Effect of delay in a Musca domestica houseflies model: Stability and Global Hopf Bifurcation

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Effect of delay in a Musca domestica houseflies model: Stability and Global Hopf Bifurcation

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Abstract. The houseflies model with discrete delay is studied in both theoretical and numerical ways. The solution of the delayed system is positive and bounded. Choosing the delay as the bifurcation parameter, stability analysis for the positive equilibrium of the delayed model, local and global Hopf bifurcation are given in theoretical aspect. Dynamical behaviors such as supercritical Hopf bifurcation is detected by computer simulations. The theoretical analysis and numerical observations in this work are interesting in biomathematics research area.

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1. Introduction

To describe the population growth of houseflies Musca domestica, Taylor and Sokal (1976) proposed the delay equation [7]

\[
\frac{dx}{dt} = -dx(t) + bx(t - \tau)[k - bzx(t - \tau)],
\]

where \( x(t) \) is the number of adults at time \( t \), \( d \) denotes the death rate of adults, the time delay \( \tau \) is the length of the developmental period between oviposition and eclosion of adults. The number of eggs laid is assumed to be proportional to the number of adults, so at time \( t - \tau \) the number of new eggs would be \( bx(t - \tau) \), where \( b \) is the number of eggs laid per adult [6]. The term \( k - bzx(t - \tau) \) represents the egg-to-adult survival rate, where \( k \) is the maximum egg-adult survival rate, and \( z \) is the reduction in survival produced by each additional egg. All parameters include time delay are nonnegative.
The initial value of equation (1.1) is given by
\[ x(\theta) = \phi(\theta) > 0, \quad \theta \in [-\tau, 0] \] (1.2)
where \( \phi \) is continuous on \([-\tau, 0]\).

For the delayed houseflies model (1.1), Deng [1] taken the inhomogeneous distribution of the houseflies in different spatial locations into account, considered a modified houseflies model with diffusive term. By employing the upper-lower solution method and monotone iteration technique, obtained the sufficient conditions for the existence of traveling wave fronts.

Analytical analysis of equation houseflies model (1.1) has never been carried out [6], therefore, our manuscript mainly study the stability, local and global Hopf bifurcation of delayed houseflies model (1.1). The results show that with the increase of time delay, the stable positive equilibrium loses its stability, and stable periodic solutions can be obtained through the supercritical Hopf bifurcation. As the delay continues to increase, the system with delay bifurcates into unstable periodic solutions. The houseflies model (1.1) exhibits asymptotic convergence to the unique positive equilibrium for all choices of the parameters when delay is ignored. Therefore, any rich dynamic behaviors such as supercritical Hopf bifurcation will be completely attributed to the introduction of time delay terms.

Our manuscript mainly emphasizes the influence of time delay \( \tau \) on the dynamics of houseflies model (1.1). It’s organized as follows. In Sec. 2, we use the delay \( \tau \) as the bifurcation parameter, the analysis of Hopf bifurcation is given for the positive equilibrium for system (1.1) in theoretical aspect. Furthermore, we prove the global existence of Hopf bifurcation. In Sec. 3, using the numerical continuation software DDE-BIFTOOL, numerical explorations are carried out in order to support the obtained theoretical predictions. Simulations show that as the delay increases, the positive equilibrium loses its stability and bifurcated a family of orbitally asymptotically stable periodic solutions. Rich dynamics such as Hopf bifurcation has been demonstrated when choose the delay as the bifurcation parameter. In Sec. 4, we will give our conclusions and main contributions, in particular on the impact of time delay from the biological aspect.

2. Local and Global Hopf Bifurcation Analysis

When delay doesn’t exist for the houseflies model (1.1), it exhibits asymptotic convergence to the unique positive equilibrium for all choices of the parameters. Therefore, any rich dynamic behaviors will be completely attributed to the introduction of delay terms.

**Theorem 2.1.** The solutions of system (1.1) with initial data (1.2) are positive and bounded for \( t > 0 \).

*Proof.* See Appendix I. \( \Box \)
Effect of delay in a Musca domestica houseflies model

System (1.1) admits equilibria
\[ x_0 = 0 \text{ and } x^* = \frac{bk - d}{b^2z} \text{ when } bk > d. \]
Small perturbations from \( x_0 = 0 \) satisfy the linear equation
\[ \frac{dx}{dt} = -dx(t) + bkkx(t - \tau), \]
which shows that \( x_0 = 0 \) is unstable when \( bk > d \).

