Stochastic dual epidemic hypothesis model with Ornstein-Uhlenbeck process: Analysis and numerical simulations with SARS-CoV-2 variants

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Research Article

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Posted Date: March 13th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2669967/v1

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Additional Declarations: No competing interests reported.
Stochastic dual epidemic hypothesis model with Ornstein-Uhlenbeck process: Analysis and numerical simulations with SARS-CoV-2 variants

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Abstract

As it is widely known, the spread of infectious diseases can result in significant socioeconomic consequences and pose a threat to public health. However, biologically plausible models that incorporate stochastic interference and dual epidemic hypotheses have received limited attention. This paper aims to bridge this gap by examining a stochastic dual epidemic hypothesis model that incorporates Ornstein-Uhlenbeck processes to perturb the nonlinear incidence rate. We provide a rigorous analysis of the model, first proving the existence and uniqueness of global solution. We also analyze sufficient conditions for the extinction or persistence of each disease. Additionally, we establish the existence of an ergodic stationary distribution and derive expressions for the normal distribution followed by the global solution around the endemic equilibrium. Finally, using several numerical experimental examples, we validate our theoretical results with the case of dual variants of SARS-CoV-2 that can simultaneously infect humans. This paper contributes to the understanding of stochastic dual epidemic hypotheses and provides a foundation for future research in this field.

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1. Introduction

Epidemiological models play a vital role in studying the transmission of diseases in populations. They are a powerful tool for qualitative and quantitative analysis of the spread and control of infectious diseases \cite{1, 2, 3, 4}. The research results obtained from epidemiological models help in predicting the trends in the development of infectious diseases, identifying the key factors influencing their spread, and finding optimal strategies for prevention and control of infectious diseases. In summary, epidemiological models serve as a valuable tool for understanding and addressing the challenges posed by infectious diseases. Li et al. investigated an SIVS epidemic model with age structure and discussed the local and global stability of the equilibria \cite{5}. Lu et al. studied a SIRS epidemic model, in which the infection function first increases to its maximum value when a new infectious disease appears, and then decreases due to psychological effects, ultimately reaching a saturated level due to psychological effect \cite{6}.

In particular, Meng et al. \cite{7} proposed the following mathematical model with nonlinear incidence rate and dual epidemic hypothesis, involving susceptible individuals $S$ and infected individuals $I_1$ and $I_2$ caused by different viruses:

\begin{equation}
\begin{aligned}
\frac{dS(t)}{dt} &= A - dS(t) - \frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} + r_1 I_1(t) + r_2 I_2(t), \\
\frac{dI_1(t)}{dt} &= \frac{\beta_1 S(t) I_1(t)}{a_1 + I_1(t)} - (d + \alpha_1 + r_1) I_1(t), \\
\frac{dI_2(t)}{dt} &= \frac{\beta_2 S(t) I_2(t)}{a_2 + I_2(t)} - (d + \alpha_2 + r_2) I_2(t),
\end{aligned}
\end{equation}

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with initial value  
\[ S(0) > 0, \quad I_1(0) > 0, \quad \text{and} \quad I_2(0) > 0, \]
where the associated state variables and parameters are well described in Table 1.1. According to the [7], for
deterministic model (1.1), the relevant theoretical results are as follows

- If \( R_1 = \frac{A \beta_1}{a_1 d + \alpha_1 + r_1} < 1 \) and \( R_2 = \frac{A \beta_2}{a_2 d + \alpha_2 + r_2} < 1 \), then the two diseases go extinct and system (1.1) has a unique stable diseases-extinction equilibrium point \( E_0 = \left( \frac{A}{d}, 0, 0 \right) \).

- If \( R_1 > 1 \) and \( R_2 < 1 \), then the disease \( I_2 \) goes extinct and system (1.1) has a unique stable equilibrium \( E_1 = \left( \frac{(d + \alpha_1 + r_1)(\alpha_1 + I_1)}{\beta_1}, I_1^*, 0 \right) \) with \( I_1^* = \frac{\beta_1 A - a_1 d + \alpha_1 + r_1}{d + \alpha_1 + r_1 + \beta_1} \).

- If \( R_1 < 1 \) and \( R_2 > 1 \), then the disease \( I_1 \) goes extinct and system (1.1) has a unique stable equilibrium \( E_1 = \left( \frac{(d + \alpha_2 + r_2)(\alpha_2 + I_2)}{\beta_2}, 0, I_2^* \right) \) with \( I_2^* = \frac{\beta_2 A - a_2 d + \alpha_2 + r_2}{d + \alpha_2 + r_2 + \beta_2} \).

- If \( R_1 > 1, R_2 > 1, P_1 > 0 \) and \( P_2 > 0 \), then system (1.1) a unique stable positive equilibrium

\[
E^{**} = (S^{**}, I_1^{**}, I_2^{**}) = \left( \frac{A + a_1 (d + \alpha_1) + a_2 (d + \alpha_2)}{d + \frac{a_1 (d + \alpha_2)}{\beta_1 d + \alpha_1 + r_1} + \frac{a_2 (d + \alpha_2)}{\beta_2 d + \alpha_2 + r_2}}, \frac{P_1}{P_3}, \frac{P_2}{P_1} \right),
\]

where

\[
\begin{align*}
P_1 &= \beta_1 (d + \alpha_2 + r_2) \left[ A + a_2 (d + \alpha_2) \right] - a_1 d (d + \alpha_1 + r_1) (d + \alpha_2 + r_2) - a_1 \beta_2 (d + \alpha_1 + r_1) (d + \alpha_2), \\
P_2 &= \beta_2 (d + \alpha_1 + r_1) \left[ A + a_1 (d + \alpha_1) \right] - a_2 d (d + \alpha_2 + r_2) (d + \alpha_1 + r_1) - a_2 \beta_1 (d + \alpha_1 + r_2) (d + \alpha_1), \\
P_3 &= d (d + \alpha_1 + r_1) (d + \alpha_2 + r_2) + \beta_1 (d + \alpha_1) (d + \alpha_2 + r_2) + \beta_2 (d + \alpha_2) (d + \alpha_1 + r_1).
\end{align*}
\]

Despite the widespread use of deterministic models by numerous scholars to study the dynamics of disease transmission, these models may not accurately reflect the complex and unpredictable nature of real-world epidemics. In reality, populations are subject to continuous interference spectra if the environment is subject to random variations [8]. Environmental fluctuations may involve changes in climate, hygiene habits, healthcare quality, and other factors that could affect natural birth and death rates, as well as disease transmission rates [9]. Therefore, it is imperative to consider the potential benefits of incorporating environmental noise into disease modeling research.

In recent years, several epidemiological models have been proposed to describe the impact of environmental noise on the dynamics of infectious diseases [10, 11, 12, 13]. For instance, Cai et al. [14] investigated the dynamics of a stochastic SIRS model with general infection force under intervention policy in which linear white noise was assumed to perturb the disease transmission coefficient. Lahrouz et al. [15] studied the extinction and persistence of stochastic SIS epidemic models with degenerate diffusion matrix. In [7], Meng et al. have conducted a thorough investigation of the potential influence of linear white noise on disease transmission coefficients. Their analysis has yielded dynamic behaviors for the associated stochastic model. Incorporating environmental noise interference into disease modeling research can enable researchers to more accurately capture the stochastic nature of disease transmission, which, in turn, can provide a more nuanced understanding of the spread of infectious diseases. Moreover, this approach has the potential to yield valuable insights into the effectiveness of public health interventions aimed at controlling and mitigating outbreaks, ultimately enabling policymakers to make more informed decisions regarding disease control measures.
Research conducted by Cai et al. has demonstrated that introducing an Ornstein-Uhlenbeck (OU) process is a more advantageous approach to incorporating stochastic perturbations into infectious disease models compared to the commonly used linear white noise. In light of these findings, we will consider the use of an OU process to perturb the disease transmission rates $\beta_1$ and $\beta_2$ in model (1.1). Additionally, to avoid the possibility of perturbations resulting in negative parameter values [9], we will consider perturbing the natural logarithm of $\beta_1$ and $\beta_2$, i.e.,

$$
\begin{align*}
d\ln\beta_1(t) &= r_1 \left[ \ln \hat{\beta}_1 - \ln \beta_1(t) \right] dt + \sigma_1 dB_1(t), \\
d\ln\beta_2(t) &= r_2 \left[ \ln \hat{\beta}_2 - \ln \beta_2(t) \right] dt + \sigma_2 dB_2(t),
\end{align*}
$$

where for $i = 1, 2$, $\hat{\beta}_i$ denote the long-run mean level of the infection rate, $r_i$ are the speed of reversion, $\sigma_i$ is noise intensities and $B_i(t)$ are mutually independent of the standard Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Denote

$$
\begin{align*}
x_1(t) &= \ln \beta_1(t), \quad \hat{x}_1 = \ln \hat{\beta}_1, \\
x_2(t) &= \ln \beta_2(t), \quad \hat{x}_2 = \ln \hat{\beta}_2,
\end{align*}
$$

we then obtain the following stochastic model

$$
\begin{align*}
dS(t) &= \left[ A - dS(t) - \frac{e^{x_1(t)}S(t)I_1(t)}{a_1 + I_1(t)} - \frac{e^{x_2(t)}S(t)I_2(t)}{a_2 + I_2(t)} + r_1 I_1(t) + r_2 I_2(t) \right] dt, \\
dI_1(t) &= \left[ \frac{e^{x_1(t)}S(t)I_1(t)}{a_1 + I_1(t)} - (d + \alpha_1 + r_1) I_1(t) \right] dt, \\
dI_2(t) &= \left[ \frac{e^{x_2(t)}S(t)I_2(t)}{a_2 + I_2(t)} - (d + \alpha_2 + r_2) I_2(t) \right] dt, \\
dx_1(t) &= \theta_1 (\ln \hat{x}_1 - \ln x_1(t)) dt + \sigma_1 dB_1(t), \\
dx_2(t) &= \theta_2 (\ln \hat{x}_2 - \ln x_2(t)) dt + \sigma_2 dB_2(t).
\end{align*}
$$

This study presents the first analysis of the stochastic dual epidemic hypothesis model using OU process. The aim of this research is to investigate how OU process affects transmission rate coefficients and the resulting dynamics of the model. To achieve this objective, we examined the dynamic behavior of the corresponding stochastic model (1.2).

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries and establish the existence and uniqueness of the global solution for stochastic model (1.2). In Section 3, we analyze the sufficient conditions for eradicating both diseases, eradicating disease $I_1$ while disease $I_2$ persists, and eradicating disease $I_2$ while disease $I_1$ persists. In Section 4, we derive sufficient conditions for the existence of the stationary distribution of the stochastic model. In Section 5, we further discuss the probability density function of the stochastic model near the positive equilibrium point. Finally, we validate our theoretical results by providing several numerical simulation examples, which take into account the simultaneous infection of two variants of SARS-CoV-2, and conclude our study with a discussion.

2. Preliminaries and the existence and uniqueness of global solution

2.1. Preliminaries

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets), and $B_1(t)$ and $B_2(t)$ are defined on this complete probability space. We also let $\mathbb{R}_+^n = \{x = (z_1, \ldots, z_n) \in \mathbb{R}^n : z_i > 0, i = 1, \ldots, n\}$. Denote $X = \mathbb{R}_+^{n+1} \times \mathbb{R}^n$. If $M$ is a matrix, its transpose is denoted by $M^T$. If $k$ is a positive integer, let $\mathbb{N}_k$ denote a $k$-dimensional normal distribution.

