf{G}_{\mathscr{C}}(\bar{y})(v)$.
Next, for all $v_1, v_2 \in \mathcal{S}$, we get

$$\begin{aligned}&\mathbf{G}_{\mathscr{C}}(\bar{y})(v_1 + v_2) \\ &= \limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \frac{1}{t} \odot (\mathbf{G}(y + t(v_1 + v_2)) \ominus_{gH} \mathbf{G}(y)) \\ &\not\succ \limsup_{\substack{y \to \bar{y} \\ t \to 0+}} \frac{1}{t} \odot \Big[\mathbf{G}(y + tv_1 + tv_2) \ominus_{gH} \mathbf{G}(y + tv_2) \oplus \mathbf{G}(y + tv_2) \ominus_{gH} \mathbf{G}(y)\Big], \\ &\quad \text{by (i) of Lemma 2} \\ &= \mathbf{G}_{\mathscr{C}}(\bar{y})(v_1) \oplus \mathbf{G}_{\mathscr{C}}(\bar{y})(v_2).\end{aligned}$$

Hence, $\mathbf{G}_{\mathscr{C}}(\bar{y})$ is a sublinear IVF on $\mathcal{S}$.

4 Conclusion and Future Directions

In this article, mainly three concepts on IVFs have been studied—limit superior of IVF (Definition 5), upper gH-Clarke derivative (Definition 7), and sublinear IVF (Definition 9). One can trivially notice that in the degenerate case, each of the Definitions 5, 7, and 9 reduces to the respective conventional definition for the real-valued functions. It has been observed that for a gH-Lipschitz continuous IVF, the upper gH-Clarke derivative always exists (Theorem 1). Also, for a gH-Lipschitz continuous IVF, it has been found that the gH-directional derivative of a convex IVF coincides with the upper gH-Clarke derivative (Theorem 2). It has been noticed that the upper gH-Clarke derivative at an interior point of the domain of an IVF is a sublinear IVF (Theorem 3).

In analogy to the current study, future research can be carried out for other generalized directional derivatives for IVFs, e.g., Dini, Hadamard, Michel-Penot, etc., and their relationships [8]. In parallel to the proposed analysis of IVFs, another promising direction of future research can be the analysis of the fuzzy-valued functions (FVFs) as the alpha-cuts of fuzzy numbers are compact intervals [13]. Hence, in future, one can attempt to extend the proposed idea of gH-Clarke derivative for fuzzy-valued functions.

The proposed upper gH-Clarke derivative can be dealt with control systems and differential equations in a noisy or uncertain environment in the future. A noisy and uncertain environment in control system or a differential equation appears due to the incomplete information of demand for a product and changes in the climate. In future, we will try to solve a control problem in a noisy or uncertain environment with the help of gH-Clarke and gH-Fréchet derivatives.

Appendix A Proof of Lemma 1

Proof Let $\mathbf{Z} = [\underline{z},\ \overline{z}]$.

(i) If $\mathbf{Z} \succeq \mathbf{0}$, then

$$\begin{aligned} &\underline{z} \geq 0 \text{ and } \overline{z} \geq 0 \\ \Longrightarrow\ &|a|\underline{z} + |b|\underline{z} \geq |a+b|\underline{z} \text{ and } |a|\overline{z} + |b|\overline{z} \geq |a+b|\overline{c} \\ \Longrightarrow\ &|a+b| \odot \mathbf{Z} \preceq |a| \odot \mathbf{Z} \oplus |b| \odot \mathbf{Z}. \end{aligned}$$

(ii) If $\mathbf{Z} \preceq \mathbf{0}$, then

$$\begin{aligned} &\underline{z} \leq 0 \text{ and } \overline{z} \leq 0 \\ \Longrightarrow\ &|a|\underline{z} + |b|\underline{z} \leq |a+b|\underline{z} \text{ and } |a|\overline{z} + |b|\overline{z} \leq |a+b|\overline{z} \\ \Longrightarrow\ &|a+b| \odot \mathbf{Z} \succeq |a| \odot \mathbf{Z} \oplus |b| \odot \mathbf{Z}. \end{aligned}$$

Appendix B Proof of Lemma 2

Proof Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, $\mathbf{C} = [\underline{c}, \overline{c}]$ and $\mathbf{D} = [\underline{d}, \overline{d}]$.

