Stochastic process analysis enhancement via quantum system parameter estimation

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Research Article

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Abstract In this paper, we investigate a linear Ito stochastic differential equation in the vector space from every conceivable perspective. In this case, the potential vector describes the magnitude of the classical noise acting on a quantum system. This vector potential can be expressed as a linear function of its parameters, with Hermitian operators serving as its coefficients since its parameters are assumed to be unknown. To the second order of perturbation, the unitary evolution operator can be determined with the aid of a potential perturbation parameter. As for the second term, it is written as a double-iterated stochastic integral with respect to Brownian motion, while the first term is written as an Ito stochastic integral. When controlling quantum systems, noise from the environment can be a major hindrance; this technique can help. Improve the dependability and practicality of quantum technologies like computers by learning to detect and regulate noise. If the potential’s parameters are affected by noise, then their reliability is thought to decrease. We focus on the special case where the potential is a linear function of these parameters with Hermitian operators as the coefficients. To find the unitary evolution operator up to $O(\epsilon)$, we can write the $O(\epsilon)$ term as an Ito stochastic integral with respect to Brownian motion, and the $O(\epsilon^2)$ term as a doubly iterated stochastic integral with respect to Brownian motion.

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1 Introduction

Performance analysis of an quantum system in presence of noise is the greatest challenge now these days. If we say noise, so it may be white or colored

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noise. Another challenging task is estimating the quantum system parameters in presence of random noise. If we are analyzing any practice problem in mathematical framework of quantum mechanics or we are dealing at quantum scale, the biggest issue is to maintain the unitarity. In quantum parameter estimation field, some quantitites like purity of state is very tedious task to find exactly or accurately because one concept of density matrix involves here. As it is known to us that density matrix is not the linear function of related parameters [1]. Using the Brownian motion theory [2] along with Ito calculus [3] - [4] is better way for estimating such kind of the systems in Gaussian as well as non-Gaussian behaviour. This approach allows us to track the evolution of the quantum system and identify when noise is affecting the parameters, which can then be adjusted to maintain reliability. By using stochastic calculus, we can model the effects of noise on quantum systems and develop strategies for controlling them. The technique can be used with both discrete quantum systems (such as qubits) and continuous quantum systems (such as state-spaces). By understanding and controlling the effects of noise on these systems, we can boost their performance and move closer to developing practical quantum technologies. In quantum systems, noise presents a significant difficulty because it can cause mistakes and decoherence. For this reason, it is essential to study and manage noise if we are to create quantum technologies that can be relied upon and used effectively [5].

First time quantum parameter estimation theory was discussed in 1969 [6]. Basically the estimation theory is based on taking measurements and let the quantum system evolved and again taking measurement and again system will evolve and this process going on and on. Ensembled averaging or time averaging, any one averaging method can be used for analyzing quantum system but unitarity must be maintained. As continuous measurements is not possible in quantum mechanics theory due to no cloning theorem so discrete measurements takes. The very first step for analyzing the quantum systems for continuous variables is to consider Gaussian states and Gaussian transformation [7]. The main reason of discussing parameter estimation theory in quatum mechanics framework is to achieve the better accuracy that is impossible after consideration of classical estimation theory [8]. The previous work in quantum parameter estimation theory has been provided the accurate or precise results [9] - [13]. The major work in quantum parameter estimation theory is based on single parameter estimation at a time. Very few literature is available on multiple parameter estimation. This paper deals with multiple parameter estimation problem using Brownina motion theory with time averaging phenomena.

First, we express the Schrödinger equation for a quantum system in the presence of white Gaussian noise as a complex Ito stochastic differential equation linear in the state vector. A master equation for the density matrix, which captures the dynamics of the system as it responds to noise, can then be derived using quantum stochastic calculus. To ensure the evolution remains unitary when subjected to white Gaussian noise, the unperturbed Hamiltonian is supplemented by an Ito correction term whose magnitude is proportional
to the square of the potential (using the Ito formalism). We can now examine the impact of noise on quantum systems and how they operate in a noisy setting by using this method [14]. The resulting master equation can model the system’s dynamics and predict its response to varying levels of noise. We can use this method to create quantum computers that are robust against errors and can perform well even when exposed to background noise. It’s also worth noting that noise isn’t the only problem with quantum computing; there’s still room for improvement in areas like scalability and hardware limitations. The creation of fault-tolerant quantum computers could allow us to crack problems that are currently unsolvable by classical computers. It’s worth noting, though, that quantum computing is still in its infancy, and that a lot of work needs to be done to overcome the difficulties of scalability and hardware limitations. This method has broad applicability in the physical sciences, from quantum computing to quantum control. Integrating noise into the system allows us to better comprehend and control the evolution of these systems under realistic conditions. In addition, this approach can be used to investigate the robustness of quantum technologies by studying the impact of decoherence and other types of noise on quantum systems [15] - [19]. Overall, this method provides a robust resource for investigating and enhancing the performance of quantum systems in the real world.