Lemma 2.1. ([16],[17],[18]) If there exists a bounded closed domain $\mathbb{D} \subset \mathbb{R}^d$ with a regular boundary, for any initial value $X(0) \in \mathbb{R}^d$, if

$$
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau, X(0), \mathbb{D}) d\tau > 0, \quad \text{a.s.,}
$$

then...
where $P(\tau, X(0), \mathbb{D})$ is the transition probability of $X(t)$. Then system (1.2) will possesses a solution which has the Feller property. In addition, system (1.2) admits at least one invariant probability measure on $\mathbb{R}^d$, which means system (1.2) has at least one ergodic stationary distribution on $\mathbb{R}^d$.

Lemma 2.2. For a stochastic equation
\[
dx(t) = r(x - x(t)) + \sigma B(t),
\] (2.1)
where $\tilde{x}$ and $\sigma$ are positive constants and $B(t)$ is the standard Brownian motion. For $n > 0$, then
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{nx(\tau)} d\tau = e^{n\tilde{x} + \frac{n^2 \tilde{x}^2}{2\sigma^2}}.
\]
Proof. Since
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{nx(\tau)} d\tau = \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{\sqrt{\frac{n}{\sigma^2}} (x(\tau) - \tilde{x})} \frac{\sigma}{\sqrt{n}} e^{\frac{n\tilde{x}}{\sigma^2}} d\tau.
\]
Let $v(t) = \frac{\sqrt{\frac{n}{\sigma^2}} (x(t) - \tilde{x})}{\sigma}$, then it is obvious that the stationary distribution of $v(t)$ obeys $\mathbb{N}(0, 1)$. Therefore, we have
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{nx(\tau)} d\tau = e^{n\tilde{x}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} dv = e^{n\tilde{x}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(v - \frac{n\tilde{x}}{\sigma^2})^2}{2\sigma^2}} dv = e^{n\tilde{x}} e^{\frac{n^2 \tilde{x}^2}{2\sigma^2}}.
\]

Then, we present a lemma about semi-positive definite matrix.

Lemma 2.3. ([19]) For a symmetric matrix $\Omega_0$, if $\Omega_0$ satisfies $\Xi_0^2 + A_0 \Omega_0 + \Omega_0 A_0^T = 0$, where
\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad A_0 = \begin{pmatrix}
-c_1 & -c_2 & -c_3 & -c_4 & c_5 \\
1 & 0 & 0 & 0 & c_7 \\
0 & 1 & 0 & 0 & c_8 \\
0 & 0 & 1 & 0 & c_9 \\
0 & 0 & 0 & 0 & c_{10}
\end{pmatrix},
\]

with $c_1, c_3, c_4 > 0$, $c_1(c_2 c_3 - c_1 c_4) - c_3^2 > 0$ and $c_k (k = 6, 7, 8, 9, 10)$ are constant abbreviations, then
\[
\Omega_0 = \begin{pmatrix}
\frac{c_2 c_3 c_4 - c_1 c_3 c_4}{c_1 c_2 c_3 - c_1 c_3 c_4} & 0 & -\frac{c_3}{c_1} & 0 & 0 \\
0 & -\frac{c_1}{c_3} & 0 & 0 & 0 \\
\frac{c_2 c_3 c_4 - c_1 c_3 c_4}{c_1 c_2 c_3 - c_1 c_3 c_4} & 0 & \frac{c_3}{c_1} & 0 & 0 \\
0 & 0 & -\frac{c_1}{c_3} & 0 & 0 \\
0 & 0 & 0 & -\frac{c_1 c_2 c_3 - c_1 c_3 c_4}{c_1 c_2 c_3 - c_1 c_3 c_4} & 0
\end{pmatrix}
\]
is a positive semi-definite matrix.

2.2. The existence and uniqueness of global solution

Firstly, we give the following fundamental theorem with respect to a unique global solution of stochastic model (1.2).

Theorem 2.1. For any initial value $(x_1(0), x_2(0), S(0), I_1(0), I_2(0)) \in \mathbb{R}^2 \times \mathbb{R}^3_+$, there exists a unique solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t))$ of model (1.2) on $t \geq 0$, and the solution will remain in $\mathbb{M}$ with probability one almost surely (a.s.).

Proof. It is noted that the coefficients of model (1.2) are locally Lipschitz continuous, so for any given initial value $(x_1(0), x_2(0), S(0), I_1(0), I_2(0)) \in \mathbb{R}^2 \times \mathbb{R}^3_+$, there is a unique maximal local solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t))$ on $t \in [0, \tau_e)$, where $\tau_e$ is the explosion time. Denote $\mathbb{Q}_n = \{(-n, n) \times (-n, n) \times (-n, n) \times (-n, n) \times (-n, n)\}$.
Then let $k_0$ be sufficiently large such that $(x_1(0), x_2(0), \ln S(0), \ln I_1(0), \ln I_2(0)) \in \mathbb{Q}_{k_0}$. For each integer $k \geq k_0$, define the stopping time

$$
\tau_k = \inf \left\{ t \in [0, \tau_c) : S(t) \notin \left( \frac{1}{k}, k \right), I_1(t) \notin \left( \frac{1}{k}, k \right), I_2(t) \notin \left( \frac{1}{k}, k \right) \right\}.
$$

Clearly, $\tau_k$ is non-decreasing as $k \to \infty$. We get $\tau_\infty = \lim_{k \to \infty} \tau_k$, whence $\tau_\infty \leq \tau_c$ a.s. In order to show local solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t))$ is global, we only need to verify $\tau_\infty = \infty$ a.s.

Define the nonnegative $C^2$-Lyapunov function as follows

$$
V_0 = \left( e^{x_1} - x_1 - 1 \right) + \left( e^{x_2} - x_2 - 1 \right) + \left( S - 1 - \ln S \right) + (I_1 - 1 - \ln I_1) + (I_2 - 1 - \ln I_2),
$$

where the non-negativity of $V_0$ can be obtained through the inequality $y - 1 - \ln y \geq 0$ for $y > 0$. Applying Itô’s formula to $V_0$, we have

$$
L V_0 = \left( e^{x_1} - 1 \right)(\hat{x}_1 - x_1) + \frac{\sigma_1^2}{2} e^{x_1} + \frac{\sigma_2^2}{2} e^{x_2} + \left( 1 - \frac{1}{S} \right) \left[ A - dS - \frac{e^{x_1} SI_1}{a_1 + I_1} - \frac{e^{x_2} SI_2}{a_2 + I_2} + r_1 I_1 + r_2 I_2 \right]
$$

$$
+ \left( 1 - \frac{1}{I_1} \right) \left[ \frac{e^{x_1} SI_1}{a_1 + I_1} - (d + \alpha_1 + r_1) I_1 \right] + \left( 1 - \frac{1}{I_2} \right) \left[ \frac{e^{x_2} SI_2}{a_2 + I_2} - (d + \alpha_2 + r_2) I_2 \right]
$$

$$
\leq f_1(x_1) + f_2(x_2) + A + 3d + \alpha_1 + r_1 + \alpha_2 + r_2,
$$

where

$$
f_1(x_1) = \theta_1 \left( e^{x_1} - 1 \right)(\hat{x}_1 - x_1) + \frac{\sigma_1^2}{2} e^{x_1},
$$

$$
f_2(x_2) = \theta_2 \left( e^{x_2} - 1 \right)(\hat{x}_2 - x_2) + \frac{\sigma_2^2}{2} e^{x_2}.
$$

Note that $f_1(-\infty) = f_1(+\infty) = f_2(-\infty) = f_2(+\infty) = -\infty$. Therefore, on the real number domain, $f_1(x_1)$ and $f_2(x_2)$ have upper bounds, respectively. Thus we have

$$
L V_0 \leq \sup_{x_1 \in \mathbb{R}} \{ f_1(x_1) \} + \sup_{x_2 \in \mathbb{R}} \{ f_2(x_2) \} + A + 3d + \alpha_1 + r_1 + \alpha_2 + r_2 := K_0,
$$

where $K_0$ is a positive constant. A similar proof of Theorem 3.1 of Zhou et al. [20], thus the rest of the proof is omitted here. This completes the proof. \[\square\]

**Remark 2.1.** Theorem 2.1 demonstrates that for any initial value $(x_1(0), x_2(0), S(0), I_1(0), I_2(0)) \in \mathbb{R}^2 \times \mathbb{R}^3$, there exists a unique solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t)) \in \mathbb{R}^2 \times \mathbb{R}^3$ a.s. of system (1.2). Notice that

$$
(S + I_1 + I_2)' = A - d(S + I_1 + I_2) - \alpha_1 I_1 - \alpha_2 I_2 \leq A - d(S + I_1 + I_2).
$$

Hence one gets

$$
S(t) + I_1(t) + I_2(t) \leq \frac{A \theta}{d} + e^{-dt} \left( S(0) + I_1(0) + I_2(0) - \frac{A \theta}{d} \right),
$$

Thus if $S(0) + I_1(0) + I_2(0) < \frac{A \theta}{d}$, then $S(t) + I_1(t) + I_2(t) < \frac{A \theta}{d}$ a.s.. Thus the region

$$
\Gamma = \left\{ (x_1, x_2, S, I_1, I_2) \in \mathbb{R}^2 \times \mathbb{R}^3 : S + I_1 + I_2 < \frac{A \theta}{d} \right\}
$$

is a positively invariant set of system (1.2). From now on, we always assume that the initial value $(x_1(0), x_2(0), S(0), I_1(0), I_2(0)) \in \Gamma$. 

5
3. Extinction and persistence of the disease

Define

\[ R_1^c = \frac{A \beta_1 e^{\frac{x_1^2}{\sigma^2_1}}}{a_1 (d + \alpha_1 + r_1)}, \quad R_2^c = \frac{A \beta_2 e^{\frac{x_2^2}{\sigma^2_2}}}{a_2 (d + \alpha_2 + r_2)}. \]

**Theorem 3.1.** (i) If \( R_1^c < 1 \) and \( R_2^c < 1 \), then

\[ \lim_{t \to \infty} I_1(t) = 0, \quad \lim_{t \to \infty} I_2(t) = 0, \text{ a.s.,} \]

which implies that the two infectious diseases of system (1.2) go to extinction in a long term.

(ii) If \( R_1^c > 1 \) and \( R_2^c < 1 \), then

\[ \liminf_{t \to \infty} \langle I_1 \rangle_t \geq \frac{a_1 (d + \alpha_1 + r_1)}{\beta_1 e^{\frac{x_1^2}{\sigma^2_1}} + d + \alpha_1 + r_1} (R_1^c - 1) > 0, \quad \lim_{t \to \infty} I_2(t) = 0, \text{ a.s.,} \]

which implies that the disease \( I_1 \) will persist in a long term, while \( I_2 \) will go to extinct in a long term.

(iii) If \( R_1^c < 1 \) and \( R_2^c > 1 \), then

\[ \lim_{t \to \infty} I_1(t) = 0, \quad \liminf_{t \to \infty} \langle I_2 \rangle_t \geq \frac{a_2 (d + \alpha_2 + r_2)}{\beta_2 e^{\frac{x_2^2}{\sigma^2_2}} + d + \alpha_2 + r_2} (R_2^c - 1) > 0, \text{ a.s.,} \]

which implies that the disease \( I_1 \) will go to extinct in a long term, while \( I_2 \) will persist in a long term.