(i) We have the following four possible cases.
- **Case** 1. Let $\overline{a} - \overline{c} \geq \underline{a} - \underline{c}$ and $\overline{c} - \overline{b} \geq \underline{c} - \underline{b}$. Then, $\overline{a} - \overline{b} \geq \underline{a} - \underline{b}$ and

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\underline{a} - \underline{c}, \overline{a} - \overline{c}] \oplus [\underline{c} - \underline{b}, \overline{c} - \overline{b}] = [\underline{a} - \underline{b}, \overline{a} - \overline{b}] = \mathbf{A} \ominus_{gH} \mathbf{B}.$$

- **Case** 2. Let $\overline{a}-\overline{c} \leq \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} \leq \underline{c}-\underline{b}$. Therefore, $\overline{a}-\overline{b} \leq \underline{a}-\underline{b}$ and

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{c}, \underline{a}-\underline{c}] \oplus [\overline{c}-\overline{b}, \underline{c}-\underline{b}] = [\overline{a}-\overline{b}, \underline{a}-\underline{b}] = \mathbf{A} \ominus_{gH} \mathbf{B}.$$

- **Case** 3. Let $\overline{a}-\overline{c} < \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} > \underline{c}-\underline{b}$. Therefore,

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) = [\overline{a}-\overline{c}, \underline{a}-\underline{c}] \oplus [\underline{c}-\underline{b}, \overline{c}-\overline{b}] = [\overline{a}-\overline{c}+\underline{c}-\underline{b}, \underline{a}-\underline{c}+\overline{c}-\overline{b}].$$

If possible, let

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) \prec \mathbf{A} \ominus_{gH} \mathbf{B}. \qquad \text{(Appendix B.1)}$$

If $\overline{a}-\overline{b} \geq \underline{a}-\underline{b}$, then from (Appendix B.1) we get

$$\begin{aligned} & [\overline{a}-\overline{c}+\underline{c}-\underline{b}, \underline{a}-\underline{c}+\overline{c}-\overline{b}] \prec [\underline{a}-\underline{b}, \overline{a}-\overline{b}] \\ \Longrightarrow\ & \underline{a}-\underline{c}+\overline{c}-\overline{b} \leq \overline{a}-\overline{b} \\ \Longrightarrow\ & \underline{a}-\underline{c} \leq \overline{a}-\overline{c}, \quad \text{which is an impossibility.} \end{aligned}$$

Further, if $\overline{a}-\overline{b} \leq \underline{a}-\underline{b}$, then from (Appendix B.1), we have

$$\begin{aligned} & [\overline{a}-\overline{c}+\underline{c}-\underline{b}, \underline{a}-\underline{c}+\overline{c}-\overline{b}] \prec [\overline{a}-\overline{b}, \underline{a}-\underline{b}] \\ \Longrightarrow\ & \underline{a}-\underline{c}+\overline{c}-\overline{b} \leq \underline{a}-\underline{b} \\ \Longrightarrow\ & \overline{c}-\overline{b} \leq \underline{c}-\underline{b}, \quad \text{which is an impossibility.} \end{aligned}$$

Thus, (Appendix B.1) is not true.

- **Case** 4. Let $\overline{a}-\overline{c} > \underline{a}-\underline{c}$ and $\overline{c}-\overline{b} < \underline{c}-\underline{b}$. Proceeding as in **Case** 3 of (i) we can prove that (Appendix B.1) is not true. Hence,

$$(\mathbf{A} \ominus_{gH} \mathbf{C}) \oplus (\mathbf{C} \ominus_{gH} \mathbf{B}) \nprec \mathbf{A} \ominus_{gH} \mathbf{B}.$$

(ii) As $\|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = \max\{|\underline{b}-\underline{a}|, |\overline{b}-\overline{a}|\}$, we break the proof in two cases.

- **Case** 1. If $(L =)\ \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = |\underline{b}-\underline{a}|$, then

$$|\underline{b}-\underline{a}| \geq |\overline{b}-\overline{a}| \implies |\underline{b}-\underline{a}| \geq \overline{b}-\overline{a} \implies \overline{b} \leq \overline{a}+L. \qquad \text{(Appendix B.2)}$$

Since $\underline{b}-\underline{a} \leq |\underline{b}-\underline{a}|$, then

$$\underline{b} \leq \underline{a}+L. \qquad \text{(Appendix B.3)}$$

From (Appendix B.2) and (Appendix B.3), we have $\mathbf{B} \preceq \mathbf{A} \oplus [L, L]$.

- **Case** 2. If $(L =)\ \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})} = |\overline{b}-\overline{a}|$, then

$$|\underline{b}-\underline{a}| \leq |\overline{b}-\overline{a}| \implies \underline{b}-\underline{a} \leq |\overline{b}-\overline{a}| \implies \underline{b} \leq \underline{a}+L. \qquad \text{(Appendix B.4)}$$

Since $\overline{b}-\overline{a} \leq |\overline{b}-\overline{a}|$,

$$\overline{b} \leq \overline{a}+L. \qquad \text{(Appendix B.5)}$$

From (Appendix B.4) and (Appendix B.5), we obtain $\mathbf{B} \preceq \mathbf{A} \oplus [L, L]$, where $L = \|\mathbf{B} \ominus_{gH} \mathbf{A}\|_{I(\mathbb{R})}$.

