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A simplified RHSS iteration method for saddle point problems

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Abstract

Recently, Bai and Benzi [Regularized HSS iteration methods for saddle-point linear systems, *Bit Numer. Math.*, 57 (2017) 287-311] have presented a class of regularized Hermitian and skew-Hermitian splitting methods to solve saddle point problems. In this paper, we establish a simplified RHSS (SRHSS) preconditioner which is much closer to the saddle point matrix than the RHSS preconditioner. We prove the convergence of the proposed method (SRHSS) under suitable restrictions on the iteration parameters. We also study the spectral properties of the preconditioned matrix and eigenvector distribution. Lastly, numerical experiments are carried out and experimental results show that the proposed SRHSS preconditioner method is feasible and effective.

Key words: Saddle point problems; preconditioning; shift-splitting; iterative method; spectral properties.

AMSC: 65F10; 65F15; 65F50

1 Introduction

We consider the solution of the saddle point problem

\[
\begin{pmatrix}
B & E \\
-E^* & O
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} =
\begin{pmatrix}
f \\
g
\end{pmatrix} \quad \text{or} \quad Au = b,
\]

where \(B \in \mathbb{C}^{m \times m}\) is a Hermitian positive definite matrix, \(E \in \mathbb{C}^{m \times n}\) is a rectangular matrix of full column rank, \(E^* \in \mathbb{C}^{n \times m}\) is the conjugate transpose of \(E\), \(O \in \mathbb{C}^{n \times n}\) is the zero matrix, and \(f \in \mathbb{C}^m, g \in \mathbb{C}^n\). As we know, the saddle point

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problem (1) arise frequently in a variety of scientific computing and engineering applications. See [1-3] for more background information and references therein.

In recent years, many efficient iteration methods and preconditioners have been studied for solving large scale linear system (1) such as SOR iteration methods [4], HSS-like preconditioners methods [5-6], constraint iteration methods [7], augmented Lagrangian preconditioners methods, the indefinite iterative method and so on. Recently, a class of regularized HSS methods was presented in [8] to solve saddle point problems.

In this paper, we propose a simplified regularized HSS method, which was much closer to the generalized saddle point problems than the regularized HSS preconditioner in [8]. Furthermore, we discuss theoretical analysis of the proposed iterative method and provide numerical experiments to validate the effectiveness of the proposed method.

2 The simplified regularized HSS preconditioners

In this section, we propose a simplified regularized HSS preconditioners and introduce some special simplified regularized HSS (SRHSS) preconditioners.

Suppose that \( Q \in \mathbb{C}^{n \times n} \) are Hermitian positive semidefinite matrix. For the coefficient matrix \( A \) of Eq. (2), Bai and Benzi in [8] discuss the following regularized Hermitian and skew-Hermitian splitting (RHSS):

\[
A = \begin{pmatrix} B & O \\ O & Q \end{pmatrix} + \begin{pmatrix} O & E \\ -E^* & -Q \end{pmatrix} = H_+ + S_- \\
= \begin{pmatrix} O & E \\ -E^* & Q \end{pmatrix} + \begin{pmatrix} B & O \\ O & -Q \end{pmatrix} = S_+ + H_-.
\]

Then the RHS splitting leads to the following HSS iteration scheme:

\[
\begin{aligned}
(\alpha I + H_+)x_{k+\frac{1}{2}} &= (\alpha I - S_-)x_k + b, \\
(\alpha I + S_+)x_{k+\frac{1}{2}} &= (\alpha I - H_-)x_k + b.
\end{aligned}
\]

