Haar wavelet collocation method for solving hyperbolic type double interfaces problem with discontinuous coefficients

Muhammad Asif (✉ asif.tangi@uop.edu.pk)  
University of Peshawar  
Muhammad Umar Farooq  
University of Peshawar  
Nadeem Haider  
University of Peshawar

Research Article

Keywords: Heterogeneous media, Hyperbolic Problems, Haar wavelets, Collocation Method, Finite Difference Method

Posted Date: March 6th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2635180/v1

License: ☋ ☀ This work is licensed under a Creative Commons Attribution 4.0 International License. Read Full License

Additional Declarations: No competing interests reported.
Haar wavelet collocation method for solving hyperbolic type double interfaces problem with discontinuous coefficients

Muhammad Asif\textsuperscript{a},∗, Umar Farooq\textsuperscript{a}, Nadeem Haider\textsuperscript{a}

\textsuperscript{a}Department of Mathematics, University of Peshawar, Pakistan

Abstract

In this article, we considered wave propagation problems through heterogeneous media or hyperbolic type interface problems. A hybrid numerical technique is presented for the numerical solution (NS) of these type of problems. The proposed method based on Haar wavelet collocation method (HWCM) and finite difference method (FDM). In this technique, the second order spatial partial derivative is approximated by truncated Haar wavelet series and temporal derivative is approximated by FDM. In case of linear hyperbolic interface problems, the resulting algebraic systems are solved by the Gauss elimination method. While in the case of nonlinear, the nonlinearity of the problem by using quasi-Newton linearization technique. The maximum absolute errors (MAEs), root mean square errors (RMSEs) and computational convergence rate ($R_cN$) are calculated by utilizing distinct collocation points (CPs). The convergence and stability analysis of the proposed technique are also discussed. Both the theoretical and numerical results affirms that the approximate solution catched the exact solution very well.

Keywords: Heterogeneous media, Hyperbolic Problems, Haar wavelets, Collocation Method, Finite Difference Method.

1 Introduction

Interface models perform a vital role in many fields such as biological systems, material science, fluid dynamics, and electromagnetic wave propagation [1]. Interface model is defined as the model formed by two different or same materials at different states having a common boundary is called interface model. Oil and water are examples of interface model with different material while ice and water are examples of interface model with the same materials [2]. Mostly interface model contain equations with a highly changing coefficient [3]. The importance of such types of models are discussed in the literature. Meshless and multi collocation method [4]. Finite element method (FEM) [5]. Finite difference method (FDM) [6]. The immersed and interface boundary method and matched interface method [7, 8]. Higher order immersed interface technique [9]. The ghost fluid method [10].

In the current study, we focus only on one-dimensional (1D) hyperbolic interface model. Hyperbolic interface model have a broad role in science and engineering such as fluid dynamics, propagation of sound, electrodynamics and electrostatics etc [11]. An outstanding literature can be found on the modification and advancement of distinct numerical techniques for the solution of hyperbolic interface model. The numerical simulation of hyperbolic interface model has got

∗The author to whom all the correspondence should be addressed. Email: asif.tangi@uop.edu.pk
great value and a considerable number of numerical method have been introduced. Piraux et al. in 2001, [12] Suggested a new interface method for the solution of the 1D hyperbolic interface models. Using finite element method (FEM), Adewole approximated linear hyperbolic interface models [13]. The author also explored a numerical algorithm for solving hyperbolic interface models [14]. Droubi et al. [15] gave a numerical approach to hyperbolic interface models using energy method. FEM is used by Deka and Sinha for approximating linear hyperbolic interface models [16]. In the present work we suggested a new collocation technique based on HWCM and FDM for numerical solution (NS) of hyperbolic interface models [17].

Wavelets analysis has a huge variety of applications in numerical approximations [18]. wavelets have simple and fast algorithms. These are simple and give better approximations [19]. Haar wavelets technique gives better accuracy for small number of grid points [20]. Among these wavelets, Haar wavelet is the simplest one. [21]. Haar wavelet utilized piecewise constant function [22]. Different types of Haar wavelets functions are introduced for approximations [23]. Haar wavelets contain constant box functions. They have a compact support and Haar attain only three values in a given interval, 1, −1 and 0 [24]. The Haar technique is used for several engineering and scientific models [25, 26]. These wavelets have a major contribution to the approximate solution of different complicated integral equations, integro-differential equations [27] numerical integration [28, 29], and differential equations [30].

In this paper, we will introduced a new approach for the NS of 1D linear and nonlinear hyperbolic interface model based on HWCM and FDM are given as under:

\[
w_{tt}(z, t) = (k(z)w_{z}(z, t))_{z} + W(z, t), \quad 0 < z < 1, \quad t > 0, \quad (1)
\]

\[
(\phi(w_{tt}(z, t)), w_{zz}(z, t), w_{z}(z, t), w(z, t), k(z), z, t) = W(z, t), \quad 0 < z < 1, \quad t > 0. \quad (2)
\]

For double interface model the given interval \(I = [0, 1]\) can be categorized into three subintervals \(I_1 = [0, \alpha]\), \(I_2 = [\alpha, \beta]\) and \(I_3 = [\beta, 1]\) at interface point \(0 < z = \alpha < 1\) and \(0 < z = \beta < 1\). The function contained in Eq. (1) and Eq. (2) become as:

\[
(k(z), w(z, t), W(z, t)) = \begin{cases} 
k_1(z), w_1(z, t), W_1(z, t) & z \in I_1, 
k_2(z), w_2(z, t), W_2(z, t) & z \in I_2, 
k_3(z), w_3(z, t), W_3(z, t) & z \in I_3,
\end{cases} \quad (3)
\]

