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Zhonglin Chai (✉ chaizhonglin@163.com)

China Jiliang University <https://orcid.org/0000-0003-4675-7589>

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A study on orderings based on fuzzy quasi-order relations

Zhonglin Chai^{a,b}

^aCollege of Sciences, ^bKey Laboratory of Intelligent Manufacturing Quality Big Data Tracing and Analysis of Zhejiang Province, China Jiliang University, Hangzhou 310018, China

Abstract

This paper further studies orderings based on fuzzy quasi-order relations using fuzzy graph. Firstly, a fuzzy relation on a finite set is represented equivalently by a fuzzy graph. Using the graph, some new results on fuzzy relations are derived. In ranking those alternatives, we usually obtain a quasi-order relation, which often has inconsistencies, so it cannot be used for orderings directly. We need to remake it into a reasonable partial order relation for orderings. This paper studies these inconsistencies, and divides them into two types: framework inconsistencies and degree inconsistencies. For the former, a reasonable and feasible method is presented to eliminate them. To eliminate the latter, the concept of complete partial order relation is presented, which is more suitable than partial order relation to rank the alternatives. A method to obtain a reasonable complete partial order relation for a quasi-order relation is given also. An example is given as well to illustrate these discussions. Lastly, the paper discusses the connection between quasi-order relations and preference relations for orderings and some other related problems.

Keywords: Fuzzy quasi-order relation; Ordering; Partial order relation; Fuzzy graph

1 Introduction

In the real world, many things have connections or relationships between them, so we need to study them (Rosen 2008). When the connections have uncertain or fuzzy trait, we should use fuzzy theories to study them (Bodenhofer and Klawonn 2004). Fuzzy relations are a generalization of crisp relations, and form one of the primary substances of fuzzy theory. Zadeh is the pioneer of fuzzy theory, and the first man who studied fuzzy relations (Zadeh 1965, 1971). Since then, a lot of scholars have joined the queue of studying fuzzy relations (González and Marín 1997; Kundu 2000; Tan 2013; Zhang et al. 2020).

Relations and graphs both belong to discrete mathematics, possessing many common characteristics, so graphs can be used to study relations (Rosen 2008). Studying fuzzy relations needs fuzzy graphs, which were introduced by Rosenfeld (1975), and a lot of people have then interested in them (Bhutani and Rosenfeld 2003; Mathew and Sunitha 2009; Sameena and Sunitha 2006). Many results on fuzzy graphs have been gotten, and they have been used in many domains (Peng and Sun 2007; Hu 2014; Binu et al. 2019). Naturally, some scholars have used graphs to study fuzzy relations (Imai et al. 2000; Kundu 2000; Mathew and Sunitha 2010). Although as it is, some connections between fuzzy graphs and relations still need to study.

In fuzzy relations, partial order relations are an important content of study as they can be used to rank alternatives (Kundu 2000; Bodenhofer and Klawonn 2004; Peng and Sun 2007). In practical problems, however, the relation obtained often is a quasi-order relation rather than a partial order relation (Liang et al. 2005; Ma et al. 2006), so how to use the relation to obtain a reasonable partial order relation for orderings has long been a problem. Besides, many scholars have focused their interest on studying preference relations and using them for orderings (Viedma et al. 2004; Xu et al. 2013; Zhang 2019).

The paper aims to use fuzzy graph to study fuzzy quasi-order relations, and then use the derived results to study orderings based on them. The paper is organized as follows. Section 2 reviews some basic definitions and results of fuzzy relations. Section 3 gives some basic definitions and results of fuzzy graphs. The graphic representation of fuzzy relations is also given. Section 4 studies fuzzy quasi-order relations using fuzzy graph. Section 5 studies how to obtain a reasonable fuzzy partial order relation for ordering by a quasi-order relation. Firstly, it gives a method to obtain a reasonable partially ordered fuzzy quasi-order relation for a non-partially ordered quasi-order relation. Second, it gives a method to obtain a reasonable fuzzy partial order relation for a partially ordered quasi-order relation. An example is given as well to show the studies. Lastly, some discussions about the connection between quasi-order relations and preference relations for orderings and some other related problems are given.

2 Definitions and some results on fuzzy relations

In this paper, X always denotes the finite set of alternatives $\{x_1, x_2, \dots, x_n\}$ ($n \geq 2$), where x_i denotes the i th alternative; moreover, any relation is fuzzy in the following.

Let R be a relation on X , which can be represented by a fuzzy matrix $R=(r_{ij})_{n \times n} \in F(X \times X)$ (simply by (r_{ij})). Thus, some well known definitions and results about relations are given according to fuzzy matrices.