This RHSS iteration methods can also result from the splitting \( A = M_\alpha - N_\alpha \) with

\[
M_\alpha = \frac{1}{2\alpha} \begin{pmatrix} \alpha I + B & 0 \\ 0 & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & E \\ -E^* & \alpha I + Q \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha I + B & (I + \frac{1}{\alpha} B)E \\ -E^* & \alpha I + Q \end{pmatrix},
\]

and

\[
N_\alpha = \frac{1}{2\alpha} \begin{pmatrix} \alpha I - B & O \\ O & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & -E \\ E^* & \alpha I + Q \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \alpha I - B & (\frac{1}{\alpha}B - I)E \\ E^* & \alpha I + Q \end{pmatrix}.
\]
The splitting matrix $M_{\alpha}$ can be employed as the RHSS preconditioners for the saddle point problem (1). Note that the pre-factor has no effect on the preconditioned system. For the sake of convenience, in this paper, we propose a simplified RHSS (SRHSS) preconditioner $P_{\alpha}$ which is defined as follows:

$$P_{\alpha} = \frac{1}{\alpha} \begin{pmatrix} B & O & \alpha I & E \\ O & \alpha I & -E^* & Q \end{pmatrix} = \begin{pmatrix} B & \frac{1}{\alpha}BE \\ -E^* & Q \end{pmatrix},$$

(6)

where $Q \in \mathbb{C}^{n \times n}$ is a Hermitian positive definite matrix. Since $B, Q \in \mathbb{C}^{n \times n}$ are Hermitian positive definite matrices, $E \in \mathbb{C}^{p \times q}$ is a rectangular matrix of full column rank, the matrix $P_{\alpha}$ is nonsingular [9].

Obviously, the difference between the preconditioner $P_{\alpha}$ and the generalized saddle point matrix $A$ can be given by

$$L_{\alpha} = P_{\alpha} - A = \begin{pmatrix} O & (\frac{1}{\alpha}B - I)E \\ O & Q \end{pmatrix}.$$

(7)

From (5) and (7), we can easily find that our proposed preconditioner $P_{\alpha}$ in (6) is a better approximation to the matrix $A$ than the preconditioner $M_{\alpha}$ in (4), so it may be easier to analyze some properties of the preconditioned matrix $P_{\alpha}^{-1}A$. By the special splitting of matrix $A = P_{\alpha} - L_{\alpha}$, we obtain the following simplified RHSS splitting method (SRHSS):

**The simplified RHSS (SRHSS) splitting iteration method:** Let $\alpha$ be a positive real number, $Q \in \mathbb{C}^{n \times n}$ be a symmetric positive definite matrix. Given an initial guess $(x^{(0)T}, y^{(0)T})^T$,

$$\begin{pmatrix} B & \frac{1}{\alpha}BE \\ -E^* & Q \end{pmatrix} \begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} O & (\frac{1}{\alpha}B - I)E \\ O & Q \end{pmatrix} \begin{pmatrix} x^k \\ y^k \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix}.$$ 

(8)

Hence, this special splitting method can also be written in the following fixed point form

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = J(\alpha) \begin{pmatrix} x^k \\ y^k \end{pmatrix} + c,$$

(9)

where

$$J(\alpha) = P_{\alpha}^{-1}L_{\alpha} = \begin{pmatrix} B & \frac{1}{\alpha}BE \\ -E^* & Q \end{pmatrix}^{-1} \begin{pmatrix} O & (\frac{1}{\alpha}B - I)E \\ O & Q \end{pmatrix} = \begin{pmatrix} O & (I - \frac{1}{\alpha}ED^{-1}E^*)(\frac{1}{\alpha}I - B^{-1})E - \frac{1}{\alpha}ED^{-1}Q \\ O & I - D^{-1}F \end{pmatrix},$$

(10)

is the iteration matrix and

$$c = \begin{pmatrix} B & \frac{1}{\alpha}BE \\ -E^* & Q \end{pmatrix}^{-1} \begin{pmatrix} f \\ -g \end{pmatrix}.$$
and
\[ D = Q + \frac{1}{\alpha}E^*E, \quad F = Q + E^*B^{-1}E. \] (11)