Subject to the Interface Conditions at \(z = \alpha\) and \(z = \beta\) are:

\[
\gamma_1w_1(\alpha, t) - \gamma_2w_2(\alpha, t) = q_1(t) \quad (4)
\]

\[
\gamma_3w_1(\alpha, t) - \gamma_4w_2(\alpha, t) = q_2(t) \quad (5)
\]

\[
\gamma_5w_2(\beta, t) - \gamma_6w_3(\beta, t) = q_3(t) \quad (6)
\]

\[
\gamma_7w_2(\beta, t) - \gamma_8w_3(\beta, t) = q_4(t) \quad (7)
\]

Initial Conditions:

\[
w(z, 0) = w_0(z) \quad (8)
\]

and

Dirichlet boundary Condition:

\[
w_1(0, t) = s(0, t), \quad w_3(1, t) = s(1, t), \quad t > 0. \quad (9)
\]

The function \(k_1(z), k_2(z)\) and \(k_3(z)\) and \(W_1(z, t), W_2(z, t)\) and \(W_3(z, t)\) are the sufficiently smooth function defined on corresponding subinterval.
2 Haar Wavelet

The \(i\)th Haar wavelet can be written as:

\[
h_i(\zeta) = \begin{cases} 
1 & \text{for } \zeta \in [\zeta_1, \zeta_2), \\
-1 & \text{for } \zeta \in [\zeta_2, \zeta_3), \\
0 & \text{somewhere}, 
\end{cases}
\]

(10)

where

\[
\zeta_1 = \frac{p}{m}, \quad \zeta_2 = \frac{p + 1/2}{m}, \quad \zeta_3 = \frac{p + 1}{m}.
\]

Where integer \(m = 2^g\). The value \(g = 0, 1, \ldots, G\), where \(g\) denotes the dilation parameter and \(G\) is the maximum resolution level. Similarly \(p = 0, 1, \ldots, m - 1\) indicates the translation parameter. The value of \(i\) can be determined as \(i = p + m + 1\).

We will introduce the following notations for the Haar integrals as:

\[
p_{k,1}(\zeta) = \int_0^{\zeta} h_i dt, \quad K = 1, 2, \ldots, 2N.
\]

(11)

The value of \(p_{k,m}(\zeta)\) is given by (10),

\[
p_{k,m}(\zeta) = \begin{cases} 
0 & \text{for } \zeta \in [0, \zeta_1), \\
\frac{1}{m} (\zeta - \zeta_1)^m & \text{for } \zeta \in [\zeta_1, \zeta_2), \\
\frac{1}{m} [((\zeta - \zeta_1)^m - 2(\zeta - \zeta_2)^m] & \text{for } \zeta \in [\zeta_2, \zeta_3), \\
\frac{1}{m} [((\zeta - \zeta_1)^m - 2(\zeta - \zeta_2)^m + (\zeta - \zeta_3)^m] & \text{for } \zeta \in [\zeta_3, 1), m = 1, 2, \ldots.
\end{cases}
\]

(12)

3 Numerical method

We approximate the time derivative by using the forward difference formula:

\[
W_t(z, t) = \frac{w(z, t_l) - w(z, t_{l-1})}{dt}
\]

and

\[
W(z, t_l) = w(z, t_{l-1}) - w(z, t_{l-1})dt
\]

We approximate the time derivative by using the central difference formula:

\[
W_{tt}(z, t) = \frac{w(z, t_{l+1}) - 2w(z, t_l) + w(z, t_{l-1})}{dt^2}
\]

and

\[
W_{tt}(z, t) = \frac{w(z, t_l) - 2(w(z, t_{l-1}) - w(z, t_{l-1})dt) + w(z, t_{l-1})}{dt^2}
\]

where \(t_{l+1}, t_l\) and \(t_{l-1}\) are Initial time, Current and next time.

Suppose that the function \(w_{1zz}(z, t)\) can be approximated over the subinterval \(I_1 = [0, \alpha]\) using Haar wavelets as:

\[
w_{1zz}(z, t) \approx \sum_{i=1}^{2N} a_i(t) h_i(z), \quad z \in I_1
\]

(17)
Integrating the above (17)

\[ w_{1z}(z, t) \approx w_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,1}(z), \ z \in I_1 \] (18)

and

\[ w_1(z, t) \approx s(0, t) + zw_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,2}(z) - zp_{i,1}(\alpha)), \ z \in I_1 \] (19)

By utilizing Haar wavelet, the function \( w_{2zz}(z, t) \) may be approximated over the subinterval \( I_2 = [\alpha, \beta] \) as:

\[ w_{2zz}(z, t) \approx \sum_{i=1}^{2N} b_i(t)h_i(z), \ z \in I_2 \] (20)

Integrating the above (20)

\[ w_{2z}(z, t) \approx w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,1}(z), \ z \in I_2 \] (21)

and

\[ w_2(z, t) \approx w_2(\beta, t) - (\beta, z)w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,2}(\alpha) - p_{i,2}(\beta), \ z \in I_2 \] (22)

While, the function \( w_{3zz}(z, t) \) is approximated over the subinterval \( I_3 = [\beta, 1) \) in this form.

\[ w_{3zz}(z, t) \approx \sum_{i=1}^{2N} c_i(t)h_i(z), \ z \in I_3 \] (23)

Integrating the above (23)

\[ w_{3z}(z, t) \approx w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t)(p_{i,1}(z), \ z \in I_3 \] (24)

and

\[ w_3(z, t) \approx s(1, t) - (1 - z)w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t)(p_{i,2}(z) - p_{i,2}(1)), \ z \in I_3 \] (25)