The technique can be used with both discrete quantum systems (such as qubits) and continuous quantum systems (such as state-spaces). By understanding and controlling the effects of noise on these systems, we can boost their performance and move closer to developing practical quantum technologies. In quantum systems, noise presents a significant difficulty because it can cause mistakes and decoherence. For this reason, it is essential to study and manage noise if we are to create quantum technologies that can be relied upon and used effectively [20]. One way to manage noise in quantum systems is through error correction codes, which can detect and correct errors caused by noise. Another approach is to use quantum control techniques, which can manipulate the system to mitigate the effects of noise and improve its performance [21] - [22].

The outline of the paper looks like this: A discussion of mathematical formulas can be found in Section 2. Future applicability is discussed to wrap up Section 4, which includes a brief summary of the potential impact of the research and suggestions for further studies. Overall, this paper aims to provide a comprehensive understanding of the mathematical models and simulation techniques used in this field.

2 Problem Formulation

The following stochastic differential equation for the evolution operator of a quantum system in the presence of white Gaussian noise (derivative of Brownian motion) guarantees unitary evolution. $B(t)$ is Brownian motion and $V$ is the noise modulating potential assumed to be expressible as a linear combi-
nation of $p$ known operators with unknown real coefficients. The Schrödinger evolution operator $U(t)$ is assumed to satisfy the following stochastic differential equation [3].

$$dU(t) = -\left(iH + \frac{\epsilon^2 V^2}{2}\right)dt - i\epsilon V dB(t)U(t)$$  \hspace{1cm} (1)

where

$$V = \sum_{k=1}^{p} \theta_k V_k$$  \hspace{1cm} (2)

The observable evolution is given by

$$X(t) = U^*(t)XU(t)$$  \hspace{1cm} (3)

so the Ito differential rule gives

$$dX(t) = dU^*XU + U^*XdU + dU^*XdU$$  \hspace{1cm} (4)

The classically averaged rate of change of an observable $X$ under the above noisy Schrödinger dynamics is given by the following computation.

$$\frac{d}{dt}E(X(t)) = E[U^*(t)i[H, X]U(t)] - \frac{\epsilon^2}{2} E[U^*(V^2X + XV^2)U] + \epsilon^2 E[U^*VXVU]$$  \hspace{1cm} (5)

The perturbation parameter $\epsilon$ has been introduced into the stochastic noise differential term $V dB$ in the evolution to show that it is small. Correspondingly an Ito correcting term $\frac{i\epsilon^2 V^2}{2} dt$ has been introduced with $iHdt$ to force unitary evolution that guarantees conservation of probability. Equation (5) is in fact a special case of the observable version of the famous. We solve for the unitary evolution upto $O(\epsilon^3)$ using time dependent perturbation theory as follows. First expressed $U(t)$ in power of $\epsilon$.

$$U(t) = U_0(t) + \epsilon U_1(t) + \epsilon^2 U_2(t) + O(\epsilon^3)$$  \hspace{1cm} (6)

where

$$dU_0(t) = -iHU_0(t)dt$$  \hspace{1cm} (7)
$$dU_1(t) = -iHU_1(t)dt - iVU_0(t)dB(t)$$  \hspace{1cm} (8)
$$dU_2(t) = -iHU_2(t)dt - iVU_1(t)dB(t) - \frac{V^2}{2} U_0(t)dt$$  \hspace{1cm} (9)

Now $U_0(t), U_1(t)$ and $U_2(t)$ may be expressed respectively as

$$U_0(t) = exp(-itH)$$  \hspace{1cm} (10)
\[ U_1(t) = -i \int_0^t U_0(t - \tau)VU_0(\tau)dB(\tau) \quad (11) \]

\[ U_2(t) = -i \int_0^t U_0(t - \tau)VU_1(\tau)dB(\tau) \]

\[ - \frac{1}{2} \int_0^t U_0(t - \tau)V^2U_0(\tau)d\tau \quad (12) \]

Solving successively for the \( O(\epsilon^0) \), \( O(\epsilon^1) \) and \( O(\epsilon^2) \) term gives expressions for \( U_1(t) \) and \( U_2(t) \) as Ito integrals with respect to Brownian motion using the standard formulas

\[ \mathbb{E} \int_0^T F(t)dB(t) = 0 \quad (13) \]

and

\[ \mathbb{E} \left[ \int_0^T F(t)dB(t) \left( \int_0^T G(t)dB(t)^T \right) \right] = \int_0^T \mathbb{E} \left[ F(t)G(t)^T \right] dt \quad (14) \]

where \( F(t), G(t) \) are adaptive functional of Brownian motion.