When the preconditioner \( P_\alpha \) of SRHSS method is used to accelerate the convergence rate of Krylov subspace methods (such as GMRES), we need to solve a linear system with \( P_\alpha \) as the coefficient matrix. That is to say, we need to solve the residual equation \( P_\alpha Z = r \), i.e.,

\[
\begin{pmatrix}
B & \frac{1}{\alpha}BE \\
-E^* & Q
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} =
\begin{pmatrix}
r_1 \\
r_2
\end{pmatrix},
\] (12)

where \( z_1, z_2, r_1, r_2 \in \mathbb{R}^n \). Obviously, the matrix \( P_\alpha \) has the following factorization

\[
P_\alpha = \begin{pmatrix}
I & \frac{1}{\alpha}BEQ^{-1} \\
O & I
\end{pmatrix}
\begin{pmatrix}
B + \frac{1}{\alpha}BEQ^{-1}E^* & 0 \\
0 & Q
\end{pmatrix}^{-1}
\begin{pmatrix}
I & O \\
-Q^{-1}E^* & I
\end{pmatrix},
\] (13)

and

\[
P_\alpha^{-1} = \begin{pmatrix}
I & O \\
-Q^{-1}E^* & I
\end{pmatrix}
\begin{pmatrix}
S^{-1} & O \\
0 & Q^{-1}
\end{pmatrix}
\begin{pmatrix}
I & -\frac{1}{\alpha}BEQ^{-1} \\
O & I
\end{pmatrix},
\] (14)

where
\[
S = B + \frac{1}{\alpha}BEQ^{-1}E^* = B + \frac{1}{\alpha}EQ^{-1}E^*.
\]

Then by (11), we have

\[
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} =
\begin{pmatrix}
I & O \\
-Q^{-1}E^* & I
\end{pmatrix}
\begin{pmatrix}
S^{-1} & O \\
0 & Q^{-1}
\end{pmatrix}
\begin{pmatrix}
I & -\frac{1}{\alpha}BEQ^{-1} \\
O & I
\end{pmatrix}
\begin{pmatrix}
r_1 \\
r_2
\end{pmatrix},
\] (15)

Therefore, we can obtain the following algorithmic version of solving linear system (11).

**Algorithm 2.1** For a given vector \( r = (r_1^T, r_2^T)^T \), the vector \( z = (z_1^T, z_2^T)^T \) can be computed by

- **Step 1:** Solve \( Qw = r_2 \) for \( w \);
- **Step 2:** Let \( w_1 = r_1 - \frac{1}{\alpha}BEw \);
- **Step 3:** Solve \( Bt = w_1 \) for \( t \);
- **Step 4:** Solve \( (I + \frac{1}{\alpha}EQ^{-1}E^*)z_1 = t \) for \( z_1 \);
- **Step 5:** Solve \( Qv = E^*z_1 \) for \( v \);
- **Step 6:** Let \( z_2 = w + v \).

From Algorithm 2.1, we can observe that the linear systems with the coefficient matrix \( Q, B \) and \( I + \frac{1}{\alpha}EQ^{-1}E^* \) need to be solved at each iteration.
However, this may be very costly and impractical in actual implementations because of the sparsity pattern of \( I + \frac{1}{\alpha} EQ^{-1} E^* \). Fortunately, the matrices \( Q, B \) and \( I + \frac{1}{\alpha} EQ^{-1} E^* \) are all symmetric positive definite for all \( \alpha > 0 \). Therefore, the conjugate gradient (CG) or the preconditioned conjugate gradient (PCG) method can be employed to solve the sub-system of linear equations with the coefficient matrix \( I + \frac{1}{\alpha} EQ^{-1} E^* \) by a prescribed accuracy in practice. In addition, the sub-linear equations with the coefficient matrix \( I + \frac{1}{\alpha} EQ^{-1} E^* \) can also be solved by some iterative methods such as the Cholesky or LU factorization in combination with AMD or column AMD reordering. We address here that this problem will be solved by the CG method in this paper.