The above interface conditions become

\[ \gamma_1 \left( s(0, t) + zw_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,2}(z) - zp_{i,1}(\alpha)) \right) - \gamma_2 \left( w_2(\beta, t) - (\beta - z)w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,2}(\alpha) - p_{i,2}(\beta), \ z \in I_2 \right) = q_1(t) \] (26)

and

\[ \gamma_3 \left( w_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,1}(z) - p_{i,1}(\alpha)) \right) - \gamma_4 \left( w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,1}(\alpha)) \right) = q_2(t) \] (27)
and
\[ \gamma_5 \left( w_2(\beta, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,2} (\alpha) - p_{i,2} (\beta)) \right) - \gamma_6 \left( s(1, t) \right) \]
\[ -(1 - \beta)w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t)(p_{i,2} (\beta) - p_{i,2} (1)) = q_3(t) \]

Now
\[ \gamma_7 \left( w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,1} (\beta)) \right) - \gamma_8 \left( w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t)(p_{i,1} (\beta)) \right) = q_4(t) \]

In the aforementioned system of equation will be discussed for linear and nonlinear cases separately.

### 3.1 Linear Case:

For the linear case Eq. (1) become as:
\[ s(0, t) + zw_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,2} (z) - zp_{i,1} (\alpha)) - 2(w_1(z, t_1) - w_1(z, t_0)dt) + w_1(z, t_0) \]
\[ = \beta_1(z) \sum_{i=1}^{2N} a_i h_i(z)dt^2 + \beta_{1z}(z) \sum_{i=1}^{2N} a_i(p_{i,1} (z) - p_{i,1} (\alpha))dt^2 + W_1(z, t_0)dt^2, \quad z \in I_1 \]
\[ w_2(\beta, t) - (\beta - z)w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,2} (z) - p_{i,2} (\beta)) - 2(w_2(z, t_1) - w_2(z, t_0)dt) + w_2(z, t_0) \]
\[ = \beta_2(z) \sum_{i=1}^{2N} b_i h_i(z)dt^2 + \beta_{2z}(z) \sum_{i=1}^{2N} b_i(p_{i,1} (z))dt^2 + W_2(z, t_0)dt^2, \quad z \in I_2 \]
\[ s(1, t) - (1 - z)w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t)(p_{i,2} (z) - p_{i,2} (1)) - 2(w_3(z, t_0) - w_3(z, t_0)dt) + w_3(z, t_0) \]
\[ = \beta_3(z) \sum_{i=1}^{2N} c_i(t) h_i(z)dt^2 + \beta_{3z}(z) \sum_{i=1}^{2N} c_i(p_{i,1} (z))dt^2 + W_3(z, t_0)dt^2, \quad z \in I_3 \]

for double interface condition, substituting the following CPs at \( z = \alpha \) and \( z = \beta \) define as:
\[ Z_d = \begin{cases} 
\alpha(d - 0.5)/(2N) & \text{for } d = 1, 2, \ldots, 2N \\
\alpha + (\beta - \alpha)(c - 2N - 0.5)/(2N) & \text{for } d = 2N + 1, 2N + 2, \ldots, 4N \\
\beta + (1 - \beta)(d - 4N - 0.5)/(2N) & \text{for } d = 4N + 1, 4N + 2, \ldots, 6N 
\end{cases} \]

The discretization form of the equations we have:
\[ \sum_{i=1}^{2N} a_i(t)(p_{i,2} (z_j) - z_j p_{i,1} (\alpha)) - \beta_{1z}(z_j)(p_{i,1} (z_j) - p_{i,1} (\alpha))dt^2 - \beta_1(z_j) h_i(z_j)dt^2 + z_j w_{1z}(\alpha, t) \]
\[ = 2(w_1(z_j, t_0) - w_1(z_j, t_0)dt) - w_1(z_j, t_0) + W_1(z_j, t_0)dt^2 - s(0, t), \quad j = 1, 2, \ldots, 2N \]
\[ \sum_{i=1}^{2N} b_i(t)(p_{i,2}(z_j) - p_{i,2}(\beta)) - \beta_{2z}(z_j)(p_{i,1}(z_j)dt^2) - \beta_{2}(z_j)h_i(z_j)dt^2 - (\beta - z_j)w_{2z}(\alpha, t) = 2(w_2(z_j, t_0) - w_2(z_j, t_0)dt) - w_2(z_j, t_0) + W_2(z_j, t_0)dt^2 - w_2(\beta, t), \quad j = 2N + 1, 2N + 2, \ldots, 4N \] (35)

and
\[ \sum_{i=1}^{2N} c_i(t)(p_{i,2}(z_j) - p_{i,2}(1)) - \beta_{3z}(z_j)(p_{i,1}(z_j)dt^2) - \beta_{3}(z_j)h_i(z_j)dt^2 - (1 - z_j)w_{3z}(\beta, t) = 2(w_3(z_j, t_0) - w_3(z_j, t_0)dt) - w_3(z_j, t_0) + W_3(z_j, t_0)dt^2 - s(1, t), \quad j = 4N + 1, 4N + 2, \ldots, 6N \] (36)

Combining Eqs.(26), Eqs.(27), Eqs.(28) and Eqs. (29) with Eqs.(34), Eqs. (35) and Eqs.(36) give \((6N + 2) \times (6N + 2)\) order of linear system containing \(w_{1z}(\alpha, t), w_{2z}(\alpha, t)\) and \(w_{3z}(\beta, t)\) unknowns, and Haar coefficients \(a_i(t), b_i(t)\) and \(c_i(t)\) \(i=1,2,\ldots,2N\).

\[ KX = B \] (37)

Where
From Eq.(37), we obtain
\[ X = K^{-1}B \] (38)

Combining Eqs.(19),Eqs.(22) and Eqs.(25) give order of linear system.