2.1. Local Hopf bifurcation
The following we want to see how the delay \( \tau \) affects the stability of the positive equilibrium \( x^* \). Let \( X = x - x^* \), then system (1.1) becomes
\[
\frac{dX}{dt} = -d(X(t) + x^*) + b(X(t - \tau) + x^*)[k - bz(X(t - \tau) + x^*)]
= -dX(t) + (2d - kb)X(t - \tau) - b^2zX^2(t - \tau).
\]
Thus, the linearized equation at \( x^* \) is
\[ \frac{dX}{dt} = -dX(t) + (2d - kb)X(t - \tau). \]
We look for solutions of the form \( X(t) = ce^{\lambda t} \), where \( c \) is a constant. The eigenvalues \( \lambda \) are solutions of the associated characteristic equation
\[ \lambda + d + (kb - 2d)e^{-\lambda \tau} = 0, \]
which is a transcendental equation. By the linearization theory, \( x^* \) is asymptotically stable if all eigenvalues of (2.1) have negative real parts.

Notice that when there is no time delay, i.e., \( \tau = 0 \), the eigenvalue of the characteristic equation (2.1) is \( \lambda = d - kb \), which is a negative real number.

Set \( \lambda = \mu + i\nu \). Separating the real and imaginary parts of the characteristic equation (2.1), we obtain
\[
\begin{align*}
\mu + d + (kb - 2d)e^{-\mu \tau} \cos \nu \tau &= 0, \\
\nu - (kb - 2d)e^{-\mu \tau} \sin \nu \tau &= 0.
\end{align*}
\]
We seek conditions on \( \tau \) such that \( \text{Re}\lambda \) changes from negative to positive. By the continuity, if \( \lambda \) changes from \( d - kb \) to a value such that \( \text{Re}\lambda = \mu > 0 \) when \( \tau \) increases, there must be some value of \( \tau \), say \( \tau_0 \), at which \( \text{Re}\lambda(\tau_0) = \mu((\tau_0)) = 0 \). In other words, the characteristic equation (2.1) must have a pair of purely imaginary roots \( \pm i\nu_0 \), \( \nu_0 = \nu(\tau_0) \). Suppose such is the case. Then we have
\[ \tau_k = \frac{1}{\nu_0} \left( \arcsin \frac{\nu_0}{kb - 2d} + 2k\pi \right), k = 0, 1, 2, \ldots, \]
(2.2)
where \( \nu_0 = \sqrt{(kb - 3d)(kb - d)} \). Therefore, when \( \tau = \tau_k \), equation (2.1) has a pair of purely imaginary roots \( \pm i\nu_0 \), which are simple and all other roots have negative real parts. When \( 0 < \tau < \tau_0 \), all roots of equation (2.1) have strictly negative real parts.

Denote \( \lambda(\tau) = \mu(\tau) + i\nu(\tau) \) the root of equation (2.1) near \( \tau = \tau_k \) satisfying \( \mu(\tau_k) = 0 \) \( \nu(\tau_k) = \nu_0, k = 0, 1, 2, \ldots \). By the theory of delay differential
equations, for \( \forall \tau_k, \exists \varepsilon > 0 \) s.t. \( \lambda(\tau) \) in \( |\tau - \tau_k| < \varepsilon \) about \( \tau \) is continuous and differentiable. Furthermore, we have the transversality condition

\[
\left. \frac{d \text{Re} \lambda(\tau)}{d\tau} \right|_{\tau = \tau_j} = \frac{1}{(kb - 2d)^2} > 0
\]  

(2.3)

We have just shown the following theorem, which is due to Hassard et al. [2].

**Theorem 2.2.** For the houseflies model (1.1), we have

(i) If \( 0 \leq \tau < \tau_0 \), then the positive equilibrium \( x^* = \frac{bk - d}{b^2z} \) is locally asymptotically stable.

(ii) If \( \tau > \tau_0 \), then \( x^* \) is unstable.