3 Convergence of SRHSS iteration method

In this section, we will give some properties and convergence of the corresponding iterative method, we also present some underlying theoretical results. We shall use the following notations and definitions. For a vector \( x \), \( x^* \) denotes the complex conjugate transpose of the vector \( x \). We use \( \lambda(A) \) and \( \rho(A) \) to represent the eigenvalues and the spectral radius of the matrix \( A \) respectively. Note that the iteration matrix of the proposed method is \( J(\alpha) \), therefore, the modified generalized shift-splitting iteration method is convergent if and only if the spectral radius of the matrix \( J(\alpha) \), defined in (10) is less than one, i.e., \( J(\alpha) < 1 \).

Lemma 3.1. ([10]) Let \( W, T \in \mathbb{R}^{m \times m} \) be SPD and symmetric, respectively. Then the eigenvalues of the matrix \( W^{-1}T \) are all real.

Theorem 3.2. Let \( B \in \mathbb{R}^{m \times m} \) and \( Q \in \mathbb{R}^{n \times n} \) be symmetric positive definite matrices, \( E \in \mathbb{R}^{m \times n} \) be of full column rank, and \( \alpha > 0 \). Assume \( J(\alpha) \) is defined in (10). Then the iterative method (9) is convergent if the parameter \( \alpha \) satisfy the following inequalities:

\[
\alpha < \frac{2\lambda_m}{\gamma_1},
\]

where \( \lambda_m \) is the minimum eigenvalue of the matrix \( Q^{-\frac{1}{2}} E^* B^{-1} EQ^{-\frac{1}{2}} \), \( \gamma_1 \) is the minimum eigenvalue of the matrix \( Q^{-\frac{1}{2}} E^* B^{-1} EQ^{-\frac{1}{2}} \). That is to say that all eigenvalues of the iteration matrix \( J(\alpha) \) locate in the open circle \( \{ \lambda \in \mathbb{C} : |\lambda| < 1 \} \).

Proof. By (10), we have

\[
J(\alpha) = \begin{pmatrix}
O & (I - \frac{1}{\alpha} ED^{-1} E^*)(\frac{1}{\alpha} I - B^{-1})E - \frac{1}{\alpha} ED^{-1}Q \\
O & I - D^{-1} F
\end{pmatrix}.
\]

Here, \( D = Q + \frac{1}{\alpha} E^* E \) and \( F = Q + \frac{1}{\alpha} E^* B^{-1} E \).

Let \( G = I - D^{-1} F \). Denote \( \widetilde{G} := I - \widetilde{D}^{-1} \widetilde{F} \) with \( \widetilde{D} := I + \frac{1}{\alpha} Q^{-\frac{1}{2}} E^* E Q^{-\frac{1}{2}} \) and \( \widetilde{F} := I + \frac{1}{\alpha} Q^{-\frac{1}{2}} E^* B^{-1} EQ^{-\frac{1}{2}} \). We can easily find that the matrix \( G \) is similar to the matrix \( \widetilde{G} \). Let \( \gamma_1 \) be the eigenvalue of the matrix \( Q^{-\frac{1}{2}} E^* B^{-1} EQ^{-\frac{1}{2}} \) and \( \lambda_i \)
be the eigenvalue of the matrix \( Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}} \). Then we can obtain
\[
\rho(J(\alpha)) = \rho(G) = \rho(\tilde{G}) \leq \| \tilde{G} \|_2
\]
(17)