**Nonlinear case**

Similarly, for the nonlinear case Eq.(2) is linearized by using the formula.
\[ \left( w \frac{\partial w}{\partial z} \right)^{k+1} = w^k \left( \frac{\partial w}{\partial z} \right)^{k+1} + w^{k+1} \left( \frac{\partial w}{\partial z} \right)^k - w^k \left( \frac{\partial w}{\partial z} \right)^k \] (39)

Now by substitution in Eq.(2),Then discretization yields become as:
\[ \phi \left( \frac{1}{dt^2} \left( s(0, t) + z_jw_{1z}(\alpha, t) + \sum_{i=1}^{2N} a_i(t)(p_{i,2}(z_j) - z_jp_{i,1}(\alpha)) - 2(w_1(z_j, t_0) - w_1(z_j, t_0)dt) \right) + w_1(z_j, t_0), \right) + \sum_{i=1}^{2N} a_i(t)(p_{i,1}(z_j) - p_{i,1}(\alpha), s(0, t) + z_jw_{1z}(\alpha, t) \right) \right) = W_1(z_j, t), \quad j = 1, 2, \ldots, N \] (40)

\[ \phi \left( \frac{1}{dt^2} \left( w_2(\beta, t) - (\beta - z_j)w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,2}(z_j) - p_{i,2}(\beta)) - 2(w_2(z_j, t_0) - w_2(z_j, t_0)dt) \right) + w_2(z_j, t_0), \right) + \sum_{i=1}^{2N} b_i(t)(p_{i,1}(z_j), w_{2z}(\alpha, t) + \sum_{i=1}^{2N} b_i(t)(p_{i,1}(z_j)), w_2(\beta, t) - (\beta - z_j)w_{2z}(\alpha, t) \right) = W_2(z_j, t), \quad j = 2N + 1, 2N + 2, \ldots, 4N \] (41)
\[
\phi \left( \frac{1}{dt^2} \left( s(1, t) - (1 - z_j)w_{3z}(\beta, t) + \sum_{i=1}^{2N} c_i(t) (p_{i,2}(z_j) - p_{i,2}(1)) - 2(w_3(z_j, t_0) - w_3(z_j, t_0))dt \right) \\
+w_3(z_j, t_0) \right) + \sum_{i=1}^{2N} c_i(t)h_i(z_j), w_{3z}(\alpha, t) + \sum_{i=1}^{2N} c_i(p_{i,1}(z_j)), (s(1, t) - (1 - z_j)w_{3z}(\beta, t) \\
+ \sum_{i=1}^{2N} c_i(t)(p_{i,2}(z_j) - p_{i,2}(1)), \beta_3(z_j), z_j, t) = W_3(z_j, t), \quad j = 4N + 1, 4N + 2, \ldots, 6N
\]

(42)

Combining Eqs. (26), Eqs. (27), Eqs. (28) and Eqs. (29) with Eqs. (40), Eqs. (41) and Eqs. (42) give \((6N + 2) \times (6N + 2)\) order of linear system containing \(w_{1z}(\alpha, t), w_{2z}(\alpha, t)\) and \(w_{3z}(\beta, t)\) and Haar coefficients \(a(t), b(t)\) and \(c_i(t) \forall i = 1, 2, \ldots, 2N\) unknowns. Finally we substitute these unknowns Eq. (19), Eq. (22) and Eq. (25). The solution can be obtained by using these unknowns.

4 Convergence

Lemma 1. [33] Assume that \(s \in C^2(-\infty, \infty)\) with \(|s'| \leq K, \forall \eta \in (a, b); K > 0\) and \(s = \sum_{i=0}^{\infty} \lambda_i h_i(x)\), then \(|\lambda_i| \leq K 2^{-(3j-2)/2}\).

Lemma 2. [33] Let \(s \in C^2(-\infty, \infty)\) be a continuous on \((a, b)\). Then the error norm at J-th level satisfies

\[ ||E_J||^2 \leq \frac{K^2}{12} 2^{-2J}, \]

where \(|s'| \leq K, \forall \eta \in (a, b)\) and \(K > 0\), \(M\) is a positive real number related to the J-th level resolution of the wavelet given by \(M = 2^J\).

Theorem 1. If \(s(\eta)\) is the exact solution and \(s^{2M}(\eta)\) is the approximate solution of the Eq. (1), the error norm at J-th level resolution is given by

\[ ||E_J||_\eta = ||s - s^{2M}|| = \mathcal{O} \left( 2^{-3(2^J)} \right). \]

Proof. The error estimate of the proposed method at J-th level resolution is given as

\[ ||E_J||_\eta = ||s - s^{2M}|| = \left| \sum_{i=2M+1}^{\infty} a_i(p_{i,2}(\eta) - \eta p_{i,1}(\zeta)) \right|, \quad (43) \]

which implies

\[ ||E_J||^2_\eta = \left| \int_{-\infty}^{\infty} \left( \sum_{i=2M+1}^{\infty} a_i(p_{i,2}(\eta) - \eta p_{i,1}(\zeta)), \sum_{l=2M+1}^{\infty} a_l(p_{l,2}(\eta) - \eta p_{l,1}(\zeta)) \right) d\eta \right| \]

\[ = \left| \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} \int_a^b a_i a_l (p_{i,2}(\eta) - \eta p_{i,1}(\zeta))(p_{l,2}(\eta) - \eta p_{l,1}(\zeta)) d\eta \right| \]

\[ \leq \left| \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} a_i a_l K_{i,l} \right|, \quad (44) \]
where \( K_{i,l} = \sup \int_{a}^{b} (p_{i,2}(\eta) - \eta p_{i,1}(\zeta))(p_{l,2}(\eta) - \eta p_{l,1}(\zeta))d\eta \). Now Eq. (44) can be written as

\[
||E_J||_{\eta}^2 \leq \sum_{i=2M+1}^{\infty} |a_i(a_{2M+1}K_{i,2M+1} + a_{2M+2}K_{i,2M+2} + ...)|
\]

\[
\leq \sum_{i=2M+1}^{\infty} |a_iK_{i}(a_{2M+1} + a_{2M+2} + ...)|, \quad \text{where} \quad K_i = \sup K_{i,l}
\]

\[
\leq \sum_{i=2M+1}^{\infty} (|a_iK_{i}a_{2M+1}| + |a_iK_{i}a_{2M+2}| + ...)
\]

\[
\leq \sum_{i=2M+1}^{\infty} (|a_iK_{i}a_{2M+1}| + |a_iK_{i}a_{2M+2}| + ...).
\]  