Definition 2.1(Zadeh 1971;Hu 2014). Let $R=(r_{ij})$ be a fuzzy relation on X , R is called

- (1) Reflexive iff $r_{ii}=1$ for any i ,
- (2) Symmetric iff $r_{ij}=r_{ji}$ for any i, j ,
- (3) Antisymmetric iff for any $i \neq j$ there is $r_{ji}=0$ when $r_{ij} > 0$,
- (4) Transitive iff $R^2 \subseteq R$.

Notice: here the operators in computing R^2 are the two famous operators \wedge and \vee .

Definition 2.2(Zadeh 1971;Hu 2014). Let R be a fuzzy relation on X ,

- (1) R is a quasi-order relation iff it is reflexive and antisymmetric,
- (2) R is a partial order relation iff it is reflexive, antisymmetric, and transitive.

Definition 2.3(Zadeh 1971;Hu 2014). Let R be a fuzzy relation on X , a relation A containing R is transitive, and for any relation B containing R with transitivity there always is $B \supseteq A$, then A is called the transitive closure of R , denoted by R^t .

Theorem 2.1(Imai et al. 2000;Peng and Sun 2007;Hu 2014). Let R be a fuzzy relation on X with reflexivity, then

- (1) R^k is reflexive for any positive integer k ,
- (2) There is $R^s \subseteq R^k$ for any two positive integers $s \leq k$,
- (3) There exists an integer $k \leq n-1$ such that $R^k = R^{k+1} = \dots = R^t$.

From Definitions 2.1, 2.2 and Theorem 2.1 there is

Corollary 2.1. A sufficient and necessary condition that a quasi-order relation R on X is partial order is that $R=R^2$.

Let R be a quasi-order relation. If R^t still is antisymmetric, R^t is a partial order relation by Definitions 2.1, 2.2 and Theorem 2.1, so

Definition 2.4(Peng and Sun 2007;Hu 2014). Let R be a fuzzy quasi-order relation on X , if R^t still is antisymmetric, R is called partially ordered.

Notice: As this paper mainly discusses order relations for the given matrix $R=(r_{ij})$, so we need

further give some explanations about it. Suppose r_{ij} ($i \neq j$) denotes the degree of x_i superior to x_j when comparing them, i.e., $r_{ij}=0$ when x_i is not superior (or is inferior) to x_j or they can not be compared, $r_{ij}>0$ when x_i is superior to x_j (denoted by $x_i > x_j$), and the greater the value, the greater the degree, and $r_{ij}=1$ means x_i is completely superior to x_j .

3 Definitions and some results of fuzzy graphs

In this section, we give the graphic representation of fuzzy relations and review some definitions of fuzzy graphs.

Definition 3.1(Peng and Sun 2007;Rosen 2008;Ghosh et al. 2010). A fuzzy graph is a weighted graph, which can be undirected or directed, but the weight must be in $(0, 1]$.

For a fuzzy relation $R=(r_{ij})$ on X , if some r_{ij} are different from r_{ji} , it can be represented equivalently by a directed graph, where X is the vertex set. If $r_{ij} \neq 0$, an arrow is used pointing from x_i to x_j to indicate the directed edge from x_i to x_j , and r_{ij} is the weight. If $r_{ij} = 0$, such an edge does not exist.

For two vertices x_i and x_j of the graph of a relation $R=(r_{ij})$, if x_i is adjacent to x_j , the edge is denoted by (x_i, x_j) . Sometimes (x_i, x_j) is represented by r_{ij} when there is no danger of confusion.

When R is symmetric, we need not differentiate between the two edges r_{ij} and r_{ji} , so R can be represented equivalently by an undirected graph, where X is the vertex set, x_i is adjacent x_j when $r_{ij} > 0$ or $r_{ji} > 0$, and the edge is denoted by $\{x_i, x_j\}$ or $\{x_j, x_i\}$.

Conversely, a fuzzy graph G can be represented equivalently by the form (X, R) (here we do not consider the situation of multi-graphs), where X is the vertex set, R is the relation derived from G . If G is directed, the R is undoubted by the graphic representation of fuzzy relations. If G is undirected, we use a symmetric matrix to represent the relation, so the R is also clear. Thus in the following, we always use $G=(X, R)$ (sometimes G is used only) to represent the fuzzy graph (simply called graph). Besides, as this paper mainly discusses order relations, so any graph is directed in the following.

For a relation R , its graph G is called the graph of R , denoted by G_R ; for a graph $G=(X, R)$, R is called the relation of graph G , denoted by R_G .

Definition 3.2(Rosen 2008;Mathew and Sunitha 2009;Ghosh et al. 2010). Let $G=(X, R)$ be a fuzzy graph, the vertex sequence x_0, x_1, \dots, x_k ($k > 0$), in which x_{i-1} is adjacent to x_i ($i=1, 2, \dots, k$), is called a walk, where x_0 and x_k are called the initial vertex and terminal vertex of the walk, respectively, k is called the length of the walk.