**Proof.** Firstly, integrating of the model (1.2), we obtain

\[
\frac{S(t) - S(0)}{t} + \frac{I_1(t) - I_1(0)}{t} + \frac{I_2(t) - I_2(0)}{t} = A - d \langle S \rangle_t - (d + \alpha_1) \langle I_1 \rangle_t - (d + \alpha_2) \langle I_2 \rangle_t \tag{3.1}
\]

such that

\[
\langle S \rangle_t = \frac{A}{d} - \frac{d + \alpha_1}{d} \langle I_1 \rangle_t - \frac{d + \alpha_2}{d} \langle I_2 \rangle_t. \tag{3.2}
\]

(i) Applying the Itô’s formula to \( \ln I_1 \), we have

\[
\frac{d \ln I_1}{dt} = \frac{e^{x_1 S}}{a_1 + I_1} - (d + \alpha_1 + r_1) \leq \frac{e^{x_1 S}}{a_1} - (d + \alpha_1 + r_1) \leq \frac{e^{x_1 A}}{a_1 d} - (d + \alpha_1 + r_1). \tag{3.3}
\]

Integrating (3.3) from 0 to \( t \), then dividing by \( t \) on both sides and combining Lemma 2.2, we get

\[
\frac{\ln I_1(t) - \ln I_1(0)}{t} \leq \frac{A}{a_1 d} \left( \frac{1}{t} \int_0^t e^{x_1 \tau} d\tau \right) - (d + \alpha_1 + r_1) = \frac{A \beta_1 e^{\frac{x_1^2}{\sigma^2_1}}}{a_1 d} - (d + \alpha_1 + r_1) = (R_1^c - 1)(d + \alpha_1 + r_1).
\]

Hence, if \( R_1^c < 1 \), then we have

\[
\lim_{t \to \infty} I_1(t) = 0, \text{ a.s.,} \tag{3.4}
\]

which means that that disease \( I_1 \) of system (1.2) go to extinction in a long term. For disease \( I_2 \), similarly using the same analysis as above, we can derive that

\[
\frac{\ln I_2(t) - \ln I_2(0)}{t} \leq \frac{A}{a_2 d} \left( \frac{1}{t} \int_0^t e^{x_2 \tau} d\tau \right) - (d + \alpha_2 + r_2) = (R_2^c - 1)(d + \alpha_2 + r_2).
\]

Thus if \( R_2^c < 1 \),

\[
\lim_{t \to \infty} I_2(t) = 0, \text{ a.s.,} \tag{3.5}
\]

which implies that that disease \( I_2 \) of system (1.2) go to extinction in a long term. Then, if \( R_1^c < 1 \) and \( R_2^c < 1 \), one has

\[
\lim_{t \to \infty} I_1(t) = 0, \quad \lim_{t \to \infty} I_2(t) = 0, \text{ a.s.}
\]
(ii) According to (3.1), we have
\[ \frac{S(t) - S(0)}{t} + \frac{I_1(t) - I_1(0)}{t} + \frac{I_2(t) - I_2(0)}{t} \geq A - d(S)_t - (d + \alpha_1)(I_1)_t - \kappa_1, \]
thus
\[ \langle S \rangle_t \geq \frac{A}{d} - \frac{d + \alpha_1}{d} \langle I_1 \rangle_t - \frac{\kappa_1}{d}, \]
where \( \kappa_1 \) is a sufficiently small positive constant. Applying Itô’s formula to \( I_1 + a_1 \ln I_1 \), we have
\[ \frac{d(I_1 + a_1 \ln I_1)}{dt} = e^{x_1}S - (d + \alpha_1 + r_1)I_1 - a_1(d + \alpha_1 + r_1). \tag{3.6} \]
Integrating (3.6) from 0 to \( t \), then dividing by \( t \) on both sides, we have
\[ \frac{I_1(t) - I_1(0)}{t} + \frac{a_1 \ln I_1(t) - a_1 \ln I_1(0)}{t} \geq \left( \frac{1}{t} \int_0^t e^{x_1(r)}dr \right) \left( \frac{A}{d} - \frac{d + \alpha_1}{d} \langle I_1 \rangle_t - \frac{\kappa_1}{d} \right) - (d + \alpha_1 + r_1)\langle I_1 \rangle_t - a_1(d + \alpha_1 + r_1). \]
Since \( I_1(t) \leq \frac{A}{a_1} \), letting \( t \) tends to infinity then combining Lemma 2.2, thus we have
\[
\liminf_{t \to \infty} \langle I_1 \rangle_t \geq \frac{1}{\beta_1 e^{\frac{e^x}{\tau}} + d + \alpha_1 + r_1} \left[ \frac{A\beta_1 e^{\frac{e^x}{\tau}}}{d} - a_1(d + \alpha_1 + r_1) - \frac{\kappa_1 \beta_1 e^{\frac{e^x}{\tau}}}{d} \right]
\]
\[ = \frac{1}{\beta_1 e^{\frac{e^x}{\tau}} + d + \alpha_1 + r_1} \left[ a_1(d + \alpha_1 + r_1)(R_1^* - 1) - \frac{\kappa_1 \beta_1 e^{\frac{e^x}{\tau}}}{d} \right], \text{ a.s..} \]
Since \( \kappa_1 \) is arbitrarily small, letting \( \kappa_1 \to 0^+ \), one obtains that if \( R_1^* > 1 \),
\[ \liminf_{t \to \infty} \langle I_1 \rangle_t \geq \frac{a_1(d + \alpha_1 + r_1)}{\beta_1 e^{\frac{e^x}{\tau}} + d + \alpha_1 + r_1} (R_1^* - 1) > 0, \text{ a.s..} \]
Combining (3.5), it can be obtained that if \( R_1^* > 1 \) and \( R_2^* < 1 \), then disease \( I_1 \) will continue to spread while disease \( I_2 \) will become extinct for a long time.

(iii) Similar to the proof process in (ii), combining (3.4), one can obtain that if \( R_1^* < 1 \) and \( R_2^* > 1 \),
\[ \lim_{t \to \infty} I_1(t) = 0, \quad \liminf_{t \to \infty} \langle I_2 \rangle_t \geq \frac{a_2(d + \alpha_2 + r_2)}{\beta_2 e^{\frac{e^x}{\tau_2}} + d + \alpha_2 + r_2} (R_2^* - 1) > 0, \text{ a.s..} \]
This completes the proof. \( \square \)

4. Stationary distribution

For the stochastic virus infection model, we are concerned mainly with the persistence of the virus and the infected cells. In this section, on the basis of the theory of Lemma 2.1, we will give sufficient conditions for the existence of a unique ergodic stationary distribution for the system (1.2). Furthermore, the local probability density function is given in the neighborhood of the positive equilibrium point of the model (1.2). Define
\[ R_1^* = \frac{A\beta_1 e^{\frac{e^x}{\tau}}}{a_1 d(d + \alpha_1 + r_1)}, \quad R_2^* = \frac{A\beta_2 e^{\frac{e^x}{\tau_2}}}{a_2 d(d + \alpha_2 + r_2)}. \]

**Theorem 4.1.** Assume that \( R_1^* > 1 \) and \( R_2^* > 1 \), then the stochastic system (1.2) admits at least one ergodic stationary distribution.

**Proof.** We divide the proof of Theorem 4.1 into three steps: (i) Construct a stochastic Lyapunov function; (ii) Construct a compact set; (iii) Give the existence and ergodicity of the solution of system (1.2).
Step 1. (A stochastic Lyapunov function): Applying Itô’s formula to \(-\ln S, -\ln I_1, -\ln I_2, -\frac{I_1}{d+\alpha_1+r_1}\), and \(-\frac{I_2}{d+\alpha_2+r_2}\) respectively, we have

\[
\mathcal{L}(-\ln S) = \frac{-A}{S} + d + \frac{e^{x_1}I_1}{a_1 + I_1} + \frac{e^{x_2}I_2}{a_2 + I_2} - \frac{r_1I_1}{S} - \frac{r_2I_2}{S} \leq -\frac{A}{S} + d + \frac{e^{x_1}I_1}{a_1} + \frac{e^{x_2}I_2}{a_2}, \quad (4.1)
\]

\[
\mathcal{L}(-\ln I_1) = -\frac{e^{x_1}S}{a_1 + I_1} + d + \alpha_1 + r_1, \quad (4.2)
\]

\[
\mathcal{L}(-\ln I_2) = -\frac{e^{x_2}S}{a_2 + I_2} + d + \alpha_2 + r_2, \quad (4.3)
\]

\[
\mathcal{L} \left( -\frac{I_1}{d+\alpha_1+r_1} \right) = \frac{e^{x_1}SI_1}{(a_1 + I_1)(d + \alpha_1 + r_1)} - I_1 \leq \frac{e^{x_1}S}{d+\alpha_1+r_1} - (I_1 + a_1) + a_1, \quad (4.4)
\]

and

\[
\mathcal{L} \left( -\frac{I_2}{d+\alpha_2+r_2} \right) = \frac{e^{x_2}SI_2}{(a_2 + I_2)(d + \alpha_2 + r_2)} - I_2 \leq \frac{e^{x_2}S}{d+\alpha_2+r_2} - (I_2 + a_2) + a_2. \quad (4.5)
\]

Then denote a \(C^2\)-function \(V_1\) as follows

\[
V_1 = -\ln I_1 - \ln I_2 - (p_1 + p_2)\ln S - \frac{p_1I_1}{d+\alpha_1+r_1} - \frac{p_2I_2}{d+\alpha_2+r_2}
\]

with \(p_k (k = 1, 2, 3, 4)\) are determined later. Combining (4.1), (4.2), (4.3), (4.4) and (4.5), then one has

\[
\mathcal{L}V_1 \leq -\frac{e^{x_1}S}{a_1 + I_1} - \frac{p_1A}{S} - p_3(a_1 + I_1) + d + \alpha_1 + r_1 - \frac{e^{x_2}S}{a_2 + I_2} - \frac{p_2A}{S} - p_4(a_2 + I_2) + d + \alpha_2 + r_2
\]

\[
+ (p_1 + p_2) \left( d + \frac{e^{x_1}I_1}{a_1} + \frac{e^{x_2}I_2}{a_2} \right) + p_3 \left( \frac{e^{x_1}S}{d+\alpha_1+r_1} + a_1 \right) + p_4 \left( \frac{e^{x_2}S}{d+\alpha_2+r_2} + a_2 \right)
\]

\[
\leq -3 \left( p_1p_3\Phi\right)^\frac{1}{2} + d + \alpha_1 + r_1 - 3 \left( p_2p_4\Phi\right)^\frac{1}{2} + d + \alpha_2 + r_2 + (p_1 + p_2) \left( d + \frac{e^{x_1}I_1}{a_1} + \frac{e^{x_2}I_2}{a_2} \right)
\]

\[
+ p_3 \left( \frac{e^{x_1}S}{d+\alpha_1+r_1} + a_1 \right) + p_4 \left( \frac{e^{x_2}S}{d+\alpha_2+r_2} + a_2 \right) + 3 \left( p_1p_3\Phi\right)^\frac{1}{2} \left( \frac{e^{x_2}I_2}{a_1} - \frac{e^{x_2}I_2}{a_2} \right)
\]

\[
+ 3 \left( p_2p_4\Phi\right)^\frac{1}{2} \left( \frac{e^{x_2}I_2}{a_1} - \frac{e^{x_2}I_2}{a_2} \right), \quad (4.6)
\]

where \(p_k (k = 1, 2, 3, 4)\) take the values as

\[
p_1d = p_3a_1 = \frac{A\hat{\beta}_1e^{x_1^2}}{a_1d}, \quad p_2d = p_4a_2 = \frac{A\hat{\beta}_2e^{x_1^2}}{a_2d}.
\]
As a result, we get
\[
\mathcal{L}V_1 \leq -\frac{A\tilde{b}_1 e^{\frac{r_1^2}{2a_1 d}}}{a_1 d} + d + \alpha_1 + r_1 - \frac{A\tilde{b}_2 e^{\frac{r_1^2}{2a_2 d}}}{a_2 d} + d + \alpha_2 + r_2 + (p_1 + p_2) \left( \frac{e^{x_1} I_1}{a_1} + \frac{e^{x_2} I_2}{a_2} \right) \\
+ \frac{p_3 e^{x_1} S}{d + \alpha_1 + r_1} + \frac{p_4 e^{x_2} S}{d + \alpha_2 + r_2} + 3 (p_1 p_3 A) \frac{1}{3} \left( e^{x_1} - \tilde{b}_1 e^{\frac{r_1^2}{2a_1 d}} \right) + 3 (p_2 p_4 A) \frac{1}{3} \left( e^{x_2} - \tilde{b}_2 e^{\frac{r_2^2}{2a_2 d}} \right)
\]

\[
= - (R^*_1 - 1)(d + \alpha_1 + r_1) - (R^*_2 - 1)(d + \alpha_2 + r_2) + (p_1 + p_2) \left( \frac{e^{x_1} I_1}{a_1} + \frac{e^{x_2} I_2}{a_2} \right)
\]

\[
+ \frac{p_3 e^{x_1} S}{d + \alpha_1 + r_1} + \frac{p_4 e^{x_2} S}{d + \alpha_2 + r_2} + f_1(x_1) + f_2(x_2),
\]

where
\[
f_1(x_1) = 3 (p_1 p_3 A) \frac{1}{3} \left( e^{x_1} - \tilde{b}_1 e^{\frac{r_1^2}{2a_1 d}} \right), \quad f_2(x_2) = 3 (p_2 p_4 A) \frac{1}{3} \left( e^{x_2} - \tilde{b}_2 e^{\frac{r_2^2}{2a_2 d}} \right).
\]