Since
\[
\| \tilde{G} \|_2 \leq \max_{1 \leq i \leq m} \left| 1 - \frac{1 + \gamma_i(Q^{-\frac{1}{2}}E^*B^{-1}EQ^{-\frac{1}{2}})}{1 + \frac{1}{\alpha} \lambda_i(Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}})} \right|
\]
\[
= \max_{1 \leq i \leq m} \left| \frac{\frac{1}{\alpha} \lambda_i(Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}}) - \gamma_i(Q^{-\frac{1}{2}}E^*B^{-1}EQ^{-\frac{1}{2}})}{1 + \frac{1}{\alpha} \lambda_i(Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}})} \right|
\]
\[
\leq \max_{1 \leq i \leq m} \left| \frac{\frac{1}{\alpha} \lambda_i(Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}}) - \gamma_i(Q^{-\frac{1}{2}}E^*B^{-1}EQ^{-\frac{1}{2}})}{1 + \frac{1}{\alpha} \lambda_i(Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}})} \right|
\]
\[=: \sigma(\alpha). \quad (18)\]

Since the matrices \( Q \) and \( B \) are SPD, \( E \) is of full column rank, by Lemma 3.1, we have that the eigenvalues of the matrices \( Q^{-\frac{1}{2}}E^*EQ^{-\frac{1}{2}} \) and \( Q^{-\frac{1}{2}}E^*B^{-1}EQ^{-\frac{1}{2}} \) are all real and positive. Since \( \alpha < \frac{2 \lambda_m \gamma_1}{\gamma_1} \) by (15), we have \( \sigma(\alpha) < 1 \), i.e., \( \rho(J(\alpha)) < 1 \), the theorem is proved. Next, we will give the spectral properties of the preconditioned matrix \( P_\alpha^{-1}A \).

**Theorem 3.3.** Let \( B \in \mathbb{R}^{m \times m} \) and \( Q \in \mathbb{R}^{n \times n} \) be symmetric positive definite matrices, \( E \in \mathbb{R}^{m \times n} \) be of full column rank, and \( \alpha > 0 \). Then the preconditioned matrix \( P_\alpha^{-1}A \) has eigenvalue 1 of multiplicity at least \( n \) and the remaining eigenvalues are the eigenvalues of the matrix \((Q + \frac{1}{\alpha}E^*E)^{-1}(Q + E^*B^{-1}E)\). Especially, all eigenvalues of preconditioned matrix \( P_\alpha^{-1}A \) are 1 if \( B = \alpha I \).

**proof.** Since \( P_\alpha^{-1}A = I - J(\alpha) \), by (10), we have
\[
P_\alpha^{-1}A = I - J(\alpha) = \begin{pmatrix} I & (I - \frac{1}{\alpha}ED^{-1}E^*)(\frac{1}{\alpha}I - B^{-1})E - \frac{1}{\alpha}ED^{-1}Q \\ 0 & D^{-1}F \end{pmatrix}, \quad (19)
\]
here, matrices \( D \) and \( F \) are defined in (11). By (19), we can easily obtain the result of the theorem. If \( B = \alpha I \), then \( D = F \). Obviously, all eigenvalues of matrix \( P_\alpha^{-1}A \) are 1, the theorem is proved.