Definition 3.3(Rosen 2008;Mathew and Sunitha 2009;Ghosh et al. 2010). Let $G=(X, R)$ be a fuzzy graph, x_0, x_1, \dots, x_k a walk in G , if the x_i are distinct (x_0 and x_k are possible the same), it is called a path from x_0 to x_k . If x_0 and x_k are the same and $k=1$, it is called a loop; if x_0 and x_k are the same and $k \geq 2$, it is called a closed path or cycle.

Definition 3.4(Peng and Sun 2007;Mathew and Sunitha 2009;Hu 2014). Let $G=(X, R)$ be a fuzzy graph, x_0, x_1, \dots, x_k a walk W in G , denote $S(W) = \bigwedge_{i=1}^k r_{i-1i}$, $S(W)$ is called the strength of W .

Definition 3.5(Peng and Sun 2007;Hu 2014). Let $G=(X, R)$ be a fuzzy graph, $x_i, x_j \in X$, P_1, P_2, \dots, P_l are all paths in G from x_i to x_j , denote $S_{ij} = \bigvee_{i=1}^l S(P_i)$, S_{ij} is called the connectivity strength from x_i to x_j in G .

Definition 3.6(Peng and Sun 2007;Hu 2014). Let $G=(X, R)$ be a fuzzy graph, P a path from x_i to x_j , if $S(P)=S_{ij}$, P is called a strong path from x_i to x_j in G .

Besides, quasi-order relations are reflexive, indicating that their graphs have a loop at each vertex weighting 1.

4 Study fuzzy relations using fuzzy graph

Now we use fuzzy graphs to study quasi-order relations.

Theorem 4.1. A sufficient and necessary condition that fuzzy quasi-order relation $R=(r_{ij})$ on X is partially ordered is that G_R has no cycles.

Proof. As a cycle has at least two different vertices from Definition 3.3, so we only consider any two different alternatives x_i, x_j .

Sufficiency. Suppose there is a positive integer $k \leq n-1$ such that $R^k=R^t$ from Theorem 2.1(3). Let $R^k=(r_{ij}^k)$, then there is (Zadeh 1965;Peng and Sun 2007)

$$r_{ij}^k = \bigvee_{j_1, j_2, \dots, j_{k-1}} (r_{ij_1} \wedge r_{j_1 j_2} \wedge \dots \wedge r_{j_{k-1} j}) \quad (4.1)$$

From Definition 2.1(3) if $r_{ji} > 0$, then $r_{ij} = 0$. From Theorem 2.1(2) there still is $r_{ji}^k > 0$. If $r_{ij}^k > 0$, then by (4.1) there are edges $r_{ij_1}, r_{j_1 j_2}, \dots, r_{j_{k-1} j}$ such that $r_{ij_1} > 0, r_{j_1 j_2} > 0, \dots, r_{j_{k-1} j} > 0$. These edges together with r_{ji} form a closed walk, so a cycle exists, a contradiction, thus $r_{ij}^k = 0$. If $r_{ij} = r_{ji} = 0$, we consider r_{ij}^k and r_{ji}^k , by a similar way we see that they cannot be both greater than 0. These show that R^t still is antisymmetric, hence R is partially ordered from definition 2.4.

Necessity. Suppose R is partially ordered. If G_R had a cycle, in which the edges are $r_{ij_1}, r_{j_1 j_2}, \dots, r_{j_{k-1} j}$ ($1 \leq k \leq n-1$ by Definition 3.3), we consider R^k . From Theorem 2.1(2), (4.1), and $k \leq n-1$ there are $r_{ij_1}^k > 0, r_{j_1 j_2}^k > 0, \dots, r_{j_{k-1} j}^k > 0$. This shows that R^t is not antisymmetric from Theorem 2.1(3), a contradiction. This completes the proof. \square

Theorem 4.2. A sufficient and necessary condition that quasi-order relation $R=(r_{ij})$ on X is partial order is that if there are paths from x_i to x_j in G_R , then x_i is adjacent to x_j , and the edge (x_i, x_j) is a strong path from x_i to x_j in G_R .

Proof. Obviously, we only need to consider the case that $x_i \neq x_j$.

Sufficiency. For any x_i and x_j , if there are no paths from x_i to x_j , then there are $r_{ij} = 0$ and $\bigvee_{k=1}^n (r_{ik} \wedge r_{kj}) = 0$. If there are paths from x_i to x_j , as the edge (x_i, x_j) is a strong path, then for any k , there is $r_{ij} \geq r_{ik} \wedge r_{kj}$ from Definitions 3.4-3.6. Hence for both cases there is $r_{ij} \geq \bigvee_{k=1}^n (r_{ik} \wedge r_{kj})$, thus $R^2 \subseteq R$. So R is partial order from Definitions 2.1, 2.2.