From the Inequality of arithmetic and geometric means, we have
\[
e^{x_1} \leq \frac{p_5 e^{2x_1}}{4p_5}, \quad e^{x_2} \leq \frac{p_6 e^{2x_2}}{4p_6},
\]

(4.8)

where \( p_5 \) and \( p_6 \) are positive constants to be determined later. Combining \( \Gamma \) and (4.8), we have
\[
\mathcal{L}V_1 \leq - (R^*_1 - 1)(d + \alpha_1 + r_1) - (R^*_2 - 1)(d + \alpha_2 + r_2) + (p_1 + p_2) \left[ \left( \frac{p_5 e^{2x_1}}{4p_5} \right) \frac{I_1}{a_1} + \left( \frac{p_6 e^{2x_2}}{4p_6} \right) \frac{I_2}{a_2} \right]
\]

\[
+ \frac{p_3 S \left( p_5 e^{2x_1} + \frac{1}{4p_5} \right)}{d + \alpha_1 + r_1} + \frac{p_4 S \left( p_6 e^{2x_2} + \frac{1}{4p_6} \right)}{d + \alpha_2 + r_2} + f_1(x_1) + f_2(x_2)
\]

\[
\leq - (R^*_1 - 1)(d + \alpha_1 + r_1) - (R^*_2 - 1)(d + \alpha_2 + r_2) + (p_1 + p_2) \left[ \frac{p_5 e^{2x_1} I_1}{a_1} + \frac{p_6 e^{2x_2} I_2}{a_2} + \left( \frac{1}{4p_5} + \frac{1}{4p_6} \right) \frac{A}{d} \right]
\]

\[
+ \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) S + \frac{p_3 A}{4p_5 d(d + \alpha_1 + r_1)} + \frac{p_4 A}{4p_6 d(d + \alpha_2 + r_2)} + f_1(x_1) + f_2(x_2).
\]

(4.9)

Let
\[
p_5 = \left( p_1 + p_2 + \frac{p_3}{d + \alpha_1 + r_1} \right) \frac{A}{2d(R_1^* - 1)(d + \alpha_1 + r_1)},
\]

\[
p_6 = \left( p_1 + p_2 + \frac{p_4}{d + \alpha_2 + r_2} \right) \frac{A}{2d(R_2^* - 1)(d + \alpha_2 + r_2)}
\]

such that
\[
\left( p_1 + p_2 + \frac{p_3}{d + \alpha_1 + r_1} \right) \frac{A}{4p_5 d} = \frac{(R_1^* - 1)(d + \alpha_1 + r_1)}{2}, \quad \left( p_1 + p_2 + \frac{p_4}{d + \alpha_2 + r_2} \right) \frac{A}{4p_6 d} = \frac{(R_2^* - 1)(d + \alpha_2 + r_2)}{2}.
\]

Thus (4.9) becomes
\[
\mathcal{L}V_1 \leq - \frac{1}{2} (R_1^* - 1)(d + \alpha_1 + r_1) - \frac{1}{2} (R_2^* - 1)(d + \alpha_2 + r_2) + (p_1 + p_2) \left( \frac{p_5 e^{2x_1} I_1}{a_1} + \frac{p_6 e^{2x_2} I_2}{a_2} \right)
\]

\[
+ \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) S + f_1(x_1) + f_2(x_2).
\]

(4.10)

Then define
\[
V_2 = V_1 + e^{x_1} - x_1 - 1 + e^{x_2} - x_2 - 1,
\]

Applying Itô’s formula to \( V_2 \) and combining (4.10), we have
\[
\mathcal{L}V_2 \leq F(x_1, x_2, S, I_1, I_2) + f_1(x_1) + f_2(x_2),
\]

(4.11)
where

\[
F(x_1, x_2, S, I_1, I_2) \leq -\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2) + (p_1 + p_2)\left(\frac{p_5 e^{2x_1} I_1}{a_1} + \frac{p_6 e^{2x_2} I_2}{a_2}\right)
+ \left(\frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2}\right)S + \theta_1(e^{x_1} - 1)(\bar{x}_1 - x_1) + \frac{\sigma_1^2 e^{x_1}}{2}
+ \theta_2(e^{x_2} - 1)(\bar{x}_2 - x_2) + \frac{\sigma_2^2 e^{x_2}}{2}.
\]

Step 2. (A compact set): Then, we construct a compact set \( \mathbb{D} \subset \Gamma_1 \) as follows

\[
\mathbb{D} = \left\{(x_1, x_2, S, I_1, I_2) \in \Gamma : \epsilon \leq e^{x_1} \leq \frac{1}{\epsilon}, \epsilon \leq e^{x_2} \leq \frac{1}{\epsilon}\right\}
\]

such that \( F(x_1, x_2, S, I_1, I_2) \leq -1 \) for any \( (x_1, x_2, S, I_1, I_2) \in \Gamma \setminus \mathbb{D} \). Then let \( \mathbb{D}^c = \bigcup_{i=1}^4 \mathbb{D}_i^c \), where

\[
\mathbb{D}_1^c = \left\{(x_1, x_2, S, I_1, I_2) \in \Gamma : 0 < e^{x_1} < \epsilon\right\}, \quad \mathbb{D}_2^c = \left\{(x_1, x_2, S, I_1, I_2) \in \Gamma : \frac{1}{\epsilon} < e^{x_1}\right\},
\]

\[
\mathbb{D}_3^c = \left\{(x_1, x_2, S, I_1, I_2) \in \Gamma : 0 < e^{x_2} < \epsilon\right\}, \quad \mathbb{D}_4^c = \left\{(x_1, x_2, S, I_1, I_2) \in \Gamma : \frac{1}{\epsilon} < e^{x_2}\right\},
\]

with \( \epsilon \in (0, 1) \) is a small enough constant satisfying the following inequalities

\[
-\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2) + \theta_1\left(1 - \frac{1}{\epsilon}\right)(\ln \epsilon - \bar{x}_1) + \sup_{x_1 \in \mathbb{R}}\{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}}\{g_4(x_2)\} \leq -1,
\]

\[
-\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2) + \theta_1\left(1 - \frac{1}{\epsilon}\right)(\ln \epsilon - \bar{x}_2) + \sup_{x_1 \in \mathbb{R}}\{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}}\{g_4(x_2)\} \leq -1,
\]

\[
-\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2) + \theta_2\left(1 - \frac{1}{\epsilon}\right)(\ln \epsilon - \bar{x}_2) + \sup_{x_1 \in \mathbb{R}}\{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}}\{g_4(x_2)\} \leq -1,
\]

\[
-\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2) + \theta_2\left(1 - \frac{1}{\epsilon}\right)(\ln \epsilon - \bar{x}_2) + \sup_{x_1 \in \mathbb{R}}\{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}}\{g_4(x_2)\} \leq -1,
\]

and

\[
g_1(x_1) = \frac{1}{2}\theta_1(e^{x_1} - 1)(\bar{x}_1 - x_1) + (p_1 + p_2)\frac{p_5 e^{2x_1} A}{a_1 d} + \frac{p_3 p_5 e^{2x_1} A}{d(d + \alpha_1 + r_1)},
\]

\[
g_2(x_2) = \theta_2(e^{x_2} - 1)(\bar{x}_2 - x_2) + (p_1 + p_2)\frac{p_6 e^{2x_2} A}{a_2 d} + \frac{p_4 p_6 e^{2x_2} A}{(d + \alpha_2 + r_2)d},
\]

\[
g_3(x_1) = \theta_1(e^{x_1} - 1)(\bar{x}_1 - x_1) + (1 + M_0 \alpha_1) L_3 f_1(0) e^{x_1} + \frac{1}{2} \sigma_1^2 e^{x_1},
\]

\[
g_4(x_2) = \frac{1}{2}\theta_2(e^{x_2} - 1)(\bar{x}_2 - x_2) + (1 + M_0 \alpha_4) L_3 f_2(0) e^{x_2} + \frac{1}{2} \sigma_2^2 e^{x_2}.
\]
Case 1. If \((x_1, x_2, S, I_1, I_2) \in \mathbb{D}_1^c\), i.e. \(x_1 \in (-\infty, \ln \epsilon)\), from (4.12) and (4.13), we have

\[
F(x_1, x_2, S, I_1, I_2) \leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ (p_1 + p_2) \left( \frac{p_5 e^{2x_1}}{a_1} + \frac{p_6 e^{2x_2}}{a_2} \right) \frac{A}{d} + \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) \frac{A}{d}
\]
\[
+ \theta_1 (e^{x_1} - 1)(\widehat{x}_1 - x_1) + \frac{\sigma_1^2 e^{x_1}}{2} + \theta_2 (e^{x_2} - 1)(\widehat{x}_2 - x_2) + \frac{\sigma_2^2 e^{x_2}}{2}
\]
\[
\leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ \frac{\theta_1}{2} (e^{x_1} - 1)(\widehat{x}_1 - x_1) + \sup_{x_1 \in \mathbb{R}} \{g_1(x_1)\} + \sup_{x_2 \in \mathbb{R}} \{g_2(x_2)\}
\]
\[
\leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ \frac{\theta_1}{2} (1 - \epsilon)(\ln \epsilon - \widehat{x}_1) + \sup_{x_1 \in \mathbb{R}} \{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}} \{g_4(x_2)\}
\]
\[
\leq - 1.
\]