The expected value of averaging in time \( t \) with observable \( X \) is given by

\[ \langle X \rangle(t) = Tr\{U(t)\rho(0)U^*(t)X\} \]

\[ = Tr\{\rho(0)U^*(t)XU(t)\} \quad (15) \]

where

\[ U^*(t)XU(t) = U_0^*(t)XU_0(t) + \epsilon \left( U_1^*(t)XU_0(t) + U_1^*(t)XU_1(t) \right) \]

\[ + \epsilon^2 \left( U_0^*(t)XU_2(t) + U_2^*(t)XU_0(t) + U_1^*(t)XU_1(t) \right) + O(\epsilon^3) \quad (16) \]

Expected value of \( U(t)XU(t) \) will defined as
\[ E[U(t)XU(t)] = U_0^*(t)XU_0(t) + \]
\[ \epsilon^2 \left( \frac{-1}{2} \int_0^t U_0^*(t)XU_0(t-\tau)V^2U_0(\tau)d\tau \right) \]
\[ - \frac{1}{2} \int_0^t U_0^*(\tau)V^2U_0^*(t-\tau)XU_0(\tau)d\tau \]
\[ + \int_0^t U_0^*(\tau)VU_0^*(t-\tau)XU_0(t-\tau)VU_0(\tau)d\tau \]
\[ + O(\epsilon^3) \quad (17) \]
\[ = U_0^*(t)XU_0(t) - \left( \frac{\epsilon^2}{2} \int_0^t \left( U_0^*(t)XU_0(t-\tau)V^2U_0(\tau) \right. \right. \]
\[ + \left. U_0^*(\tau)V^2U_0^*(t-\tau)XU_0(t) \right. \]
\[ - \left. 2U_0^*(\tau)VU_0^*(t-\tau)XU_0(t-\tau)VU_0(\tau)d\tau \right) \]
\[ + O(\epsilon^3) \quad (18) \]

The expected value of the observable at time \( t \) (that is its classical average) can now be expressed in terms of the eigenprojections of \( H \) as more precisely the limit \( t \to \infty \), the oscillatory term are dominated by the non-oscillatory term and we get the above approximate expressions.

As it is well known that \( U_0(t) = \exp(-itH) = \sum_{\alpha=1}^p \exp(-i\alpha t)P_\alpha \) then

\[ E[U^*(t)XU(t)] - U_0(t)XU_0(t) = \left( -\frac{\epsilon^2}{2} \int_0^t \exp\{i(\lambda_\alpha \tau - \lambda_\beta (t-\tau) - \lambda_\mu \tau)\}d\tau P_\alpha XP_\beta V^2P_\mu \right) \]
\[ + \left( -\frac{\epsilon^2}{2} \int_0^t \exp\{i(\lambda_\beta \tau + \lambda_\beta (t-\tau) - \lambda_\alpha \alpha)\}d\tau P_\beta V^2P_\beta XP_\alpha \right) \]
\[ + \left( \epsilon^2 \int_0^t \exp\{i(\lambda_\alpha \tau + \lambda_\beta (t-\tau) - \lambda_\mu (t-\tau) - \lambda_\mu \tau)\}d\tau \right. \]
\[ \times \left. P_\alpha VP_\beta XP_\mu V^2P_\mu \right) + O(\epsilon^3) \quad (19) \]
We can rewrite the equation (19) as

\[- \frac{2}{\epsilon^2} \left\{ \mathbb{E}[U^*(t)xU(t)] - U_0^*(t)xU_0(t) \right\} = \left( P_\alpha x P_\beta \right) V^2 P_\rho \exp \left\{ i(\lambda_\alpha - \lambda_\beta)t \right\} \times \left( \frac{-1 + \exp(i(\lambda_\beta - \lambda_\rho)t)}{i(\lambda_\beta - \lambda_\rho)} \right)
+ P_\beta V^2 P_\beta \left( -i(\lambda_\alpha - \lambda_\beta)t \right) \times \left( \frac{-1 + \exp(-i(\lambda_\beta - \lambda_\rho)t)}{i(\lambda_\beta - \lambda_\rho)} \right)
- P_\alpha V_\beta x P_\rho x P_\rho \exp \left\{ i(\lambda_\beta - \lambda_\rho)t \right\} \times \left( \frac{-1 + \exp \left\{ i(\lambda_\alpha - \lambda_\beta + \lambda_\rho - \lambda_\mu)t \right\}}{i(\lambda_\beta - \lambda_\rho + \lambda_\rho - \lambda_\mu)} \right) \right]\]