Necessity. If R is partial order, from Corollary 2.1 there is $R=R^{n-1}$, i.e.,

$$r_{ij} = \bigvee_{j_1, j_2, \dots, j_{n-2}} (r_{ij_1} \wedge r_{j_1 j_2} \wedge \dots \wedge r_{j_{n-2} j}) \quad (4.2)$$

For any x_i and x_j , if there are no paths from x_i to x_j , then $r_{ij} = 0$ by (4.2) obviously. Now consider the case that there are paths from x_i to x_j . For a path in G_R , the connectivity strength of a walk which is obtained from it by adding some vertices through loops is unchanged. So for a path, there

is a walk $r_{i_1} \wedge r_{j_1 j_2} \wedge \dots \wedge r_{j_{n-2} j}$ whose strength is equal to that of the path. Thus, $r_{ij} > 0$ from (4.2).

One sees that the strength of any path is not greater than r_{ij} , so x_i is adjacent to x_j , and the edge (x_i, x_j) is a strong path from x_i to x_j . This completes the proof. \square

5 Find a reasonable partial order relation for a quasi-order relation

In real problems, we often need to rank some alternatives, so partial order relations are needed as they can be used for orderings. When we are deriving such a relation, however, due to the availability and uncertainty of information as well as the vagueness of human feeling and recognition, the entry in the obtained relation often is inaccurate or approximate (Liang et al. 2005; Ma et al. 2006). Thus, we need a partial order relation but the harvest often is a quasi-order relation. Moreover, in the relation, the phenomenon of inconsistencies may appear. Ma et al. (2006) and Xu et al. (2013) discussed it mainly for the case that $r_{ij} > 0$, $r_{jk} > 0$, and $r_{ki} > 0$. In fact it has no essential difference with the case that $r_{ij} > 0$ and $r_{ji} > 0$, and that $r_{ij} > 0$, $r_{jk} > 0$, $r_{ks} > 0$, and $r_{si} > 0$ is only a more complex case. In real problems, the case that $r_{ij} > 0$ and $r_{ji} > 0$ can be avoided, but the phenomenon of inconsistencies is difficult to avoid. Inconsistency is a contradiction phenomenon among the alternatives. Now, let's discuss it.

Definition 5.1. For quasi-order relation $R=(r_{ij})$ on X , if there are some different alternatives x_i of X such that $r_{i_1} > 0, r_{j_1 j_2} > 0, \dots, r_{j_k i} > 0 (k > 0)$, R is called inconsistency.

How to test that a quasi-order relation R is inconsistency? From Definition 5.1 when R is inconsistency, cycles in G_R exist, so from Theorem 4.1 there is

Theorem 5.1. A sufficiency and necessary condition that quasi-order relation $R=(r_{ij})$ on X is inconsistent is that its graph G_R has cycles, it is non-partially ordered, or its transitive closure R^t is not antisymmetric.

In orderings by quasi-order relation $R=(r_{ij})$ on X , a common used method (Peng and Sun 2007; Hu 2014) is computing its transitive closure R^t firstly. If it still is antisymmetric, it is partial order, so it can be used for orderings. Otherwise, use a new relation, denoted by $A=(a_{ij})$, where

$$a_{ij} = \begin{cases} r^t_{ij} - r^t_{ji}, & \text{if } i \neq j, r^t_{ij} > r^t_{ji} \\ 0, & \text{if } i \neq j, r^t_{ij} \leq r^t_{ji} \\ r^t_{ij}, & \text{if } i = j \end{cases} \quad (5.1)$$

It is a partial order relation, can be used for orderings.

For a non-partially ordered quasi-order relation $R=(r_{ij})$ on X , i.e., its G_R has cycles, we can convert it into a partially ordered quasi-order relation by cutting some edges for each cycle, i.e., let the corresponding r_{ij} be 0. Obviously, the manner of preserving the original information of R as much as possible is cutting edges as little as possible, and the most reasonable manner of cutting edges is cutting the edge having the least weight among the edges for each cycle. Thus, the best manner of converting a quasi-order relation with inconsistencies into a quasi-order relation with consistency is cutting only one edge having the least weight for each cycle. Besides, there is another manner to eliminate those inconsistencies: let the weights of all the edges in a cycle decrease by the number of the smallest weight of those weights (so the edge having the smallest weight is cut naturally). For these two ways, the latter is poorer as the loss of the original

information of R for the former is less than that of the latter.