Case 2. If \((x_1, x_2, S, I_1, I_2) \in \mathbb{D}_2^c\), i.e. \(x_1 \in (-\ln \epsilon, \infty)\), from (4.12) and (4.14), we have

\[
F(x_1, x_2, S, I_1, I_2) \leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ (p_1 + p_2) \left( \frac{p_5 e^{2x_1}}{a_1} + \frac{p_6 e^{2x_2}}{a_2} \right) \frac{A}{d} + \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) \frac{A}{d}
\]
\[
+ \theta_1 (e^{x_1} - 1)(\widehat{x}_1 - x_1) + \frac{\sigma_1^2 e^{x_1}}{2} + \theta_2 (e^{x_2} - 1)(\widehat{x}_2 - x_2) + \frac{\sigma_2^2 e^{x_2}}{2}
\]
\[
\leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ \frac{\theta_1}{2} (e^{x_1} - 1)(\widehat{x}_1 - x_1) + \sup_{x_1 \in \mathbb{R}} \{g_1(x_1)\} + \sup_{x_2 \in \mathbb{R}} \{g_2(x_2)\}
\]
\[
\leq - \frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
\]
\[
+ \frac{\theta_1}{2} \left( \frac{1}{\epsilon} - 1 \right)(\ln \epsilon + \widehat{x}_1) + \sup_{x_1 \in \mathbb{R}} \{g_3(x_1)\} + \sup_{x_2 \in \mathbb{R}} \{g_4(x_2)\}
\]
\[
\leq - 1.
\]
Case 3. If \((x_1, x_2, S, I_1, I_2) \in \mathbb{D}_1^c\), i.e. \(x_2 \in (-\infty, \ln \epsilon)\). Similar to Case 1, from (4.12) and (4.15), we have

\[
F(x_1, x_2, S, I_1, I_2) \leq -\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
+ (p_1 + p_2) \left( \frac{p_5 e^{2x_1}}{a_1} + \frac{p_6 e^{2x_2}}{a_2} \right) \frac{A}{d} + \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) \frac{A}{d}
+ \theta_1 (e^{x_1} - 1)(\tilde{x}_1 - x_1) + \frac{\sigma_1^2 e^{x_1}}{2} + \theta_2 (e^{x_2} - 1)(\tilde{x}_2 - x_2) + \frac{\sigma_2^2 e^{x_2}}{2}
\leq -\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
+ \frac{\theta_2}{2} (e^{x_2} - 1)(\tilde{x}_2 - x_2) + \sup_{x_1 \in \mathbb{R}} \{ g_3(x_1) \} + \sup_{x_2 \in \mathbb{R}} \{ g_4(x_2) \}
\leq -1.
\]

Case 4. If \((x_1, x_2, S, I_1, I_2) \in \mathbb{D}_4^c\), i.e. \(x_1 \in (-\ln \epsilon, \infty)\). Similar to Case 2, from (4.12) and (4.16), we have

\[
F(x_1, x_2, S, I_1, I_2) \leq -\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
+ (p_1 + p_2) \left( \frac{p_5 e^{2x_1}}{a_1} + \frac{p_6 e^{2x_2}}{a_2} \right) \frac{A}{d} + \left( \frac{p_3 p_5 e^{2x_1}}{d + \alpha_1 + r_1} + \frac{p_4 p_6 e^{2x_2}}{d + \alpha_2 + r_2} \right) \frac{A}{d}
+ \theta_1 (e^{x_1} - 1)(\tilde{x}_1 - x_1) + \frac{\sigma_1^2 e^{x_1}}{2} + \theta_2 (e^{x_2} - 1)(\tilde{x}_2 - x_2) + \frac{\sigma_2^2 e^{x_2}}{2}
\leq -\frac{1}{2}(R_1^s - 1)(d + \alpha_1 + r_1) - \frac{1}{2}(R_2^s - 1)(d + \alpha_2 + r_2)
+ \frac{\theta_2}{2} (e^{x_2} - 1)(\tilde{x}_2 - x_2) + \sup_{x_1 \in \mathbb{R}} \{ g_3(x_1) \} + \sup_{x_2 \in \mathbb{R}} \{ g_4(x_2) \}
\leq -1.
\]

In summary, we have \(F(x_1, x_2, S, I_1, I_2) \leq -1\) for all \((x_1, x_2, S, I_1, I_2) \in \mathbb{D}^c\).

Step 3. (The existence and ergodicity of the solution of system (1.2)): Since the function \(V_2\) tends to \(\infty\) as \(S\), \(I_1\) or \(I_2\) approach the boundary of \(\mathbb{R}_+^2\) or as \(\|(S, I_1, I_2)\| \to \infty\). Thus, there exists a point \((\tilde{S}, \tilde{I}, \tilde{x}_1, \tilde{x}_2)\) in the interior of \(\Gamma\) which makes \(V_2(\tilde{x}_1, \tilde{x}_2, \tilde{S}, \tilde{I})\) take the minimum value. Hence

\[ V = V_2 - V_2(\tilde{S}, \tilde{I}, \tilde{x}_1, \tilde{x}_2) \]

is a non-negative \(C^2\)-function. Applying the Itô’s formula to \(V\), we have

\[ \mathcal{L}V \leq F(x_1, x_2, S, I_1, I_2) + f_1(x_1) + f_2(x_2). \]
For any initial value \((S(0), I_1(0), I_2(0), x_1(0), x_2(0)) \in \Gamma\) and a interval \([0,t]\), combining (4.11), we get

\[
0 \leq \frac{\mathbb{E}V(S(t), I_1(t), I_2(t), x_1(t), x_2(t))}{t} = \frac{\mathbb{E}V(S(0), I_1(0), I_2(0), x_1(0), x_2(0))}{t} + \frac{1}{t} \int_0^t \mathbb{E}(\mathcal{L}V(S(\tau), I_1(\tau), I_2(\tau), x_1(\tau), x_2(\tau)))d\tau
\]

\[
\leq \frac{\mathbb{E}V(S(0), I_1(0), I_2(0), x_1(0), x_2(0))}{t} + \frac{1}{t} \int_0^t \mathbb{E}(F(S(\tau), I_1(\tau), I_2(\tau), x_1(\tau), x_2(\tau)))d\tau
\]

\[
+ 3(p_1p_3A)^{\frac{1}{2}} \left( \frac{1}{t} \int_0^t e^{\frac{x_1 + x_2}{2}} d\tau - \tilde{\beta}_1 e^{-\frac{\sigma_1^2}{4}} \right) + 3(p_2p_4A)^{\frac{1}{2}} \left( \frac{1}{t} \int_0^t e^{\frac{x_1 + x_2}{2}} d\tau - \tilde{\beta}_2 e^{-\frac{\sigma_2^2}{4}} \right).
\]  

(4.17)

According to Lemma 2.2, we have

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{\frac{x_1 + x_2}{2}} d\tau = \left( \tilde{\beta}_1 \right)^{\frac{1}{2}} e^{-\frac{\sigma_1^2}{4}}, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{\frac{x_1 + x_2}{2}} d\tau = \left( \tilde{\beta}_2 \right)^{\frac{1}{2}} e^{-\frac{\sigma_2^2}{4}}.
\]

(4.18)

Substituting (4.18) into (4.17) and allowing \(t \to \infty\), we have

\[
0 \leq \frac{\mathbb{E}V(S(0), I_1(0), I_2(0), x_1(0), x_2(0))}{t} + \frac{1}{t} \int_0^t \mathbb{E}(F(S(\tau), I_1(\tau), I_2(\tau), x_1(\tau), x_2(\tau)))d\tau
\]

\[
= \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}(F(S(\tau), I_1(\tau), I_2(\tau), x_1(\tau), x_2(\tau)))d\tau, \quad a.s.
\]

On the other hand,

\[
F(x_1, x_2, S, I_1, I_2) \leq M, \quad \forall (x_1, x_2, S, I_1, I_2) \in \mathbb{R}_+^3 \times \mathbb{R}^2,
\]

where

\[
M = \sup_{(x_1, x_2, S, I_1, I_2) \in \mathbb{R}_+^3 \times \mathbb{R}^2} \left\{ p_5e^{2x_1}I_1/a_1 + p_6e^{2x_2}I_2/a_2 + \left( \frac{p_3p_5e^{2x_1}}{d + a_1 + r_1} + \frac{p_4p_6e^{2x_2}}{d + a_2 + r_2} \right) S + \theta_1(e^{x_1} - 1)(\tilde{x}_1 - x_1) + \frac{\sigma_1^2e^{x_1}}{2} + \theta_2(e^{x_2} - 1)(\tilde{x}_2 - x_2) + \frac{\sigma_2^2e^{x_2}}{2} \right\} < +\infty.
\]

Then we have

\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}(F(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)))d\tau
\]

\[
= \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}(F(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)))d\tau \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}\}}d\tau
\]

\[
+ \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{E}(F(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)))d\tau \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}^c\}}d\tau
\]

\[
\leq M \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}\}}d\tau - \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}^c\}}d\tau
\]

\[
\leq (M + 1) \lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}\}}d\tau - 1.
\]

Therefore, we have

\[
\lim \inf_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{(x_1(\tau), x_2(\tau), S(\tau), I_1(\tau), I_2(\tau)) \in \mathbb{E}\}}d\tau \geq \frac{1}{M + 1} > 0, \quad a.s..
\]

(4.19)

Let \(P(t, (x_1(t), x_2(t), S(t), I_1(t), I_2(t)), \Omega)\) be the transition probability of \((x_1(t), x_2(t), S(t), I_1(t), I_2(t))\) belongs
to the set $\Omega$. Making the use of Fatou’s lemma \cite{16}, we have

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{P}(\tau,(x_1(\tau),x_2(\tau),S(\tau),I_1(\tau),I_2(\tau)),\mathbb{D})d\tau \geq \frac{1}{M+1} > 0, \text{ a.s.} \quad (4.20)$$

According to Lemma 2.1, system (1.2) has at least one stationary distribution on $\Gamma$ which has the Feller and ergodic property. This completes the proof. \hfill \blacksquare

5. Density function around the quasi-equilibrium point

Furthermore, we will give the exact expression for the local probability density function of the stationary distribution. First, we give the quasi-equilibrium point $E^* = (\ln \tilde{\beta}_1, \ln \tilde{\beta}_2, S^*, I_1^*, I_2^*)$ of stochastic system (1.2), where

$$S^* = \frac{A + a_1(d + a_1) + a_2(d + a_2)}{d + \frac{\beta_1(d + a_1)}{d + a_1 + r_1} + \frac{\beta_2(d + a_2)}{d + a_2 + r_2}}, \quad I_1^* = \frac{M_1}{M_3}, \quad I_2^* = \frac{M_2}{M_3},$$

with

$$M_1 = \beta_1(d + a_2 + r_2)[A + a_2(d + a_2)] - a_1d(d + a_1 + r_1)(d + a_2 + r_2) - a_1\beta_2(d + a_1 + r_1)(d + a_2),$$

$$M_2 = \beta_2(d + a_1 + r_1)[A + a_1(d + a_1)] - a_2d(d + a_2 + r_2)(d + a_1 + r_1) - a_2\beta_1(d + a_2 + r_2)(d + a_1),$$

$$M_3 = d(d + a_1 + r_1)(d + a_2 + r_2) + \beta_1(d + a_1)(d + a_2 + r_2) + \beta_2(d + a_2)(d + a_1 + r_1).$$

This makes the following equations hold

$$\begin{align*}
A + r_1I_1^* + r_2I_2^* &= \left( d + \frac{\beta_1I_1^*}{a_1 + I_1^*} + \frac{\beta_2I_2^*}{a_2 + I_2^*} \right) S^*, \\
\frac{\beta_1S^*}{a_1 + I_1^*} &= (d + a_1 + r_1), \\
\frac{\beta_2S^*}{a_2 + I_2^*} &= (d + a_2 + r_2).
\end{align*} \quad (5.1)$$

Define

$$R_1^p = \frac{A\hat{\beta}_1}{a_1d(d + a_1 + r_1)}, \quad R_2^p = \frac{A\hat{\beta}_2}{a_2d(d + a_2 + r_2)}.$$  

Similar to Theorem 2.1 in \cite{7}, it is obtained that $E^*$ exists if $M_1, M_2 > 0$ and $R_1^p, R_2^p > 1$.