(45)

Now, using Lemma 1 and 2, inequality (45) can be written as

\[
||E_J||_{\eta}^2 \leq K_1K_2^{2-(3.2^J+1)} \sum_{i=2M+1}^{\infty} |a_iK_i|
\]

\[
\leq K_1K_2^{2-(3.2^J+1)} \quad \text{where} \quad K_1 = \sup K_i,
\]

which on further simplification and taking square root we get

\[
||E_J||_{\eta} \leq \sqrt{K_1K_2^{2-(3.2^J+1)}}
\]

\[
\leq O \left( 2^{-3(2^J)} \right).
\]

It is concluded that error norm is inversely proportional to level of the Haar wavelet resolution \( J \). Hence the error of the HWCM decreases as \( J \) increases i.e.,

\[
||E_J||_{\eta} \to 0 \quad \text{as} \quad J \to \infty,
\]

\[
\implies ||E_J||_{\eta} \to 0 \quad \text{as} \quad M \to \infty.
\]

**Theorem 2.** If \( s(\eta, t_p) \) is the exact solution and \( s^{2M}(\eta, t_p) \) is the approximate solution of the Eq. (1). If \( p = 0, 1, 2...P \), where \( P \) is a positive integer, then the error norm at \( J \)-th level resolution is given by

\[
\text{Error} = ||E_J||_{\eta} + ||E_J||_{t_p} = O \left( 2^{-3(2^J)} \right) + O(\Delta t).
\]

**Proof.** For \( p = 0, 1, 2...P \)

\[
||E_J||_{\eta} = O \left( 2^{-3(2^J)} \right), \quad \text{(see Theorem1)}.
\]

For time derivatives we have used first order finite difference approximation in Eq. (1), so

\[
||E_J||_{t_p} = O(\Delta t).
\]

Hence

\[
\text{Error} = ||E_J||_{\eta} + ||E_J||_{t_p} = O \left( 2^{-3(2^J)} \right) + O(\Delta t).
\]

\[ \square \]
5 Stability

In this section we studied the computational stability of the proposed technique. For this purpose we observed the maximum eigenvalues of matrix \( X \) at every time step, which represents the corresponding Haar weights. All the maximum eigenvalues of matrix \( X \) are stay away from zero (see Fig. 1) and this leads to a sufficient condition for the proposed technique to be stable.

![Example 1](image1.png)  ![Example 4](image4.png)

Figure 1: The stability analysis of the proposed method at \( N = 64, \Delta t = 0.01/32, t = 1, a = 0 \) and \( b = 1 \).

6 Numerical Validation

In this section some numerical test problems are considered to illustrate the efficiency of the proposed analytical work done in the preceding sections. The 1D linear and nonlinear hyperbolic interface models are chosen as test problems for experimentations. We also calculated, \( R_cN \) which is characterized as:

\[
R_cN = \frac{\log[E_c(N/2)/E_cN]}{\log(2)}.
\]

**Example 1.** Consider the following linear hyperbolic interface model:

\[
W_{tt}(z,t) = (k(z)w_z(z,t))_z + W(z,t),
\]

with the exact solution:

\[
w(z,t) = \begin{cases} 
  w_1(z,t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
  w_2(z,t) = e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
  w_3(z,t) = z^3e^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]

Subject to the interface Conditions at \( z=0.2 \) and \( z=0.6 \) are:

\[
w_2(0.2, t) = w_1(0.2, t),
\]

\[
k_2(z)w_2_z(0.2, t) = k_1(z)w_1_z(0.2, t), \quad t > 0,
\]

\[
w_3(0.6, t) = w_2(0.6, t)
\]
\[ k_3(z)w_{3z}(0.6, t) = k_2(z)w_{2z}(0.6, t), \quad t > 0, \]

Initial conditions:
\[
w(y, 0) = \begin{cases} 
  w_1(z, 0) = z, & 0 \leq z \leq 0.2, \\
  w_2(z, 0) = e^{-z^2}, & 0.2 \leq z \leq 0.6, \\
  w_3(z, 0) = z^3, & 0.6 \leq z \leq 1,
\end{cases}
\]

Dirichlet boundary conditions:
\[
s_1(0, t) = 0, \quad s_3(1, t) = e^{-t}, \quad t > 0,
\]

Where
\[
w(z, t) = \begin{cases} 
  w_1(z, t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
  w_2(z, t) = -8z^2e^{-z^2}e^{-t} + 5e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
  w_2(z, t) = z^3e^{-t} - 18ze^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]

and
\[
k(z) = \begin{cases} 
  k_1(z) = 1, & 0 \leq z \leq 0.2, \\
  k_2(z) = 2, & 0.2 \leq z \leq 0.6, \\
  k_3(z) = 3, & 0.6 \leq z \leq 1,
\end{cases}
\]

Table 1: Error Analysis of Example (1)