(iii) If possible, let there exist $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ in $I(\mathbb{R})$ such that

$$\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} > \|\mathbf{A} \ominus_{gH} \mathbf{C}\|_{I(\mathbb{R})} \oplus \|\mathbf{B} \ominus_{gH} \mathbf{D}\|_{I(\mathbb{R})}. \qquad \text{(Appendix B.6)}$$

According to the definition of gH-difference of two intervals,

$$\text{either} \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\underline{a}-\underline{b}, \overline{a}-\overline{b}] \quad \text{or} \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a}-\overline{b}, \underline{a}-\underline{b}],$$

$$\text{either} \quad \mathbf{C} \ominus_{gH} \mathbf{D} = [\underline{c}-\underline{d}, \overline{c}-\overline{d}] \quad \text{or} \quad \mathbf{C} \ominus_{gH} \mathbf{D} = [\overline{c}-\overline{d}, \underline{c}-\underline{d}],$$
$$\text{either} \quad \mathbf{A} \ominus_{gH} \mathbf{C} = [\underline{a}-\underline{c}, \overline{a}-\overline{c}] \quad \text{or} \quad \mathbf{A} \ominus_{gH} \mathbf{B} = [\overline{a}-\overline{c}, \underline{a}-\underline{c}],$$

and

$$\text{either} \quad \mathbf{B} \ominus_{gH} \mathbf{D} = [\underline{b}-\underline{d}, \overline{b}-\overline{d}] \quad \text{or} \quad \mathbf{B} \ominus_{gH} \mathbf{D} = [\overline{b}-\overline{d}, \underline{b}-\underline{d}].$$

Then, one of the following holds true:
(a) $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\underline{a}-\underline{b}-\underline{c}+\underline{d},\ \overline{a}-\overline{b}-\overline{c}+\overline{d}]$

(b) $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\underline{a} - \underline{b} - \overline{c} + \overline{d},\ \overline{a} - \overline{b} - \underline{c} + \underline{d}]$
(c) $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \overline{c} + \overline{d},\ \underline{a} - \underline{b} - \underline{c} + \underline{d}]$
(d) $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \underline{c} + \underline{d},\ \underline{a} - \underline{b} - \overline{c} + \overline{d}]$.

- **Case** 1. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\underline{a} - \underline{b} - \underline{c} + \underline{d},\ \overline{a} - \overline{b} - \overline{c} + \overline{d}]$.
  (a) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b} - \underline{c} + \underline{d}|$, then from equation (Appendix B.6), we have

$$|\underline{a} - \underline{b} - \underline{c} + \underline{d}| > |\underline{a} - \underline{c}| + |\underline{b} - \underline{d}| > |\underline{a} - \underline{b} - \underline{c} + \underline{d}|,$$

  which is impossible.
  (b) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\overline{a} - \overline{b} - \overline{c} + \overline{d}|$, then from equation (Appendix B.6), we have

$$|\overline{a} - \overline{b} - \overline{c} + \overline{d}| > |\overline{a} - \overline{c}| + |\overline{b} - \overline{d}| > |\overline{a} - \overline{b} - \overline{c} + \overline{d}|,$$

  which is again impossible.
- **Case** 2. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \overline{c} + \overline{d},\ \underline{a} - \underline{b} - \underline{c} + \underline{d}]$.
  For this case, two subcases are similar to the **Case** 1 of (iii) will lead to impossibilities.
- **Case** 3. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\underline{a} - \underline{b} - \overline{c} + \overline{d},\ \overline{a} - \overline{b} - \underline{c} + \underline{d}]$. Then,

$$\underline{a} - \underline{b} \leq \overline{a} - \overline{b} \text{ and } \overline{c} - \overline{d} \leq \underline{c} - \underline{d}. \qquad \text{(Appendix B.7)}$$

  (a) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\overline{a} - \overline{b} - \underline{c} + \underline{d}|$, then $\overline{a} - \overline{b} - \underline{c} + \underline{d} \geq 0$. From equation (Appendix B.6), we have

$$|\overline{a} - \overline{b} - \underline{c} + \underline{d}| > |\overline{a} - \overline{c}| + |\overline{b} - \overline{d}| \implies \overline{c} - \overline{d} > \underline{c} - \underline{d},$$

  which is contradictory to (Appendix B.7).
  (b) If $\|(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D})\|_{I(\mathbb{R})} = |\underline{a} - \underline{b} - \overline{c} + \overline{d}|$, then $\underline{a} - \underline{b} - \overline{c} + \overline{d} < 0$. From equation (Appendix B.6), we have

$$-(\underline{a} - \underline{b} - \overline{c} + \overline{d}) = |\underline{a} - \underline{b} - \overline{c} + \overline{d}| > |\underline{a} - \underline{c}| + |\underline{b} - \underline{d}| \implies \overline{c} - \overline{d} > \underline{c} - \underline{d},$$

  which is again contradictory to (Appendix B.7).
- **Case** 4. Let $(\mathbf{A} \ominus_{gH} \mathbf{B}) \ominus_{gH} (\mathbf{C} \ominus_{gH} \mathbf{D}) = [\overline{a} - \overline{b} - \underline{c} + \underline{d},\ \underline{a} - \underline{b} - \overline{c} + \overline{d}]$.
  All the two subcases for this case are similar to **Case** 3 of (iii).

Hence, (Appendix B.6) is wrong, and thus the result follows.

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Conflicts of interest/Competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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