**Theorem 3.4.** Suppose the same condition of Theorem 3.3 is hold. Let \( \sigma_m, \theta_m \) and \( \sigma_1, \theta_1 \) be the smallest and largest singular values of the matrix \( E \) and \( B \) separately, \( \epsilon_1 \) be the maximum eigenvalues of \( Q \). Then the following results hold true:
\[
\text{sp}(D^{-1}F) \subseteq \text{sp}((Q + \frac{1}{\alpha}E^*E)^{-1}(Q + E^*B^{-1}E)) = \left[ \frac{\alpha \sigma_m^2}{\theta_1(\alpha \epsilon_1 + \sigma_1)}, \frac{\alpha(\theta_m \epsilon_1 + \sigma_1^2)}{\theta_m \sigma_m} \right],
\]
i.e., the remaining non-unit eigenvalues of the matrix \( P_\alpha^{-1}A \) fall into the interval
\[
\left[ \frac{\alpha \sigma_m^2}{\theta_1(\alpha \epsilon_1 + \sigma_1)}, \frac{\alpha(\theta_m \epsilon_1 + \sigma_1^2)}{\theta_m \sigma_m} \right].
\]
Proof Suppose \( u \) is an eigenvector associated with the eigenvalue \( \lambda \) of the matrix \( P^{-1}_\alpha A \). Then we have
\[
D^{-1}F = \lambda u,
\]
i.e.,
\[
(Q + E^*B^{-1}E)u = \lambda (Q + \frac{1}{\alpha} E^*E)u,
\]
then
\[
\frac{u^*(Q + E^*B^{-1}E)u}{u^*u} = \lambda \frac{u^*(Q + \frac{1}{\alpha} E^*E)u}{u^*u}.
\]
Let
\[
\frac{u^*Qu}{u^*u} = c, \quad \frac{u^*E^*Eu}{u^*u} = \sigma_i, \quad \frac{u^*E^*B^{-1}Eu}{u^*u} = d.
\]
By (20), we have
\[
\lambda = \frac{c + d}{c + \frac{1}{\alpha} \sigma_i},
\]
so
\[
\frac{d}{c + \frac{1}{\alpha} \sigma_i} \leq \lambda \leq \frac{c + d}{\frac{1}{\alpha} \sigma_i}.
\]
Let \( y = Eu \), then
\[
d = \frac{u^*E^*B^{-1}Eu}{u^*u} = \frac{y^*B^{-1}y}{y^*y} \cdot \frac{u^*E^*Eu}{u^*u},
\]
and we can get
\[
\frac{\sigma_m^2}{\theta_1} \leq d \leq \frac{\sigma_1^2}{\theta_m}.
\]
By (21), we have
\[
\frac{\alpha \sigma_m^2}{\theta_1(\alpha c_1 + \sigma_1)} \leq \lambda \leq \frac{\alpha(\theta_m c_1 + \sigma_1^2)}{\theta_m \sigma_m}.
\]
The Theorem 3.4 is proved.

Remark 3.5. If \( Q = O \), then the non-unit eigenvalues of the matrix \( P^{-1}_\alpha A \) satisfy
\[
\frac{\alpha \sigma_m^2}{\theta_1 + \sigma_1} \leq \lambda \leq \frac{\alpha \sigma_1^2}{\theta_m \sigma_m}.
\]

Theorem 3.6. Let the SRHSS precoditioner \( P_\alpha \) be defined as in (6) and the conditions of Theorem 3.2 are satisfied. Then the degree of the minimal polynomial of the preconditioned matrix \( P^{-1}_\alpha A \) is at most \( m + 1 \).

Proof The proof is analogous to Theorem 3.3 in [21], here is omitted.

From (7), we can easily find the relationship
\[
P^{-1}_\alpha A = P^{-1}_\alpha (P_\alpha - L_\alpha) = I - J(\alpha),
\]
between the SRHSS preconditioned matrix $P^{-1}_\alpha A$ and the iteration matrix $J(\alpha)$. This yields to $\lambda(P^{-1}_\alpha A) = 1 - \lambda(J(\alpha))$. Therefore, we also have the following spectral properties of matrix $P^{-1}_\alpha A$.

**Theorem 3.7.** Let $B \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrices, $E \in \mathbb{R}^{m \times n}$ be of full column rank, and $\alpha > 0$. Let the SRHSS preconditioned matrix $P^{-1}_\alpha A$ be defined as in (25) and $\lambda$ be an eigenvalue of the SRHSS preconditioned matrix $P^{-1}_\alpha A$, then it holds that $|\lambda - 1| < 1$, i.e. all eigenvalues of the preconditioned matrix $P^{-1}_\alpha A$ are located in a circle centered at $(1,0)$ with radius 1.