Let $(y_1,y_2,y_3,y_4,y_5)^T = (x_1 - \hat{x}_1, x_2 - \hat{x}_2, S - S^*, I_1 - I_1^*, I_2 - I_2^*)^T$. Applying Itô’s integral, we obtain the corresponding linearized system of model (1.2):

$$\begin{align*}
\frac{dy_1}{dt} &= -\theta_1y_1dt + \sigma_1dB_1(t), \\
\frac{dy_2}{dt} &= -\theta_2y_2dt + \sigma_2dB_2(t), \\
\frac{dy_3}{dt} &= (-\rho_1y_1 - \rho_2y_2 - \rho_3y_3 + \rho_4y_4 + \rho_5y_5)dt, \\
\frac{dy_4}{dt} &= (\rho_1y_1 + \rho_6y_3 - \rho_7y_4)dt, \\
\frac{dy_5}{dt} &= (\rho_2y_2 + \rho_8y_3 - \rho_9y_5)dt,
\end{align*} \quad (5.2)$$

where

$$\begin{align*}
\rho_1 &= \frac{\beta_1S^*I_1^*}{a_1 + I_1^*}, & \rho_2 &= \frac{\beta_2S^*I_2^*}{a_2 + I_2^*}, & \rho_3 &= d + \frac{\beta_1I_1^*}{a_1 + I_1^*} + \frac{\beta_2I_2^*}{a_2 + I_2^*}, & \rho_4 &= r_1 - \frac{a_1\beta_1S^*}{(a_1 + I_1^*)^2}, & \rho_5 &= r_2 - \frac{a_2\beta_2S^*}{(a_2 + I_2^*)^2}, \\
\rho_6 &= \frac{\beta_1I_1^*}{a_1 + I_1^*}, & \rho_7 &= \frac{\beta_1S^*I_1^*}{(a_1 + I_1^*)^2}, & \rho_8 &= \frac{\beta_2I_2^*}{a_2 + I_2^*}, & \rho_9 &= \frac{\beta_2S^*I_2^*}{(a_2 + I_2^*)^2}.
\end{align*}$$
Let $Y = (y_1, y_2, y_3, y_4, y_5)^T$, then the model (5.2) can be equivalently written as
\[ dY(t) = AY(t)dt + \Xi dB(t), \]
where
\[ A = \begin{pmatrix} -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & -\theta_2 & 0 & 0 & 0 \\ -\rho_1 & -\rho_2 & -\rho_3 & \rho_4 & \rho_5 \\ \rho_1 & 0 & \rho_6 & -\rho_7 & 0 \\ 0 & \rho_2 & \rho_8 & 0 & -\rho_9 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

**Theorem 5.1.** If $M_1, M_2 > 0$ and $R_1^2, R_2^2 > 1$, then the stationary solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t))$ to system (1.2) around $E^*$ follows a normal distribution $N_5(E^*, \Sigma)$, where the covariance matrix $\Sigma = \Sigma_1 + \Sigma_2$ with
\[
\Sigma_1 = \begin{cases} 
\left( (\sigma_1 \rho_1 \rho_5 \rho_8)^2 (T_5 T_2 T_1)^{-1} \Omega_1 [(T_5 T_2 T_1)^{-1}]^T, \right. & \text{if } \alpha_1 (a_2 + I_1^*) = \beta_2 I_2^*, \\
\left( (\sigma_1 \rho_1 \rho_5 (\omega_1 (\rho_6 + \rho_7 - \rho_6) + \rho_5))^2 (T_5 T_2 T_1)^{-1} \Omega_1 [(T_5 T_4 T_2 T_1)^{-1}]^T, \right. & \text{if } \alpha_1 (a_2 + I_1^*) \neq \beta_2 I_2^*, \\
\end{cases}
\]
and
\[
\Sigma_2 = \begin{cases} 
\left( (\sigma_2 \rho_2 \rho_4 \rho_6)^2 (T_7 T_2 T_6)^{-1} \Omega_2 [(T_5 T_2 T_6)^{-1}]^T, \right. & \text{if } \alpha_2 (a_1 + I_1^*) = \beta_1 I_1^*, \\
\left( (\sigma_2 \rho_2 \rho_6 (\omega_2 (\rho_3 + \rho_7 - \rho_6) + \rho_4))^2 (T_5 T_8 T_2 T_6)^{-1} \Omega_2 [(T_5 T_8 T_2 T_6)^{-1}]^T, \right. & \text{if } \alpha_2 (a_1 + I_1^*) \neq \beta_1 I_1^*, \\
\end{cases}
\]
where the matrices $T_1, \ldots, T_9, \Omega_1, \Omega_2$ are defined in the following proof.

**Proof.** Similar to the analysis of Theorem 4.1 in [19], the covariance matrix $\Sigma$ can be determined by
\[ \Xi^2 + A \Sigma + \Sigma A^T = 0. \]

Due to the independent additivity of the matrix, then $\Sigma = \Sigma_1 + \Sigma_2$, where for $k = 1, 2, \Sigma_k$ are determined by the following equations, respectively
\[ \Xi_k^2 + A \Sigma_k + \Sigma_k A^T = 0. \]

Next, we will separate the cases to give the exact expressions for $\Sigma_1$ and $\Sigma_2$.

**Case 1.** For
\[ \Xi_1^2 + A \Sigma_1 + \Sigma_1 A^T = 0, \]
let
\[ T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \]
such that
\[ A_1 = T_1 A T_1^{-1} = \begin{pmatrix} -\theta_1 & 0 & 0 & 0 & 0 \\ -\rho_1 & -\rho_3 & \rho_5 & \rho_4 & -\rho_2 \\ 0 & \rho_8 & -\rho_9 & 0 & \rho_2 \\ \rho_1 & \rho_6 & 0 & -\rho_7 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 \end{pmatrix}. \]

Then let
\[ T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
such that
\[ A_2 = T_2 A_1 T_2^{-1} = \begin{pmatrix} -\theta_1 & 0 & 0 & 0 & 0 \\ -\rho_1 & -\rho_4 - \rho_3 & \rho_5 & \rho_4 & -\rho_2 \\ 0 & \rho_8 & -\rho_9 & 0 & \rho_2 \\ 0 & \omega_1 & \rho_5 & \rho_4 - \rho_7 & -\rho_2 \\ 0 & 0 & 0 & 0 & -\theta_2 \end{pmatrix}, \]
where
\[ \omega_1 = \rho_6 + \rho_7 - \rho_3 - \rho_4 = \alpha_1 - \frac{\tilde{b}_2 I_2}{a_2 + I_2}. \]

Case 1.1, \( \omega_1 = 0 \). We denote \( N_1 = (0, 0, 0, 1) \) and
\[
T_3 = \begin{pmatrix}
N_1 A_2^3 & 0 \\
N_1 A_2^2 & 0 \\
N_1 A_2 & 0 \\
N_1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
such that
\[
A_3 = T_3 A_2 T_3^{-1} = \begin{pmatrix}
a_1 & -a_2 & -a_3 & -a_4 & a_5 \\
1 & 0 & 0 & 0 & \rho_2(\rho_4 + \rho_8 + \rho_9) - (\rho_4 + \rho_7)^2 \\
0 & 1 & 0 & 0 & \rho_2(\rho_4 + \rho_8 - \rho_5) \\
0 & 0 & 1 & 0 & -\rho_2 \\
0 & 0 & 0 & 0 & -\theta_2
\end{pmatrix},
\]
where
\[
a_1 = \rho_3 + \rho_7 + \rho_9 + \theta_1 > 0,
\]
\[
a_2 = (\rho_3 + \rho_7 + \rho_9) \theta_1 + \rho_3 \rho_7 + \rho_3 \rho_9 + \rho_4 \rho_9 - \rho_5 \rho_8 - \rho_4 \rho_6 \geq (\rho_3 + \rho_7 + \rho_9) \theta_1 + \tilde{a}_2,
\]
\[
a_3 = (\rho_3 \rho_7 + \rho_3 \rho_9 + \rho_7 \rho_9 - \rho_5 \rho_8 - \rho_4 \rho_6) \theta_1 + \rho_3 \rho_7 \rho_9 - \rho_4 \rho_6 \rho_9 - \rho_5 \rho_7 \rho_8 \geq \tilde{a}_2 \theta_1 + \tilde{a}_3,
\]
\[
a_4 = (\rho_3 \rho_7 \rho_9 - \rho_4 \rho_6 \rho_9 - \rho_5 \rho_7 \rho_8) \theta_1 \geq \tilde{a}_3 \theta_1,
\]
due to \( \rho_3 + \rho_4 = \rho_6 + \rho_7 \) and \( a_5 \) is a constant abbreviation. In addition, we have
\[
\tilde{a}_2 = \frac{I_1^2 I_2}{(a_1 + I_1^2)(a_2 + I_2^2)} \left( \tilde{b}_3 (d + \alpha_2 + r_2) + (\tilde{b}_2 + 1)(d + \alpha_2 + r_1) \right) + \frac{I_1^2}{a_1 + I_1^2} \left( \tilde{b}_4 (d + \alpha_1) + d(d + \alpha_1 + r_1) \right)
\]
\[
+ \frac{I_2}{a_2 + I_2} \left( \tilde{b}_5 (d + \alpha_2 + r_2) + d(d + \alpha_2 + r_2) \right) > 0,
\]
\[
\tilde{a}_3 = \frac{I_1^2 I_2}{(a_1 + I_1^2)(a_2 + I_2^2)} \left( \tilde{b}_1 (d + \alpha_1)(d + \alpha_2 + r_2) + \tilde{b}_2 (d + \alpha_2)(d + \alpha_1 + r_1) + d(d + \alpha_1 + r_1)(d + \alpha_2 + r_2) \right) > 0,
\]
Further calculations lead to
\[
a_1 a_2 > a_3, \quad a_1(a_2 a_3 - a_1 a_4) > a_3^2,
\]
due to
\[
(\rho_3 + \rho_7 + \rho_9) \tilde{a}_2 - \tilde{a}_3
\]
\[
> \frac{I_1^2 I_2}{(a_1 + I_1^2)(a_2 + I_2^2)} \left( \tilde{b}_1 \tilde{b}_2 (2d + \alpha_1 + \alpha_2) + \tilde{b}_1 d(d + \alpha_2 + r_2) + \tilde{b}_2 d(d + \alpha_1 + r_1) + d(d + \alpha_1 + r_1)(d + \alpha_2 + r_2) \right)
\]
\[
> 0.
\]
Therefore, we obtain
\[
(T_3 T_2 T_1) \Xi_2^2 (T_3 T_2 T_1)^T + A_3^2 (T_3 T_2 T_1) \Sigma_3 (T_3 T_2 T_1)^T + \rho (T_3 T_2 T_1) \Sigma_3 (T_3 T_2 T_1)^T A_3^2 = 0.
\]
From Lemma 2.3, we obtain \((T_3T_2T_1)\Sigma_1(T_3T_2T_1)^T = (\sigma_1\rho_1\rho_5\rho_8)^2\Omega_1\), where

\[
\Omega_1 = \begin{pmatrix}
\frac{a_2a_3-a_4}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & 0 & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & 0 \\
0 & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & 0 \\
0 & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} & 0 \\
0 & 0 & 0 & \frac{a_3}{2(a_1a_2a_3-a_5^2-a_7^2a_4)} \\
\end{pmatrix}
\]

is a semi-positive definite symmetric matrix and its first 4 \(\times\) 4 elements form a matrix that is positive definite. Hence \(\Sigma_1 = (\sigma_1\rho_1\rho_5\rho_8)^2(T_3T_2T_1)^{-1}\Omega_1[(T_3T_2T_1)^{-1}]^T\) is also a semi-positive definite matrix.