Now we consider the former way: cut an edge having the smallest weight for each cycle in the graph of R . But how to know that an edge in a cycle is the one having the smallest weight to the cycle? For this, we have the following theorems.

Theorem 5.2. Let $R=(r_{ij})$ be a fuzzy quasi-order relation on X whose graph G_R has only one cycle. Then for the edge $r_{ij}(>0)$ in the cycle having the smallest weight to the cycle, there is $r'_{ij} \leq r'_{ji}$, and for any other edges r_{ij} in G_R , there are $r'_{ij} > r'_{ji}$.

Proof. Consider an edge $r_{ij}(>0)$ in G_R . From Theorem 2.1(2)(3) there is $r'_{ij} > 0$. Suppose r_{ij} is not in the cycle. If $r'_{ji} > 0$, then there will be a path from x_j to x_i from (4.1). This path together with r_{ij} forms a cycle. It is a new cycle, a contradiction, so there is $r'_{ji} = 0$, then $r'_{ij} > r'_{ji}$.

Let this cycle be $x_0, x_1, \dots, x_k (x_0=x_k, k>2)$, consider r_{ij} is in it. First consider that r_{ij} has the smallest weight to it, say r_{01} . Consider the path from x_0 to x_1 . Obviously, r_{01} is a path. If there is another path from x_0 to x_1 , then this path together with x_1, \dots, x_k forms a new cycle, a contradiction. This shows that r_{01} is the only path from x_0 to x_1 , so $r'_{01} = r_{01}$ by the proof of Theorem 4.2. Now consider r'_{10} . Obviously, x_1, \dots, x_k is a path from x_1 to x_0 , one sees that the strength of this path is not less than r_{01} , so $r'_{10} \geq r_{01} = r'_{01}$ by the proof of Theorem 4.2.

Now consider that r_{ij} is in the cycle not having the smallest weight, still suppose that it is r_{01} . Suppose that the smallest weight of edges in x_0, x_1, \dots, x_k is a , then $r_{01} > a$. There is $r'_{01} > a$ obviously. Consider r'_{10} . By a similar analysis as above we know that x_1, \dots, x_k is the only path from x_1 to x_0 , and the edge with the smallest weight in the cycle is in this path, so $r'_{10} = a < r'_{01}$. This completes the proof. \square

For quasi-order relation $R=(r_{ij})$, when G_R has many cycles but any two of them do not intersect, the conclusions of Theorem 5.2 still hold obviously. We can use $r_{ij} > 0$ and $r'_{ij} \leq r'_{ji}$ to select the edge having the smallest weight in the cycle. When G_R has many cycles and at least two cycles intersect, we have the following theorem.

Theorem 5.3. Let $R=(r_{ij})$ be a fuzzy quasi-order relation on X whose graph G_R has cycles, then there is at least one edge in the cycles such that $r_{ij} > 0$ and $r'_{ij} \leq r'_{ji}$.

Proof. Consider a cycle $x_0, x_1, \dots, x_k (x_0=x_k, k>2)$ of G_R . Obviously for each edge r_{ij} in it, there is $r_{ij} > 0$. By the proof of Theorem 5.2 there are $r'_{ij} > 0$ and $r'_{ji} > 0$. For these r'_{ij} , they have their minimum, say r'_{01} . Of course, the transitive closure of R' still is R' . This shows that for any r_{ij} in the cycle, r'_{ij} is a strong path in $G_{R'}$, the graph of R' , from x_i to x_j by (4.1). As r'_{01} is the minimum of these r'_{ij} , and they form a cycle, there must be $r'_{10} \geq r'_{01}$ by the proof of Theorem 5.2. Obviously $r_{01} > 0$, and this completes the proof. \square

Based on Theorems 5.2, 5.3, we give an algorithm to obtain a reasonable partially ordered quasi-order relation for a non-partially ordered quasi-order relation.

Algorithm 1.

Step 1. For quasi-order relation $R=(r_{ij})$, compute $R'=(r'_{ij})$.

Step 2. If R' is antisymmetric, stop. Otherwise, go to the next step.

Step 3. If r_{ij} satisfies $r_{ij} > 0$, and $r'_{ij} \leq r'_{ji}$, and it is the edge whose weight has the smallest value among those edges having these properties, replace it with 0 (the obtained relation is still denoted by R for convenience), then go to Step 1.

Now we see an example.