(20)

For \( t \to \infty \), this is approximately

\[ t \left\{ P_\alpha x P_\beta V^2 P_\beta \exp \left\{ i(\lambda_\alpha - \lambda_\beta)t \right\} \right. \]
+ P_\beta V^2 P_\beta \left( -i(\lambda_\alpha - \lambda_\beta)t \right) \times \left( \frac{-1 + \exp(-i(\lambda_\beta - \lambda_\rho)t)}{i(\lambda_\beta - \lambda_\rho)} \right)
- P_\alpha V_\beta x P_\rho x P_\rho \exp \left\{ i(\lambda_\beta - \lambda_\rho)t \right\} \times \left( \frac{-1 + \exp \left\{ i(\lambda_\alpha - \lambda_\beta + \lambda_\rho - \lambda_\mu)t \right\}}{i(\lambda_\beta - \lambda_\rho + \lambda_\rho - \lambda_\mu)} \right) \left\{ P_\alpha x P_\beta V^2 P_\rho \exp \left\{ i(\lambda_\beta - \lambda_\rho)t \right\} \right\}
- P_\alpha V_\beta x P_\rho x P_\rho \exp \left\{ i(\lambda_\beta - \lambda_\rho)t \right\} \equiv Q \text{(say)} \]
Further, the oscillatory term in this expression can be removed by constructing another time average over an infinite period of time. As a consequence, the resulting expression is a purely algebraic representation of the weighted time and ensemble average of the observable $X$. This algebraic representation is very useful in practical applications since it allows for the computation of the observable $X$ without having to perform time-consuming simulations or measurements over an infinite period of time. Additionally, it provides a clear and concise mathematical description of the behavior of $X$ under certain conditions. This expression is in terms of the eigenprojections of $H, X$ and the modulated noise potential $V$. Substituting for $V = \sum \theta_k V_k$ gives this weighted time average as a quadratic function of the parameter $\theta = \theta_k$, from which we can optimize $\theta \otimes \theta$. Estimates for the parameters $\theta_k$ can be derived from $Q$. This optimization can be used to obtain the optimal control fields that maximize or minimize the observable $X$. The resulting expression can also be used to study the dynamics of open quantum systems and their interactions with the environment.

$$Q = \sum \theta_k \theta_m \left\{ P_\alpha X P_\beta V_k V_m P_\alpha + P_\alpha V_k V_m P_\alpha X P_\alpha \
- P_\alpha V_k X P_\beta V_m P_\alpha - P_\alpha V_k P_\beta X P_\beta V_m P_\alpha \right\}$$

This gives $\theta \otimes \theta$ and the equation represents the calculation of the quantum correlation function for a given quantum state, where $\theta$ represents the measurement outcomes of two observables. The tensor product notation $\theta \otimes \theta$ indicates that the correlation function involves measurements on two separate quantum systems.

3 Conclusion

We compute the average value of an observable $X$ at time $t$ upto $O(\epsilon^2)$ and derive an approximate formula for this average in the limit of large $t$ and small $\epsilon$ using standard properties of expectation of functionals of Brownian motion. This method is useful for analyzing the behavior of physical systems in which small fluctuations are important. It allows us to make predictions about the long-term behavior of the system based on its initial conditions and the properties of Brownian motion. Quantum and classical statistical methods are combined to determine the average value of this observable at time $t$ for a variety of initial conditions. Quantum-classical averages of the observable are measured, and from this we derive an algorithm for estimating quadratic functions of the parameter vector. Moreover, by expressing the averages of these quantum observables over time as quadratic functions of the parameters, some concepts from ergodic theory are introduced. As an example, if $\theta$ is the parameter vector, then our estimate is $\theta \otimes \theta$. This approach has potential applications in fields such as machine learning and optimization, where quadratic functions
are commonly used. Additionally, the combination of quantum and classical statistical methods provides a powerful tool for studying complex systems that cannot be easily analyzed using classical methods alone.

**Ethics approval and consent to participate**

Not Applicable

**Human and Animal Ethics**

Not Applicable

**Data availability statement**

There are no associated data with my manuscript.

**Conflicts of interests**

On behalf of author, the corresponding author states that there is no conflict of interest.

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