**Case 1.2, \(\omega_1 \neq 0\).** We denote

\[
T_4 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -\frac{\omega_1}{\rho_8} & 1 \\
\end{pmatrix}.
\]

such that

\[
A_4 = T_4A_2T_4^{-1} = \begin{pmatrix}
-\theta_1 & 0 & 0 & 0 \\
-\rho_1 & -\rho_4 - \rho_3 & \frac{\rho_1\omega_1}{\rho_8} + \rho_5 & \rho_4 & -\rho_2 \\
0 & -\rho_3 & 0 & \rho_2 & -\rho_2 \\
0 & 0 & \omega_1(\rho_1 + \rho_4 - \rho_3) & \rho_4 - \rho_3 & -\rho_2(\frac{\omega_1}{\rho_8} + 1) \\
\end{pmatrix}.
\]

Further we denote

\[
T_5 = \begin{pmatrix}
N_1A_4^3 & 0 \\
N_1A_4^2 & 0 \\
N_1A_4 & 0 \\
N_1 & 0 \\
\end{pmatrix},
\]

such that

\[
A_5 = T_5A_4T_3^{-1} = \begin{pmatrix}
-a_1 & -a_2 & -a_3 & -a_4 & a_6 \\
1 & 0 & 0 & 0 & a_7 \\
0 & 1 & 0 & 0 & a_8 \\
0 & 0 & 1 & 0 & -\rho_2(\frac{\omega_1}{\rho_8} + 1) \\
0 & 0 & 0 & 0 & -\theta_2 \\
\end{pmatrix},
\]

where \(a_6, a_7\) and \(a_8\) are constant abbreviations. Similar to Case 1.1, we can obtain that

\[
(T_5T_4T_2T_1)\Xi_2^2(T_5T_4T_2T_1)^T + A_5[(T_5T_4T_2T_1)\Sigma_1(T_5T_4T_2T_1)^T] + [(T_5T_4T_2T_1)\Sigma_1(T_5T_4T_2T_1)^T]A_5^T = 0.
\]

From Lemma 2.3, we obtain \((T_3T_4T_2T_1)\Sigma_1(T_3T_4T_2T_1)^T = (\sigma_1\rho_1\rho_5\rho_8(\frac{\omega_1(\rho_1 + \rho_4 - \rho_3)}{\rho_8} + \rho_5))^2\Omega_1\) is a semi-positive definite symmetric matrix. Thus \(\Sigma_1 = (\sigma_1\rho_1\rho_8(\frac{\omega_1(\rho_1 + \rho_4 - \rho_3)}{\rho_8} + \rho_5))^2(T_3T_4T_2T_1)^{-1}\Omega_1[(T_3T_4T_2T_1)^{-1}]^T\) is also a semi-positive definite matrix.

**Case 2.** For

\[
\Xi_2^2 + A\Sigma_2 + \Sigma_2A^T = 0,
\]

Let

\[
T_6 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
such that
\[
A_6 = T_6 A_6 T_6^{-1} = \begin{pmatrix}
-\theta_2 & 0 & 0 & 0 & 0 \\
-\rho_2 & -\rho_3 & \rho_4 & \rho_5 & -\rho_1 \\
\rho_2 & 0 & -\rho_7 & 0 & \rho_1 \\
0 & \rho_6 & -\rho_7 & 0 & \rho_1 \\
0 & 0 & 0 & 0 & -\theta_2
\end{pmatrix}.
\]

Further we obtain
\[
A_7 = T_2 A_6 T_2^{-1} = \begin{pmatrix}
-\theta_2 & 0 & 0 & 0 & 0 \\
-\rho_2 & -\rho_3 & -\rho_5 & \rho_4 & \rho_5 & -\rho_1 \\
\rho_2 & 0 & -\rho_7 & 0 & \rho_1 \\
0 & \omega_2 & \rho_4 & \rho_5 & -\rho_9 & -\rho_1 \\
0 & 0 & 0 & 0 & -\theta_2
\end{pmatrix},
\]

where
\[
\omega_2 = \rho_8 + \rho_9 - \rho_3 - \rho_5 = \alpha_2 - \frac{\beta_1}{a_1 + I_4}.
\]

**Case 2.1, \( \omega_2 = 0 \).** We denote
\[
T_7 = \begin{pmatrix}
N_1 A^3_7 & 0 \\
N_1 A^2_7 & 0 \\
N_1 A_7 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

such that
\[
A_8 = T_7 A_7 T_7^{-1} = \begin{pmatrix}
-b_1 & -b_2 & -b_3 & -b_4 & b_5 \\
1 & 0 & 0 & 0 & b_6 \\
0 & 1 & 0 & 0 & \rho_1 (\rho_3 + \rho_4 - \rho_8) \\
0 & 0 & 1 & 0 & -\rho_2 \\
0 & 0 & 0 & 0 & -\theta_2
\end{pmatrix},
\]

where
\[
b_1 = \rho_3 + \rho_7 + \rho_9 + \theta_2 > 0, \\
b_2 = (\rho_3 + \rho_7 + \rho_9) \theta_2 + \rho_3 \rho_7 + \rho_3 \rho_9 + \rho_7 \rho_9 - \rho_5 \rho_8 - \rho_4 \rho_6, \\
b_3 = (\rho_3 \rho_7 + \rho_3 \rho_9 + \rho_7 \rho_9 - \rho_5 \rho_8 - \rho_4 \rho_6) \theta_1 + \rho_3 \rho_7 \rho_9 - \rho_4 \rho_6 \rho_9 - \rho_5 \rho_7 \rho_8, \\
b_4 = (\rho_3 \rho_7 \rho_9 - \rho_4 \rho_6 \rho_9 - \rho_5 \rho_7 \rho_8) \theta_1,
\]

and \( b_5 \) and \( b_6 \) are constant abbreviations. Moreover, similar to the calculation in Case 1.1, we have
\[
b_1 b_2 > b_3, \quad b_1 (b_2 b_3 - b_1 b_4) > b_3^2.
\]

Therefore, we obtain
\[
(T_7 T_2 T_6)^T \Sigma_2^2 (T_7 T_2 T_6)^T + A_8 [(T_7 T_2 T_6) \Sigma_2 (T_7 T_2 T_6)^T] + [(T_7 T_2 T_6) \Sigma_2 (T_7 T_2 T_6)^T] A_8^T = 0.
\]

From Lemma 2.3, we obtain \( (T_7 T_2 T_6)^T \Sigma_2 (T_7 T_2 T_6)^T = (\sigma_2 \rho_2 \rho_4 \rho_6)^2 \Omega_2 \), where
\[
\Omega_2 = \begin{pmatrix}
\frac{b_1}{b_3} - \frac{b_3}{b_4} & 0 & 0 & 0 & 0 \\
0 & \frac{b_1}{b_3} - \frac{b_3}{b_4} & 0 & 0 & 0 \\
0 & 0 & \frac{b_1}{b_3} - \frac{b_3}{b_4} & 0 & 0 \\
0 & 0 & 0 & \frac{b_1}{b_3} - \frac{b_3}{b_4} & 0 \\
0 & 0 & 0 & 0 & \frac{b_1}{b_3} - \frac{b_3}{b_4}
\end{pmatrix}
\]
is a semi-positive definite symmetric matrix and its first 4 \( \times 4 \) elements form a matrix that is positive definite. Hence \( \Sigma_2 = (\sigma_2 \rho_2 \rho_4 \rho_6)^2 (T_7 T_2 T_6)^{-1} \Omega_2 (T_7 T_2 T_6)^{-1} \) is also a semi-positive definite matrix.
Case 2.2, $\omega_2 \neq 0$. We denote
\[
T_8 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\frac{\omega_2}{\rho_6} & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]
such that
\[
A_9 = T_8 A_8 T_8^{-1} = \begin{pmatrix}
-\theta_1 & 0 & 0 & \frac{\omega_2}{\rho_6} + \rho_4 & 0 \\
-\rho_2 - \rho_3 - \rho_5 & \rho_5 & 0 & -\rho_1 \\
0 & \rho_6 & 0 & 0 \\
0 & \frac{\omega_2 (\rho_5 + \rho_T - \rho_6)}{\rho_6} + \rho_4 & \rho_5 - \rho_9 & -\rho_1 \left(\frac{\omega_2}{\rho_6} + 1\right) \\
0 & 0 & 0 & -\theta_1
\end{pmatrix}.
\]
Further we denote
\[
T_9 = \begin{pmatrix}
N_1 A_9^3 & 0 \\
N_1 A_9^2 & 0 \\
N_1 A_9 & 0 \\
N_1 & 0
\end{pmatrix},
\]
such that
\[
A_{10} = T_9 A_9 T_9^{-1} = \begin{pmatrix}
-b_1 & -b_2 & -b_3 & -b_4 & b_7 \\
1 & 0 & 0 & 0 & b_8 \\
0 & 1 & 0 & 0 & b_9 \\
0 & 0 & 1 & 0 & -\rho_1 \left(\frac{\omega_2}{\rho_6} + 1\right) \\
0 & 0 & 0 & 0 & -\theta_1
\end{pmatrix},
\]
where $b_7$, $b_8$ and $b_9$ are constant abbreviations. Similar to Case 1.2, we can obtain that
\[
(T_9 T_8 T_2 T_6) \Xi_2^2 (T_9 T_8 T_2 T_6)^T + A_{10} [(T_9 T_8 T_2 T_6) \Sigma_2 (T_9 T_8 T_2 T_6)^T] + [(T_9 T_8 T_2 T_6) \Sigma_2 (T_9 T_8 T_2 T_6)^T] A_{10}^T = 0.
\]
From Lemma 2.3, we obtain $(T_9 T_8 T_2 T_6) \Sigma_2 (T_9 T_8 T_2 T_6)^T = (\sigma_2 \rho_2 \rho_6 (\frac{\omega_2 (\rho_T + \rho_6 - \rho_5)}{\rho_6} + \rho_4))^2 \Omega_2$ is a semi-positive definite symmetric matrix. Thus $\Sigma_2 = (\sigma_2 \rho_2 \rho_6 (\frac{\omega_2 (\rho_T + \rho_6 - \rho_5)}{\rho_6} + \rho_4))^2 (T_9 T_8 T_2 T_6)^{-1} \Omega_2 [(T_9 T_8 T_2 T_6)^{-1}]^T$ is also a semi-positive definite matrix.

This implies that the stationary solution $(x_1(t), x_2(t), S(t), I_1(t), I_2(t))$ has a unique probability density function $\Psi(x_1, x_2, S, I_1, I_2)$ as follows:
\[
\Psi(x_1, x_2, S, I_1, I_2) = (2\pi)^{-\frac{\delta}{2}} |\Sigma|^{-\frac{\delta}{2}} \exp \left\{ -\frac{1}{2} (x_1 - x_1^*, x_2 - x_2^*, S - S^*, I_1 - I_1^*, I_2 - I_2^*) \Sigma^{-1} (x_1 - x_1^*, x_2 - x_2^*, S - S^*, I_1 - I_1^*, I_2 - I_2^*) \right\},
\]
where $\Sigma = \Sigma_1 + \Sigma_2$. This completes the proof.

6. Numerical simulations

To consider the possibility that humans could be infected with two variants of SARS-CoV-2 coronavirus simultaneously, in this section, we use the parameter values given in [21] and present several numerical experimental examples to validate the correctness of our theoretical results. By utilizing Milstein’s higher order
method [22], we obtain the discretization equation as follows,

\[
\begin{align*}
    x_1^{(k+1)} &= x_1^{(k)} + \theta_1 (x_1^{(k)} - \ln x_1^{(k)}) \Delta t + \sigma_1 \sqrt{\Delta t} \eta_k, \\
    x_2^{(k+1)} &= x_2^{(k)} + \theta_2 (x_2^{(k)} - \ln x_2^{(k)}) \Delta t + \sigma_2 \sqrt{\Delta t} \xi_k, \\
    S^{(k+1)} &= S^{(k)} + \left( A - d S^{(k)} - \frac{e^{x_1^{(k)}} S^{(k)} I_1^{(k)}}{a_1 + I_1^{(k+1)}} - \frac{e^{x_2^{(k)}} S^{(k)} I_2^{(k)}}{a_2 + I_2^{(k+1)}} + r_1 I_1^r + r_2 I_2^r \right) \Delta t, \\
    I_1^{(k+1)} &= I_1^{(k)} + \left( \frac{e^{x_1^{(k)}} S^{(k)} I_1^{(k)}}{a_1 + I_1^{(k+1)}} - (d + \alpha_1 + r_1) I_1^{(k)} \right) \Delta t, \\
    I_2^{(k+1)} &= I_2^{(k)} + \left( \frac{e^{x_2^{(k)}} S^{(k)} I_2^{(k)}}{a_2 + I_2^{(k+1)}} - (d + \alpha_2 + r_2) I_2^{(k)} \right) \Delta t,
\end{align*}
\]

where \( \Delta t > 0 \) is the time increment; \( \eta_k \) and \( \xi_k \) are two independent random variables which follow the standard normal distribution for \( k = 1, \ldots, n \); \( (x_1^{(k)}, x_2^{(k)}, S^{(k)}, I_1^{(k)}, I_2^{(k)})^T \) denotes the corresponding value of the \( k \)-th iteration of the above discretization equation. Let the initial value of the stochastic system as \( (x_1(0), x_2(0), S(0), I_1(0), I_2(0)) = (\ln \beta_1, \ln \beta_2, 30, 0.1, 0.1) \).