(iii) When \( \tau = \tau_0 \), a Hopf bifurcation occurs at \( x^* \), that is, periodic solutions bifurcate from \( x^* \). The periodic solutions exist for \( \tau > \tau_0 \) and are stable.

2.2. Global Hopf bifurcation

Base on [9], we will study the existence of global Hopf bifurcation for system (1.1) in this subsection. System (1.1) could be written as the following form

\[
\dot{x} = F(x_t, \tau, p),
\]

where \( x_t(\theta) \in ([-\tau, 0], \mathbb{R}) \). Define

\[
X = C([-\tau, 0], \mathbb{R}),
\]

\[
\Sigma = Cl\{(x(t), \tau, p) \in X \times \mathbb{R} \times \mathbb{R}^+, x(t) \text{ is a } T \text{-periodic solution of system (1.1)}\},
\]

\[
N = \{(\bar{x}, \bar{\tau}, p), F(\bar{x}, \bar{\tau}, p) = 0\}.
\]

Let \( l(x^*, \tau_k, \frac{2\pi}{v_0}) \) be the connected component of \((x^*, \tau_k, \frac{2\pi}{v_0})\) in \( \Sigma \), where \( \tau_k \) and \( v_0 \) are defined in (2.2).

**Lemma 2.3.** System (1.1) has no nontrivial solution of period \( \tau \).

**Proof.** Suppose system (1.1) has a nontrivial solution of period \( \tau \), then the following system

\[
\dot{x} = -dx + bx[k - bzx] = x[(bk - d) - b^2zx]
\]  

(2.4)

has a nontrivial solution.

We constrict function

\[
V(x) = x - x^* - x^* \ln \frac{x}{x^*}.
\]

Obviously, \( V(x) \) is positive definite. For all \( x > 0 \),

\[
\frac{dV}{dt} = -r \frac{x}{x^*}(x - x^*)^2, \text{ where } r = bk - d,
\]

hence \( \frac{dV}{dt} < 0 \) for all \( x > 0 \) and \( \lim_{t \to \infty} x(t) = x^* \), which contradict the fact that system (2.4) has a periodic solution, so we complete the proof. \( \square \)

**Theorem 2.4.** For each \( \tau > \tau_k \) \((k = 1, 2, \cdots)\), system (1.1) has at least \( k + 1 \) periodic solutions, where \( \tau_k \) is defined in (2.2).
Proof. The characteristic equation at equilibrium point $x^*$ is

$$\Delta(x^*, \tau, p)(\lambda) = \lambda + d + (kb - 2d)e^{-\lambda \tau} = 0,$$

from the discussion of Sec. 2, it can be verified $(x^*, \tau_k, \frac{2\pi}{v_0}), k = 1, 2, \cdots$ are isolated centers. Let

$$\Omega(\epsilon, \frac{2\pi}{v_0}) = \{(\eta, p) : 0 < \eta < \epsilon, | p - \frac{2\pi}{v_0} | < \epsilon\},$$

clearly if $| \tau - \tau_k | < \delta$ and $(\eta, p) \in \partial \Omega(\epsilon, \frac{2\pi}{v_0})$, the necessary and sufficient condition for $\Delta(x^*, \tau, p)(\eta + i\frac{2\pi}{p}) = 0$ are $\eta = 0, \tau = \tau_k$ and $p = \frac{2\pi}{v_0}$. Define

$$H^\pm(x^*, \tau_k, \frac{2\pi}{v_0})(\eta, p) = \Delta(x^*, \tau_k \pm \delta, p)(\eta + i\frac{2\pi}{p}).$$

By expression (2.3), we have the transversal number

$$\gamma(x^*, \tau_k, \frac{2\pi}{v_0}) = degB(H^-(x^*, \tau_k, \frac{2\pi}{v_0}), \Omega(\epsilon, \frac{2\pi}{v_0})) - degB(H^+(x^*, \tau_k, \frac{2\pi}{v_0}), \Omega(\epsilon, \frac{2\pi}{v_0})) = -1,$$

due to theorem 3.1 [9], we conclude that the connected component $l(x^*, \tau_k, \frac{2\pi}{v_0})$ through $(x^*, \tau_k, \frac{2\pi}{v_0})$ in $\Sigma$ is no empty, and we obtain

$$\Sigma(x, \tau, \bar{p}) \in l(x^*, \tau_k, \frac{2\pi}{v_0}) \cap N(x, \tau, p) < 0.$$

Since the first crossing number of each center is always $-1$, by theorem 3.3 [9], we conclude that $l(x^*, \tau_k, \frac{2\pi}{v_0})$ is unbounded.