Example 1. Let $X=\{x_1, x_2, x_3, x_4, x_5\}$, consider quasi-order relation

$$R = \begin{pmatrix} 1 & .7 & .3 & .37 & .49 \\ 0 & 1 & 0 & .05 & 0 \\ 0 & .3 & 1 & 0 & 0 \\ 0 & 0 & .16 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix} \text{ on } X. \text{ The transitive closure of } R \text{ is}$$

$$R^t = \begin{pmatrix} 1 & .7 & .36 & .45 & .49 \\ 0 & 1 & .05 & .05 & 0 \\ 0 & .3 & 1 & .05 & 0 \\ 0 & .16 & .16 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix}. R^t \text{ is not antisymmetric, showing there are cycles in } G_R. \text{ From}$$

R^t for the three elements x_2, x_3, x_4 , there are $r_{ij}^t > 0$ and $r_{ji}^t > 0$, but it is not for x_1, x_5 . So by the proof of Theorem 5.2 we know that x_1, x_5 are not in the cycle, but x_2, x_3, x_4 are in (see Fig.1). In fact for this example, the cycle is unique. From R the cycle is formed by the three edges r_{24}, r_{43} and r_{32} . For the three edges, only r_{24} satisfies $r_{ij} > 0$ and $r_{ij}^t \leq r_{ji}^t$, so r_{24} is the unique edge with the smallest weight in the cycle. Cut it, i.e., replace r_{24} with 0, and still denote the new relation by R . Now,

$$R = \begin{pmatrix} 1 & .7 & .3 & .37 & .49 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & 1 & 0 & 0 \\ 0 & 0 & .16 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix}, \text{ and it is easy to know it is partially ordered.}$$

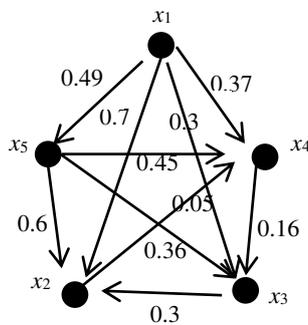


Fig. 1. Graph of R of Example 1

When quasi-order relation $R=(r_{ij})$ on X has inconsistencies, i.e., its graph has cycles, for any two distinct x_i and x_j in a cycle of G_R , there are $r_{ij}^t > 0$ and $r_{ji}^t > 0$, so all these r_{ij}^t will be changed when using (5.1). Thus the method of (5.1) is just similar to the above latter way to eliminate the inconsistency, so the original information of R is changed greatly.

Consider this method and Example 1 again, by (5.1) the obtained relation is

$$A = \begin{pmatrix} 1 & .7 & .36 & .45 & .49 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & .25 & 1 & 0 & 0 \\ 0 & .11 & .11 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix} . \quad \text{It is the transitive closure of}$$

$$R_1 = \begin{pmatrix} 1 & .7 & .3 & .37 & .49 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & .25 & 1 & 0 & 0 \\ 0 & 0 & .11 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix} . \quad \text{Compared with the initial } R, \text{ the } 0.05 \text{ is converted into } 0, \text{ at}$$

the same time 0.3 and 0.16 are converted into 0.25 and 0.11 respectively. Obviously, it is unreasonable.

Now we only consider partially ordered quasi-order relations. As given in Sec.2, we use r_{ij} ($i \neq j$) to denote the degree of x_i superior to x_j . Obviously from the point of logic, for three different elements x_i, x_j, x_k , if x_i is superior to x_j and x_j is superior to x_k , then x_i should be more superior to x_k . That is, if $r_{ij} > 0$ and $r_{jk} > 0$, then there should be $r_{ik} \geq r_{ij}$ and $r_{ik} \geq r_{jk}$, i.e., $r_{ik} \geq r_{ij} \vee r_{jk}$. Consider it generally, we give

Definition 5.2. Let $R=(r_{ij})$ be a fuzzy partial order relation on X , if for any path x_1, x_2, \dots, x_k of G_R , there always is $r_{1k} \geq \bigvee_{i=2}^k r_{i-1i}$, R is called a complete fuzzy partial order relation.

Example 1 (continued).

Consider Example1 continuously and the last R . As $r_{42}=0$, but $r_{43}=0.16$, $r_{32}=0.3$, so R is not a complete partial order relation. Even we consider its transitive closure

$$R^t = \begin{pmatrix} 1 & .7 & .36 & .45 & .49 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & 1 & 0 & 0 \\ 0 & .16 & .16 & 1 & 0 \\ 0 & .6 & .36 & .45 & 1 \end{pmatrix} , \quad \text{which is partial order, it still is not complete partial order as}$$

$r_{13}=0.36$, but $r_{15}=0.49$, $r_{53}=0.36$. This means we can not obtained complete partial order relations by transitive closure method.