<table>
<thead>
<tr>
<th>j</th>
<th>N</th>
<th>dt</th>
<th>( E_{0,N} )</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.01/6</td>
<td>(2.0000 \times 10^{-3})</td>
<td>(1.4000 \times 10^{-3})</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>0.01/12</td>
<td>(6.3370 \times 10^{-4})</td>
<td>(3.2942 \times 10^{-4})</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.01/24</td>
<td>(1.8787 \times 10^{-4})</td>
<td>(8.8981 \times 10^{-5})</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>0.01/48</td>
<td>(9.6562 \times 10^{-5})</td>
<td>(3.9642 \times 10^{-5})</td>
</tr>
<tr>
<td>4</td>
<td>96</td>
<td>0.01/96</td>
<td>(5.8569 \times 10^{-5})</td>
<td>(2.1955 \times 10^{-5})</td>
</tr>
</tbody>
</table>

This is a first linear hyperbolic interface model with constant coefficients. The proposed numerical method are implemented to this test model. The table (1) shows better accuracy and efficiency we examine in this test model. The MAEs and RMSEs are calculated. We can observe from the table that errors are decreasing as we increases the number of CPs. The errors are decreased up to order \(10^{-5}\).

Example 2. Assume the linear hyperbolic interface model:

\[
W_{tt} = \begin{cases} 
  w_{1zz}(z, t) + ze^{-t}, & 0 \leq z \leq 0.2, \\
  10w_{2zz}(z, t) - 40z^2e^{-z^2}e^{-t} + 21e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
  1000w_{3zz}(z, t) + z^3e^{-t} - 6000ze^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]

with the exact solution:

\[
w(z, t) = \begin{cases} 
  w_1(z, t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
  w_2(z, t) = e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
  w_3(z, t) = z^3e^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]
subject to the Interface Conditions at \( z=0.2 \) and \( z=0.6 \) are:

\[
\begin{align*}
    w_2(0.2, t) &= w_1(0.2, t), \\
    k_2(z)w_2z(0.2, t) &= k_1(z)w_1z(0.2, t), \quad t > 0, \\
    w_3(0.6, t) &= w_2(0.6, t) \\
    k_3(z)w_3z(0.6, t) &= k_2(z)w_2z(0.6, t), \quad t > 0,
\end{align*}
\]

Initial conditions:

\[
w(y, 0) = \begin{cases}
    w_1(z, 0) = z, & 0 \leq z \leq 0.2, \\
    w_2(z, 0) = e^{-z^2}, & 0.2 \leq z \leq 0.6, \\
    w_3(z, 0) = z^3, & 0.6 \leq z \leq 1,
\end{cases}
\]

Dirichlet boundary conditions:

\[
s_1(0, t) = 0, \quad s_3(1, t) = e^{-t}, \quad t > 0,
\]

Table 2: Error Analysis of Example (2)

<table>
<thead>
<tr>
<th>( j )</th>
<th>( \mathcal{N} )</th>
<th>( dt )</th>
<th>( E_r\mathcal{N} )</th>
<th>( \text{RMSE} )</th>
<th>( R_e\mathcal{N} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.01/6</td>
<td>( 4.4000 \times 10^{-3} )</td>
<td>( 3.0000 \times 10^{-4} )</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>0.01/12</td>
<td>( 1.3000 \times 10^{-3} )</td>
<td>( 7.5509 \times 10^{-4} )</td>
<td>1.7590</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.01/24</td>
<td>( 3.1253 \times 10^{-4} )</td>
<td>( 1.7554 \times 10^{-4} )</td>
<td>2.0564</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>0.01/48</td>
<td>( 7.1719 \times 10^{-5} )</td>
<td>( 4.0625 \times 10^{-5} )</td>
<td>2.1236</td>
</tr>
<tr>
<td>4</td>
<td>96</td>
<td>0.01/96</td>
<td>( 4.2896 \times 10^{-5} )</td>
<td>( 1.5373 \times 10^{-5} )</td>
<td>0.7415</td>
</tr>
</tbody>
</table>

This is a second linear hyperbolic interface problem with constant coefficients. The technique that is used in this problem show better accuracy and efficiency in the table (2). The numerical results are very good. The MAEs, RMSEs and \( R_e\mathcal{N} \) are calculated for this test problem. We observed from the table that both errors are reduced as we increased the number CPs.

**Example 3.** Assume the following linear hyperbolic interface model.

\[
W_{tt} = \begin{cases}
    (k_1(z)w_1z(z, t))z + ze^{-t} - 2e^{-t}, & 0 \leq z \leq 0.2, \\
    (k_2(z)w_2z(z, t))z + e^{-z^2}e^{-t} + 4ze^{-z^2}e^{-t} - 4z^3e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
    (k_3(z)w_3z(z, t))z + z^3e^{-t} - 3z^2e^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]

with the exact solution:

\[
w(z, t) = \begin{cases}
    w_1(z, t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
    w_2(z, t) = e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
    w_3(z, t) = z^3e^{-t}, & 0.6 \leq z \leq 1,
\end{cases}
\]
and
\[
    k(z) = \begin{cases} 
        k_1(z) = 2z, & 0 \leq z \leq 0.2, \\
        k_2(z) = z, & 0.2 \leq z \leq 0.6, \\
        k_3(z) = \frac{z}{3}, & 0.6 \leq z \leq 1,
    \end{cases}
\]

Subject to the interface Conditions at \(z=0.2\) and \(z=0.6\) are:
\[
    w_2(0.2, t) = w_1(0.2, t),
\]
\[
    k_2(z)w_{2z}(0.2, t) = k_1(z)w_{1z}(0.2, t), \quad t > 0,
\]
\[
    w_3(0.6, t) = w_2(0.6, t)
\]
\[
    k_3(z)w_{3z}(0.6, t) = k_2(z)w_{2z}(0.6, t), \quad t > 0,
\]