Due to expression (2.2), we have $\frac{2\pi}{v_0} < \tau_k, (k \geq 1)$. Next we shall prove the projection of $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $\tau$-space is $[\bar{\tau}, \infty)$, where $\bar{\tau} < \tau_k$, lemma 2.3 shows when $\tau = 0$, system (1.1) has no nontrivial periodic solution, so the projection of $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $\tau$-space is always from zero.

Suppose the projection of $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $\tau$-space is bounded, there exist a $\tau^* > 0$, the projection of $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $\tau$-space is in the interval $(0, \tau^*)$, from $\frac{2\pi}{v_0} < \tau_k, (k \geq 1)$ and lemma 2.3, we know $p < \tau^* \tau$ for $(x, \tau, p) \in l(x^*, \tau_k, \frac{2\pi}{v_0})$, which imply that the projection of the connected component $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $p$-space is bounded, by theorem 2.1 [9], we know that the projection of $l(x^*, \tau_k, \frac{2\pi}{v_0})$ onto $x$-space is also bounded. Thus the connected component $l(x^*, \tau_k, \frac{2\pi}{v_0})$ is bounded, which is a contradiction, so we complete the proof. □

3. Numerical Simulations

In this section, we confirm previously obtained local and global Hopf Bifurcation results concerning the dynamics at the positive equilibrium and will extend them farther with the help of numerical bifurcation analysis.
Hereafter, parameters are fixed at the following values, were reported in Taylor and Sokal (1976),

\[ b = 1.81, \quad k = 0.5107, \quad d = 0.147 \quad \text{and} \quad z = 0.000226, \]

and consider the following specific model

\[
\frac{dx}{dt} = -0.147x(t) + 1.81x(t - \tau)[0.5107 - 0.00040906x(t - \tau)], \quad (3.1)
\]

with initial data is \( x(0) = 0.1 \).

System (3.1) has two distinct equilibria, the trivial equilibrium \( x_0 = 0 \), the positive equilibrium \( x^* = 1049.9 \).

**Figure 1.** The bifurcation diagram of system (3.1) in \( \tau - x \) space.

**Figure 2.** When \( \tau = 2.8 \) and \( \tau = 3 \), trajectory of system (3.1) on \( t - x \) space.
The conclusion of Thm 2.2 can be illustrated by Fig. 1, where the blue solid line represents stable equilibrium point, while the red dotted line indicated instability. The filled green circle represents stable periodic orbit, while open blue circles are unstable periodic orbits. By Fig. 1, $E^*$ is stable when $\tau \in [0, \tau_0)$ and unstable $\tau > \tau_0$ where $\tau_0 \approx 2.9465$. The first Hopf point is at $\tau_0$ and second Hopf point is at $\tau_1 \approx 13.1966$, generated the supercritical Hopf bifurcation at $\tau_0$ and $\tau_1$. The period of the solution at the critical delay value is $\omega = \frac{2\pi}{\nu_0} = 0.6130$. In details, equilibrium $x^*$ is stable at $\tau = 0$ and remains so until $\tau_0$ when $x^*$ goes through the first Hopf bifurcation, and a pair of complex eigenvalues crosses the imaginary axis from left to the right, a stable periodic orbit appears. As the time delay $\tau$ increases to $\tau_1$, $x^*$ goes through the second Hopf bifurcation, and the pair of complex eigenvalues crosses the imaginary axis from right to the left, the stable periodic orbit remain exists and a unstable periodic orbit appears. Fig. 2 shows the trajectories of system (3.1) with $\tau = 2.8$ and $\tau = 3$, respectively. Fig. 2 shows that the positive equilibrium $x^*$ is locally asymptotical with $\tau = 2.8 < \tau_0$, then as the time delay increases, it will lose its stability and a Hopf bifurcation occurs once $\tau = 3 > \tau_0$. Similar dynamics also occur at $\tau_2, \tau_3, \cdots$. Besides, note that the stable periodic orbit always exists when $\tau > \tau_0$.