**Proof.** Suppose $\tilde{\lambda}$ is an eigenvalue of the iteration matrix $J(\alpha)$. By (25), we obtain that $\lambda = 1 - \tilde{\lambda}$. By Theorem 3.2, we know that all eigenvalues of the iteration matrix $J(\alpha)$ locate in the open circle $\{\tilde{\lambda} \in \mathbb{C} : |\tilde{\lambda}| < 1\}$. Then from the fact that $\lambda = 1 - \tilde{\lambda}$, we know that the eigenvalue $\lambda$ satisfies $|\lambda - 1| < 1$, i.e., all eigenvalues of the preconditioned matrix $P^{-1}_\alpha A$ are located inside a circle centered at $(1,0)$ with radius strictly less that 1. This completes the proof.

## 4 Numerical experiments

In this section, we provide some numerical experiments to illustrate the theoretical results obtained in Section 3 and the effectiveness of the preconditioner $P_{\alpha}$ for GMRES(m) method to solve the linear systems (1) with respect to iteration step (denoted as IT ), norm of absolution residual vectors (denoted as RES ), elapsed CPU time in seconds (denoted as CPU ).

For solving these linear systems, we employ the GMRES method with the restarting number 10 and couple GMRES method with our proposed preconditioner (denoted by SRHSS ) and preconditioner in literature [8] (denoted by RHSS ). We choose $b = Ae$, ($e$ is $(1, 1, \cdots, 1)^T \in \mathbb{C}^m$) and $Q = \gamma I$. All tests are started from the zero vector. For the following two examples, we report IT, RES and CPU for various parameters $m$ respectively. Here, RES is defined as

$$RES = \frac{\|b - Au^{(k)}\|_2}{\|b\|_2}$$

where $\|\cdot\|_2$ refers to $L_2$-norm. All numerical experiments are carried out on a PC equipped with Intel Core i3 2.3 GHz CPU and 2.00 GB RAM memory Using MATLAB R2010a. The iteration is terminated once the current iterate $x^k$ satisfies

$$\frac{\|b - (A + \alpha I)x^k\|_2}{\|b\|_2} \leq 10^{-12},$$

or if a maximum of 200 iterations is reached.
Example 4.1. (see [11]) Let the linear system (2) in which

\[ T = I \otimes V + V \otimes I, \quad \text{and} \quad W = 10(I \otimes V_c + V_c \otimes I) + 9(e_1 e_1^T + e_1 e_1^T) \otimes I, \]

where \( V = \text{tridiag}(-1, 2, -1) \in \mathbb{R}^{l \times l}, V_c = V - e_1 e_1^T - e_1 e_1^T \in \mathbb{R}^{l \times l} \), and \( e_1 \) and \( e_l \) are the first and last unit vectors in \( \mathbb{R}^l \), respectively. Here \( T \) and \( W \) correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square \([0, 1] \times [0, 1]\) with the mesh-size \( h = \frac{1}{p+1} \).

**Table 4.1.** IT, CPU and RES with RHSS and SRHSS preconditioners for Example 4.1.

<table>
<thead>
<tr>
<th>m</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>IT</td>
<td>RHSS</td>
<td>9</td>
<td>30</td>
<td>117</td>
<td>76</td>
</tr>
<tr>
<td>CPU</td>
<td>0.1400</td>
<td>0.3740</td>
<td>1.6850</td>
<td>7.7530</td>
<td>17.3620</td>
</tr>
<tr>
<td>RES</td>
<td>9.8E-013</td>
<td>9.7E-013</td>
<td>8.3E-013</td>
<td>1E-012</td>
<td>2.6E-012</td>
</tr>
<tr>
<td>IT</td>
<td>SRHSS</td>
<td>20</td>
<td>10</td>
<td>21</td>
<td>30</td>
</tr>
<tr>
<td>CPU</td>
<td>0.047</td>
<td>0.1560</td>
<td>0.8740</td>
<td>4.8820</td>
<td>13.1200</td>
</tr>
</tbody>
</table>