Initial conditions:
\[
    w(y, 0) = \begin{cases} 
        w_1(z, 0) = z, & 0 \leq z \leq 0.2, \\
        w_2(z, 0) = e^{-z^2}, & 0.2 \leq z \leq 0.6, \\
        w_3(z, 0) = z^3, & 0.6 \leq z \leq 1,
    \end{cases}
\]

Dirichlet boundary conditions:
\[
    s_1(0, t) = 0, \quad s_3(1, t) = e^{-t}, \quad t > 0,
\]

<table>
<thead>
<tr>
<th>(j)</th>
<th>(N)</th>
<th>(dt)</th>
<th>(E_{\epsilon, N})</th>
<th>(RMSE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.01/6</td>
<td>(2.7000 \times 10^{-3})</td>
<td>(1.5000 \times 10^{-4})</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>0.01/12</td>
<td>(8.9324 \times 10^{-4})</td>
<td>(5.1990 \times 10^{-4})</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.01/24</td>
<td>(3.8612 \times 10^{-4})</td>
<td>(2.2385 \times 10^{-4})</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>0.01/48</td>
<td>(2.0896 \times 10^{-4})</td>
<td>(1.0379 \times 10^{-4})</td>
</tr>
<tr>
<td>4</td>
<td>96</td>
<td>0.01/96</td>
<td>(1.1372 \times 10^{-4})</td>
<td>(5.0120 \times 10^{-5})</td>
</tr>
</tbody>
</table>

This is a third linear hyperbolic interface problem with variable coefficients. The proposed method is applied to the test problem. The MAEs, RMSEs and \(R_{\epsilon, N}\) are also calculated in this test problem as well shown in the table (3). The numerical results of this model is good as well. The \(L_{\infty}\) errors are also calculated. From the table we can examined that the newly developed numerical technique has equally better performance in this test problem as well.

**Example 4.** Suppose the nonlinear hyperbolic interface model:
\[
    W_{tt} = \begin{cases} 
        w_{1zz}(z, t) + 2w_1^2 + ze^{-t} - 2z^2 e^{-2t}, & 0 \leq z \leq 0.2, \\
        10w_{2zz}(z, t) + 2w_2^2 - 40z^2 e^{-2t} + 21e^{-t} - 2(e^{-z^2})^2 e^{-2t}, & 0.2 \leq z \leq 0.6, \\
        1000w_{3zz}(z, t) + 2w_3^2 + z^3 e^{-t} - 6000ze^{-t} - 2z^6 e^{-2t}, & 0.6 \leq z \leq 1,
    \end{cases}
\]
with the exact solution:

\[ w(z, t) = \begin{cases} 
  w_1(z, t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
  w_2(z, t) = e^{-z^2}e^{-t}, & 0.2 \leq z \leq 0.6, \\
  w_3(z, t) = z^3e^{-t}, & 0.6 \leq z \leq 1, 
\end{cases} \]

Subject to the interface Conditions at \( z=0.2 \) and \( z=0.6 \) are:

\[ w_2(0.2, t) = w_1(0.2, t), \]
\[ k_2(z)w_{2z}(0.2, t) = k_1(z)w_{1z}(0.2, t), \quad t > 0, \]
\[ w_3(0.6, t) = w_2(0.6, t) \]
\[ k_3(z)w_{3z}(0.6, t) = k_2(z)w_{2z}(0.6, t), \quad t > 0, \]

Initial conditions:

\[ w(y, 0) = \begin{cases} 
  w_1(z, 0) = z, & 0 \leq z \leq 0.2, \\
  w_2(z, 0) = e^{-z^2}, & 0.2 \leq z \leq 0.6, \\
  w_3(z, 0) = z^3, & 0.6 \leq z \leq 1, 
\end{cases} \]

Dirichlet boundary conditions:

\[ s_1(0, t) = 0, \quad s_3(1, t) = e^{-t}, \quad t > 0, \]

This is a first nonlinear problem with constant coefficients. The suggested technique is applied to this test problem. The results of this test problem is good that can be noticed from the values of \( R, N \) and RMSEs in the table (4). Different types of errors are calculated for this model at different CPs. From this table (4) we conclude that results of this numerical method is equally good in the nonlinear problem as well.

**Example 5.** Suppose the following nonlinear hyperbolic interface model:

\[ W_{tt} = \begin{cases} 
  w_{1zz}(z, t) + 2w_t^2 + ze^{-t} - 2ze^{2t}, & 0 \leq z \leq 0.2, \\
  2w_{2zz}(z, t) + w_t^2 - 2z^2e^{-2t} + z^2e^{-t} - 4e^{-t}, & 0.2 \leq z \leq 0.6, \\
  3w_{3zz}(z, t) + 2w_t^2 - z^3e^{-t} - 18ze^{-t} - 2z^6e^{-2t}, & 0.6 \leq z \leq 1, 
\end{cases} \]
with the exact solution:

\[
  w(z, t) = \begin{cases} 
    w_1(z, t) = ze^{-t}, & 0 \leq z \leq 0.2, \\
    w_2(z, t) = z^2e^{-t}, & 0.2 \leq z \leq 0.6, \\
    w_3(z, t) = z^3e^{-t}, & 0.6 \leq z \leq 1, 
  \end{cases}
\]

Subject to the interface Conditions at z=0.2 and z=0.6 are:

\[
  w_2(0.2, t) = w_1(0.2, t), \\
  k_2(z)w_{2z}(0.2, t) = k_1(z)w_{1z}(0.2, t), \quad t > 0, \\
  w_3(0.6, t) = w_2(0.6, t) \\
  k_3(z)w_{3z}(0.6, t) = k_2(z)w_{2z}(0.6, t), \quad t > 0,
\]