**Remark 3.1.** Similar dynamics have been numerically detected in the Hutchinson Model [8]

$$\frac{dx}{dt} = rx(t) \left[1 - \frac{x(t-\tau)}{K}\right],$$

where $r$ is the intrinsic growth rate and $K$ is the carrying capacity of the population, $\tau > 0$ is the time delay. As Fig. 3 shows, the positive equilibrium is $x^* = 1$. Hopf point 1 is at $\tau_0 \approx 1.5708$, Hopf point 2 is at $\tau_1 \approx 7.8540$ and third Hopf point is at $\tau_2 \approx 14.1372$ with $\omega \approx 1$.

![Figure 3. The bifurcation diagram for Hutchinson model when chose $r = K = 1$.](image-url)
Remark 3.2. Numerical simulations indicate that its dynamics are very similar to that of the Nicholson’s blowflies equation [4],

$$\frac{dx}{dt} = P x(t - \tau) \exp \left[ -\frac{x(t - \tau)}{x_0} \right] - \delta x(t). \quad (3.3)$$

However, aperiodic oscillations appear in Nicholson’s model but not in houseflies model (1.1) [5, 6].

4. Conclusions

In this manuscript, we studied a houseflies model with constant delay. Mainly discussed the effect of the time delay on dynamical behaviors of the system. The model without the delay means all solutions converge to the positive equilibrium. We incorporated the discrete delay, local and global Hopf bifurcation are given in theoretical aspect, supercritical Hopf bifurcation can take place at the positive equilibrium when we choose the delay as the bifurcation parameter. The results could be very essential for the biologists who work with delayed systems.

From the biological aspect, the most interesting results are the following. If the delay is shorter, the amount of houseflies at the fixed level, as the delay increased, the species asymptotically vary in a periodic or in a quasi-periodic way which implied the houseflies exists at a oscillatory balance behavior. As delay goes on increases, the dynamic behavior tends to the periodic solutions, it means that the species exists at the period oscillatory behaviors. This complex interesting biological phenomena in this work are interesting in both biology and mathematics research area.

Author contributions Xin Zhang: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. Renxiang Shi: Actualization, methodology, formal analysis, validation, investigation, initial draft. All authors read and approved the final manuscript.

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Availability of Data and Materials Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations Conflict of interest The authors declare that they have no competing interests.

References


Appendix I

Proof. Define space $C_+ = \{ \phi \in C([-\tau, 0], \mathbb{R}_+) \}$, where $C_+$ denotes the Banach space of continuous function mapping $[-\tau, 0]$ into $\mathbb{R}_+$. Consider system (1.1) in $x \in \mathbb{R}_+$, define function

$$H(x) = -dx + bx(t - \tau)[k - bzx(t - \tau)],$$

where $H : C_+ \to \mathbb{R}_+$ and $H \in C^\infty(\mathbb{R}_+)$, then system (1.1) becomes

$$\dot{x} = H(x_t)$$

(4.1)

with initial data $x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$. Choosing $x(\theta) \in C_+$ such that $x = 0$, we obtain

$$H(x) \big|_{x=0,x \in C_+} = H(0) \geq 0,$$

by lemma Yang et al. [10], for any solution of equation (4.1) with $x_t(\theta) \in C_+$ and $x(t) = x(t, x(0))$, we know

$$x(t) \in \mathbb{R}_+, \forall t > 0,$$

that is the solutions remain nonnegative through the region $\mathbb{R}_+$. Positivity means the cone of the solutions is invariant in the system.

Furthermore, due to system (1.1) we obtain

$$\dot{x} = -dx + bx(t - \tau)[k - bzx(t - \tau)] \leq -dx + \frac{k^2}{4z}.$$
By the comparison theory [3], hence
\[ x(t) \leq x(0) + \frac{k^2}{4dz}. \]
The above discussion implies that \( x(t) \) is ultimately bounded. Boundedness of the system ensures that there is a restriction on the growth of the species due to limited resources in the nature. \( \square \)

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