### Table 6.1: Description of variables in the model (1.1).

<table>
<thead>
<tr>
<th>Paras</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
<th>Sources</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>Birth/recruitment rate</td>
<td>0.01138</td>
<td>day(^{-1} )</td>
<td>[21, 23]</td>
</tr>
<tr>
<td>( d )</td>
<td>Natural death rate</td>
<td>( \frac{1}{76 \times 365} )</td>
<td>day(^{-1} )</td>
<td>[21, 23]</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>Strain 1 contact rate</td>
<td>0.2944</td>
<td>person(^{-1} ) day(^{-1} )</td>
<td>[21, 24]</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>Strain 2 contact rate</td>
<td>0.45</td>
<td>person(^{-1} ) day(^{-1} )</td>
<td>[21]</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>Treatment cure rate of strain 1</td>
<td>0.1</td>
<td>day(^{-1} )</td>
<td>[21, 25]</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>Treatment cure rate of strain 2</td>
<td>( \frac{1}{15} )</td>
<td>day(^{-1} )</td>
<td>[21, 25]</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>Strain 1 induced death rate</td>
<td>0.0214</td>
<td>day(^{-1} )</td>
<td>[21, 26]</td>
</tr>
<tr>
<td>( \alpha_2 )</td>
<td>Strain 2 induced death rate</td>
<td>0.0214</td>
<td>day(^{-1} )</td>
<td>[21, 26]</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>Half-saturation rate of strain 1</td>
<td>80</td>
<td>-</td>
<td>Estimated</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>Half-saturation rate of strain 2</td>
<td>168.6</td>
<td>-</td>
<td>Estimated</td>
</tr>
<tr>
<td>( \theta_1 )</td>
<td>Reversion speed of ( \ln \beta_1(t) )</td>
<td>0.3</td>
<td>-</td>
<td>[20, 27]</td>
</tr>
<tr>
<td>( \theta_2 )</td>
<td>Reversion speed of ( \ln \beta_2(t) )</td>
<td>0.3</td>
<td>-</td>
<td>[20, 27]</td>
</tr>
<tr>
<td>( \sigma_1 )</td>
<td>Noise intensity of ( \ln \beta_1(t) )</td>
<td>([0, 1])</td>
<td>-</td>
<td>[27]</td>
</tr>
<tr>
<td>( \sigma_2 )</td>
<td>Noise intensity of ( \ln \beta_2(t) )</td>
<td>([0, 1])</td>
<td>-</td>
<td>[27]</td>
</tr>
</tbody>
</table>

### Example 6.1. (Stationary distribution and density function)

First, to study the stationary distribution of model (1.2), we choose \( \sigma_1 = \sigma_2 = 0.02 \) and the other parameters are shown in (6.1). Then we compute

\[
R_1^\sigma = \frac{A \hat{\beta}_1 e^{\frac{\sigma_1^2}{2}}}{a_1 d (d + \alpha_1 + r_1)} = 9.5675 > 1, \quad R_2^\sigma = \frac{A \hat{\beta}_2 e^{\frac{\sigma_2^2}{2}}}{a_2 d (d + \alpha_2 + r_2)} = 9.5645 > 1.
\]

According to Theorem 4.1, system (1.2) admit at least an ergodic stationary distribution. The phase diagrams and frequency histograms of \( S(t), I_1(t) \) and \( I_2(t) \) are given in the left and middle columns of Fig. 6.1, respectively.

Furthermore, by Theorem 5.1, we obtain \( M_1 = 2.9335 \times 10^{-4} > 0, M_2 = 5.2781 \times 10^{-4} > 0 \) and

\[
R_1^\rho = \frac{A \hat{\beta}_1}{a_1 d (d + \alpha_1 + r_1)} = 9.5664 > 1, \quad R_2^\rho = \frac{A \hat{\beta}_2}{a_2 d (d + \alpha_2 + r_2)} = 9.5634 > 1.
\]

Then the solution \( (x_1(t), x_2(t), S(t), I_1(t), I_2(t)) \) follows the normal density function \( \Psi(x_1, x_2, S, I_1, I_2) \sim N_5(E^*, \Sigma) \),
where \( E^* = (\hat{x}_1, \hat{x}_2, S^*, I_1^*, I_2^*) = (\ln 0.2944, \ln 0.45, 33.0690, 0.1698, 0.3055) \) and

\[
\Sigma = \begin{pmatrix}
0.00066667 & 0 & -0.000048793 & 0.000045677 & -1.3206 \times 10^{-7} \\
0 & 0.00066667 & -0.000063707 & -1.3229 \times 10^{-7} & 0.000059496 \\
-0.000048793 & -0.000063707 & 0.02087 & -0.00084018 & -0.00048108 \\
0.000045677 & -1.3229 \times 10^{-7} & -0.00084018 & 0.016251 & -0.001029 \\
-1.3206 \times 10^{-7} & 0.000059496 & -0.00048108 & -0.001029 & 0.0017162
\end{pmatrix}.
\]

Therefore, one can obtain the following marginal density functions:

\[
\begin{align*}
\frac{\partial \Psi(x_1, x_2, S, I_1, I_2)}{\partial S} & \sim \mathcal{N}(33.0690, 0.02087), \\
\frac{\partial \Psi(x_1, x_2, S, I_1, I_2)}{\partial I_1} & \sim \mathcal{N}(0.1698, 0.0016251), \\
\frac{\partial \Psi(x_1, x_2, S, I_1, I_2)}{\partial I_2} & \sim \mathcal{N}(0.3055, 0.0017162),
\end{align*}
\]

The middle and right columns of Fig. 6.1 show the corresponding marginal density functions and frequency histograms for 3,000,000 iteration points, respectively.

Figure 6.1: The left column displays the quantities of susceptible individuals \( S \), and infected individuals \( I_1 \) and \( I_2 \) in system (1.2). The corresponding frequency histograms and marginal density functions are shown in the middle column. The right column displays the fitted plots of the corresponding frequency histograms and marginal density functions.

**Example 2. (Persistence of strain 1 and extinction of strain 2)** Choose \( \beta_1 = 0.025 \) and \( \sigma_1 = \sigma_2 = 0.01 \), and the other parameters are shown in Table 6.1, we have

\[ R_1^e = 0.8124 < 1, \; R_2^e = 9.5642 > 1. \]

Thus from Theorem 3.1, the disease \( I_1 \) will go to extinct in a long term, while \( I_2 \) will persist in a long term, which is supported by Fig. 6.2.

**Example 3. (Persistence of strain 2 and extinction of strain 1 in stochastic model 1)** Choose \( \beta_2 = 0.035 \) and \( \sigma_1 = \sigma_2 = 0.01 \), and the other parameters are shown in Table 6.1, we have

\[ R_1^e = 9.5672 > 1, \; R_2^e = 0.7439 < 1. \]
Thus from Theorem 3.1, the disease $I_2$ will go to extinct in a long term, while $I_1$ will persist in a long term, which is supported by Fig. 6.3.

Example 4. (Extinction of two diseases) Choose $\hat{\beta}_2 = 0.025$, $\hat{\beta}_2 = 0.035$ and $\sigma_1 = \sigma_2 = 0.4$, and the other parameters are shown in Table 6.1, we have

$$R_1^e = 0.9282 < 1, \quad R_2^e = 0.8499 < 1.$$
Thus from Theorem 3.1, the diseases $I_1$ and $I_2$ will go to extinct in a long term, which is supported by Fig. 6.4.

![Figure 6.4: Computer simulations for the stochastic solutions of $I_1$ and $I_2$ under the parameter conditions of Example 4.](image)

**Example 5.** (Variation trends of $R^c_1$, $R^s_1$, $R^p_1$, $R^c_2$, $R^s_2$ and $R^p_2$) Figure 6.5 shows the trends of $R^c_1$, $R^s_1$, $R^p_1$, $R^c_2$, $R^s_2$ and $R^p_2$ for different $\beta = \hat{\beta}_1 = \hat{\beta}_2 \in [0.02, 0.06]$. According to Theorems 3.1, 4.1 and 5.1, let $\sigma_1 = \sigma_2 = 0.5$ and the other parameters are shown in Table 6.1, and it can be found that

- the disease $I_1$ will go to extinction when $\beta \in [0.02, 0.025]$;
- the disease $I_2$ will go to extinction when $\beta \in [0.02, 0.0382]$;
- at least one ergodic stationary distribution is admitted when $\beta \in [0.0439, 0.06]$;
- the global solution $(S(t), I_1(t), I_2(t))$ follows a unique normal density function when $\beta \in [0.0471, 0.06]$.

### 7. Conclusion

The widespread outbreak of infectious diseases has caused immense damage to populations around the globe, encompassing not only high incidence rates but also significant social and economic costs. It is worth noting that environmental noise plays a pivotal role in shaping the development of epidemics. In this paper, we have thoroughly examined the dynamic behaviors of the stochastic dual epidemic hypothesis model. Our study has taken into account the potential impact of stochastic perturbations, which follow OU process, on the natural logarithm of the disease transmission coefficients $\beta_1$ and $\beta_2$. This approach serves as an improved means of introducing stochastic environmental noise to population dynamic models in biological reality, as previously demonstrated in [9]. We have summarized our key findings as follows:

- If $R^c_1 < 1$ and $R^c_2 < 1$, then the two infectious diseases of system (1.2) go to extinction in a long term.
- If $R^c_1 > 1$ and $R^c_2 < 1$, then the disease $I_1$ will persist in a long term, while $I_2$ will go to extinct in a long term.
- If $R^c_1 < 1$ and $R^c_2 > 1$, then the disease $I_1$ will go to extinct in a long term, while $I_2$ will persist in a long term.
Figure 6.5: The variation trends of $R_e^1$, $R_s^1$, $R_p^1$, $R_e^2$, $R_s^2$ and $R_p^2$ with different $\beta = \hat{\beta}_1 = \hat{\beta}_2 \in [0.02, 0.06]$ and $\sigma_1 = \sigma_2 = 0.5$. The other parameters are shown in Table 6.1.

- If $R_s^1 > 1$ and $R_s^2 > 1$, then the stochastic system (1.2) admits at least one ergodic stationary distribution
- If $M_1, M_2 > 0$ and $R_p^1, R_p^2 > 1$, then the stationary solution around $E^*$ follows a normal distribution $N_5(E^*, \Sigma)$

It is noteworthy that as the noise intensities $\sigma_1$ and $\sigma_1$ increase, diseases that were previously extinct will become persistent, with all other parameters remaining constant. Similarly, as the reversal rates $\theta_1$ and $\theta_2$ decrease, diseases that were previously extinct will become persistent, with all other parameters remaining constant. This is different from the general results of the perturbation of white noise.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11871473), Shandong Provincial Natural Science Foundation (No. ZR2019MA010) and the Fundamental Research Funds for the Central Universities, China (No. 22CX03030A).

Conflict of interest

No conflict of interests.

Data Availability

The manuscript has no associated data.

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