For example 4.1, we set \( \alpha = 1.4 \) and \( \gamma = 6 \). From Table 4.1, it shows that the number of IT for SRHSS method has not increased very much, but those for RHSS method increases exponentially as \( m \) is increasing. Although, the CPU time for both RHSS and SRHSS preconditioners become all bigger, the CPU time for SRHSS method is almost half of the RHSS method. As for the RES, the RES for the RHSS preconditioned method is increased, while those for the SRHSS preconditioned method is decreased as \( m \) is increasing. Therefore, we can easily find that the SRHSS preconditioned method is superior to the RHSS preconditioned method in terms of IT, CPU time and RES for this example.

Example 4.2. (see [12]) Let the linear system (2) in which

\[ A = \left( \begin{array}{cc} I \otimes T + T \otimes I & O \\ O & I \otimes T + T \otimes I \end{array} \right) \in R^{2p^2 \times 2p^2}, \quad B = \left( \begin{array}{c} I \otimes F \\ F \otimes I \end{array} \right) \in R^{2p^2 \times 2p^2}, \]

and

\[ T = \frac{1}{h^2} \text{tridiag}(-1, 2, -1) \in R^{p \times p}, \quad F = \frac{1}{h} \text{tridiag}(-1, 1, 0) \in R^{p \times p}, \]

with \( \otimes \) being the Kronecker product symbol, \( h = \frac{1}{p+1} \) and \( S = \text{tridiag}(a, b, c) \) being a tridiagonal matrix with \( S_{i,i} = b, S_{i-1,i} = a, S_{i,i+1} = c \) for appropriate \( i \).

**Table 4.2.** IT, CPU and RES with RHSS and SRHSS preconditioners for Example 4.2.

<table>
<thead>
<tr>
<th>m</th>
<th>6</th>
<th>8</th>
<th>12</th>
<th>16</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>IT</td>
<td>RHSS</td>
<td>9</td>
<td>30</td>
<td>117</td>
<td>76</td>
</tr>
<tr>
<td>CPU</td>
<td>0.1400</td>
<td>0.3740</td>
<td>1.6850</td>
<td>7.7530</td>
<td>17.3620</td>
</tr>
<tr>
<td>RES</td>
<td>9.8E-013</td>
<td>9.7E-013</td>
<td>8.3E-013</td>
<td>1E-012</td>
<td>2.6E-012</td>
</tr>
<tr>
<td>IT</td>
<td>SRHSS</td>
<td>20</td>
<td>10</td>
<td>21</td>
<td>30</td>
</tr>
<tr>
<td>CPU</td>
<td>0.047</td>
<td>0.1560</td>
<td>0.8740</td>
<td>4.8820</td>
<td>13.1200</td>
</tr>
</tbody>
</table>
Table 4.2 shows the numerical results for solving Example 4.2. The number of IT for RHSS method increase quickly, but those for SRHSS method is nearly around 20 as $m$ is increasing. Especially when $m = 20$, the maximum number of iterations (200) is reached while RES does not meet our experimental requirements for RHSS method. As for CPU, the CPU for RHSS method is almost about 10 times than those for SRHSS method. Moreover, the accuracy of the SRHSS preconditioned method is higher than that of the RHSS preconditioned method through the residual numerical results. So, the SRHSS preconditioned method is also superior to the RHSS preconditioned method.

5 Conclusions and remarks

For the linear systems (1), we propose a class of new preconditioned GMRES method named SRHSS preconditioned method which can efficiently accelerate the convergence rate of the preconditioned GMRES method, i.e. RHSS preconditioned method in [8] in terms of IT, CPU time and RES. Then, we analyze the convergence property of the proposed preconditioned method. Lastly, numerical experiments show that the the proposed methods are feasible, robust, and efficient than the RHSS preconditioned method in actual implementations. In the future, we will further discuss how to find a set of optimal parameters to accelerate the solution of equations, and then study the theoretical and numerical properties.

References