The inconsistency defined by Definition 5.1 for quasi-order relation $R=(r_{ij})$ on X should be called framework inconsistency as it reflects there are framework or order contradictions among some elements of X , and the inconsistency here should be called degree inconsistency as it reflects there are degree contradictions among some elements of X . A partial order relation on X means there are no framework contradictions on the elements of X . Although it can be used for orderings, it does not mean there are not degree contradictions among the elements of X . However, it is not for complete partial order relations. So for a quasi-order relation, we should find a reasonable complete partial order relation to show the order relations of the elements of X . To complete this work, we need some preparations.

Definition 5.3. Let $G=(X, R)$ be a fuzzy graph, x_0, x_1, \dots, x_k a path P in G , denote

$SS(P) = \bigvee_{i=1}^k r_{i-i}$, $SS(P)$ is called the strong strength of P .

Definition 5.4. Let $G=(X, R)$ be a fuzzy graph, $x_i, x_j \in X$, P_1, P_2, \dots, P_l are all paths in G_R from x_i to x_j , denote $SS_{i,j} = \bigvee_{i=1}^l SS(P_i)$, SS_{ij} is called the strong connectivity strength in G from x_i to x_j . If there are no paths from x_i to x_j , $SS_{ij}=0$.

Theorem 5.4. Let $R=(r_{ij})$ be a fuzzy partially ordered quasi-order relation on X , $R^s=(s_{ij})$ a fuzzy relation on X in which each s_{ij} is determined by the SS_{ij} of R , then

- (1) $R^s=(s_{ij})$ is a complete partial order relation;
- (2) It is the least complete partial order relation among complete partial order relations on X containing $R=(r_{ij})$.

Proof. (1) For any i , as $r_{ii}=1$ and loops belong to paths, so there is $s_{ii}=1$ by Definitions 5.3,5.4. Consider $i \neq j$. If $s_{ij}>0$, then $s_{ji}=0$, otherwise there will exist a contradiction with R being partially ordered or its graph having no cycles by the proof of Theorem 4.1, so R^s is quasi-order. Consider $s_{ij}>0$, suppose there are a path P of the graph of R^s from x_i to x_j and an edge (x_k, x_s) belonging to P such that $s_{ks}>s_{ij}$, by Definitions 5.3,5.4 there will exist a path P_1 of the graph of R from x_k to x_s , and a path P_2 from x_i to x_j and the former is a part of the latter such that $SS(P_1) > SS(P_2)$, a contradiction with Definition 5.3. Thus, for any path P of the graph of R^s from x_i to x_j and any edge (x_k, x_s) belonging to P , there is $s_{ks} \leq s_{ij}$. Also consider the edge s_{ij} itself, R^s is partial order by the proof of Theorem 4.2. Besides, by Definition 5.2 and $s_{ks} \leq s_{ij}$ one sees that R^s is also complete.

(2) For any $i \neq j$, if $r_{ij}=0$, $s_{ij} \geq r_{ij}$ certainly. If $r_{ij}>0$, there still is $s_{ij} \geq r_{ij}$ by Definitions 5.3,5.4 as r_{ij} is an edge of G_R , so $R^s \supseteq R$. Let $B=(b_{ij})$ be a complete partial order relation on X containing R , then there is $b_{ij} \geq r_{ij}$ for any i, j . By Definitions 5.2-5.4 there is $b_{ij} \geq s_{ij}$, so $B \supseteq R^s$. This completes the proof. \square

Thus for a quasi-order relation $R=(r_{ij})$ on X , if it is non-partially ordered, we can use Algorithm 1 to let it become a partially ordered one. For a partially ordered quasi-order relation, we should use its R^s for orderings. So the problem becomes for a partially ordered quasi-order relation $R=(r_{ij})$ on X , how to find the R^s . We introduce the following operator θ : for any $a, b \in [0,1]$, define

$$a\theta b = \begin{cases} 0, & \text{if } a \wedge b = 0 \\ a \vee b, & \text{if } a \wedge b > 0 \end{cases}$$

Definition 5.5. Let $R=(r_{ij})$ be a fuzzy relation on X , denote $R^{2(\theta)}=R \circ_{\theta} R=(r^{2(\theta)}_{ij})$, in which $R^{2(\theta)}$ is defined as follows: $r^{2(\theta)}_{ij} = \bigvee_{k=1}^n (r_{ik} \theta r_{kj})$, i.e., the composition is θ and \vee . Besides, let $R^{k(\theta)}=R \circ_{\theta} R^{k-1(\theta)}$, $k>2$.

Let $R=(r_{ij})$ be a partially ordered quasi-order relation on X , $R_0=R-I=(r_{0ij})$, in which I is the identity matrix on X . Consider $R_0^{k(\theta)}=(r_0^{k(\theta)}_{ij})$, $k>1$. By Definitions 5.3,5.4 and the proof of Theorem 4.1 one sees that $r_0^{k(\theta)}_{ij}$ is the maximum of strong strengths of all paths whose length is k from x_i to x_j ($i \neq j$), thus R^s can be obtained as follows $R^s = I \cup R_0 \cup R_0^{2(\theta)} \cup \dots \cup R_0^{n-1(\theta)}$.

Example 1 (continued).

Consider Example 1 continuously and the last R . By computation there is

$$R^s = \begin{pmatrix} 1 & .7 & .49 & .49 & .49 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & .3 & 1 & 0 & 0 \\ 0 & .3 & .16 & 1 & 0 \\ 0 & .6 & .45 & .45 & 1 \end{pmatrix}. \text{ It is the least complete partial order relation containing } R,$$

suitable for orderings for the elements of X .

6 Conclusions

This paper further studies orderings based on quasi-order relations by fuzzy graph. For a given quasi-order relation come from reality, it often has inconsistencies, not only in the framework, but also in the numbers, so we need to obtain a reasonable partial order relation close to it for orderings. This paper studies how to eliminate these inconsistencies reasonably, presents methods to derive a reasonable partial order relation about the relation. The given example shows that the results are satisfactory. In pairwise comparisons, the relation given may not be the form of quasi-order relation $R=(r_{ij})$, but the form of preference relation $A=(a_{ij})$ which satisfies additive reciprocal: $a_{ij}+a_{ji}=1$ for any i, j (Kundu 2000;Ma et al. 2006). But the two forms can be converted

$$\text{for one another by } r_{ij} = \begin{cases} 2a_{ij} - 1, & \text{if } i \neq j, a_{ij} \geq 0.5 \\ 0, & \text{if } i \neq j, a_{ij} < 0.5 \\ 1, & \text{if } i = j, \end{cases} \text{ or } a_{ij} = \begin{cases} (r_{ij} + 1) / 2, & \text{if } i \neq j, r_{ij} > 0 \\ 1 - a_{ji}, & \text{if } i \neq j, r_{ij} > 0 \\ 0.5, & \text{if } i = j \text{ or } r_{ij} = r_{ji}, \end{cases}. \text{ Thus,}$$

the orderings methods of this paper also work for preference relations. The given preference relation A may satisfies multiplicative reciprocal: $a_{ij} \cdot a_{ji} = 1$ for any i, j (Viedma et al. 2004). But the two forms of reciprocal can be converted for one another (Viedma et al. 2004), so this paper's methods also work for this type of preference relations.

Besides, the $r_{ij}(i \neq j)$ in the given relation may not represent the degree of x_i superior to x_j , but represent the degree of x_i inferior to x_j . In this case above methods still work, we only need to see that each non-zero number in the last relation denotes the degree of the first element inferior to the second one.

In giving a quasi-order relation of the alternatives, the decision maker may be unable to give the values of some r_{ij} . If such numbers are rare, he can give them zero value, by transitivity operation or Theorem 5.4 for quasi-order relations these values can be given naturally, i.e., they can be determined by other given values r_{ij} . Certainly, sometimes the given quasi-order relation may be unreasonable, in such a case we should ask the decision maker to give a new one. But how to know it is? One sees when the graph of the given relation has cycles and we want to eliminate them by cutting some edges using the above method, if the value of one cut edge is great or not small, or such edges are too many, we can deem it is. Of course, we can change it into a preference relation and use the method for it to test (Xu and Wang 2013).

Compliance with ethical standards

Conflict of interest The author declares that he has no conflicts of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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Figures

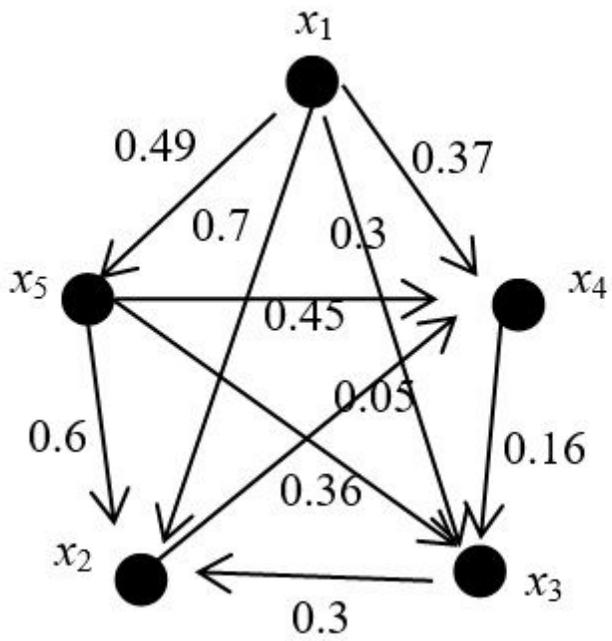


Figure 1

Graph of R of Example 1