Initial conditions:

\[
  w(y, 0) = \begin{cases} 
    w_1(z, t) = z, & 0 \leq z \leq 0.2, \\
    w_2(z, t) = z^2, & 0.2 \leq z \leq 0.6, \\
    w_3(z, t) = z^3, & 0.6 \leq z \leq 1,
  \end{cases}
\]

Dirichlet boundary conditions:

\[
  s_1(0, t) = 0, \quad s_3(1, t) = e^{-t}, \quad t > 0,
\]

<table>
<thead>
<tr>
<th>j</th>
<th>N</th>
<th>dt</th>
<th>E_\sqrt{N}</th>
<th>RMSE</th>
<th>R_c\sqrt{N}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.01/6</td>
<td>3.2000 × 10^{-3}</td>
<td>1.9000 × 10^{-3}</td>
<td>—</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>0.01/12</td>
<td>8.9154 × 10^{-4}</td>
<td>4.7406 × 10^{-4}</td>
<td>1.8437</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.01/24</td>
<td>2.0647 × 10^{-4}</td>
<td>1.2121 × 10^{-4}</td>
<td>2.1104</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>0.01/48</td>
<td>9.3521 × 10^{-5}</td>
<td>4.1399 × 10^{-5}</td>
<td>1.1426</td>
</tr>
<tr>
<td>4</td>
<td>96</td>
<td>0.01/96</td>
<td>5.5299 × 10^{-5}</td>
<td>2.0267 × 10^{-5}</td>
<td>0.7580</td>
</tr>
</tbody>
</table>

This is also nonlinear hyperbolic interface problem with constant coefficients. The proposed approximate technique is applied to this test problem. The MAEs, RMSEs and \( R_c\sqrt{N} \) are shown in table (5). This model shows better accuracy and efficiency we examine from the table. We observed that errors are decreasing as we increases the number of grids points. The errors are decreased up to order 10^{-5}.

**Example 6.** Suppose the following nonlinear hyperbolic problem:

\[
  W_{tt} = \begin{cases} 
    2w_{1zz}(z, t) + 2w_1^2 - \frac{x}{3}cost - \frac{y}{9}z^2cos^2t, & 0 \leq z \leq 0.2, \\
    2w_{2zz}(z, t) + 2w_2^2 - zcost - 2z^2cos^2t, & 0.2 \leq z \leq 0.6, \\
    3w_{3zz}(z, t) + 2w_3^2 - \frac{x}{2}cost - \frac{x^2}{2}cos^2t, & 0.6 \leq z \leq 1,
  \end{cases}
\]
with the exact solution:

\[ w(z, t) = \begin{cases} 
  w_1(z, t) = \frac{\dot{z}}{3} \cos t, & 0 \leq z \leq 0.2, \\
  w_2(z, t) = z \cos t & 0.2 \leq z \leq 0.6, \\
  w_3(z, t) = \frac{\dot{z}}{2} \cos t, & 0.6 \leq z \leq 1,
\end{cases} \]

Subject to the interface Conditions at \( z=0.2 \) and \( z=0.6 \) are:

\[ w_2(0.2, t) = w_1(0.2, t), \]

\[ k_2(z)w_{2z}(0.2, t) = k_1(z)w_{1z}(0.2, t), \quad t > 0, \]

\[ w_3(0.6, t) = w_2(0.6, t) \]

\[ k_3(z)w_{3z}(0.6, t) = k_2(z)w_{2z}(0.6, t), \quad t > 0, \]

Initial conditions:

\[ w(y, 0) = \begin{cases} 
  w_1(z, t) = \frac{\dot{z}}{3}, & 0 \leq z \leq 0.2, \\
  w_2(z, t) = z, & 0.2 \leq z \leq 0.6, \\
  w_3(z, t) = \frac{\dot{z}}{2}, & 0.6 \leq z \leq 1,
\end{cases} \]

Dirichlet boundary conditions:

\[ s_1(0, t) = 0, \quad s_3(1, t) = \frac{1}{2} \cos t, \quad t > 0, \]

<p>| Table 6: Error Analysis of Example (6) |
|---|---|---|---|---|</p>
<table>
<thead>
<tr>
<th>j</th>
<th>N</th>
<th>dt</th>
<th>( E_cN )</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
<td>0.01/6</td>
<td>( 1.2131 \times 10^{-4} )</td>
<td>( 8.1600 \times 10^{-5} )</td>
</tr>
<tr>
<td>1</td>
<td>12</td>
<td>0.01/12</td>
<td>( 6.3405 \times 10^{-5} )</td>
<td>( 4.0883 \times 10^{-5} )</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.01/24</td>
<td>( 3.1999 \times 10^{-5} )</td>
<td>( 2.0451 \times 10^{-5} )</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
<td>0.01/48</td>
<td>( 1.6227 \times 10^{-5} )</td>
<td>( 1.0234 \times 10^{-5} )</td>
</tr>
<tr>
<td>4</td>
<td>96</td>
<td>0.01/96</td>
<td>( 8.1783 \times 10^{-6} )</td>
<td>( 5.1188 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

This is nonlinear hyperbolic interface model with constant coefficients. The suggested method are implemented to the test model. In the table (6) we clearly seen that the proposed technique show better accuracy and efficiency. The MAEs, RMSEs and \( R_cN \) are calculated. The errors are decreased up to order \( 10^{-6} \). We observed that errors are deceasing as we increases the number of CPs.
Figure 2: Comparison of exact and approximate solutions for example 3.
Figure 3: Comparison of exact and approximate solutions for example 5.

Approx. solution, $N = 48$, $\Delta t = 0.01$

Approx. solution, $N = 48$, $\Delta t = 0.001$
Figure 4: Comparison of exact and approximate solutions for example 6.
7 Conclusion

In this paper, we established a new technique based on HWCM and FDM for 1D hyperbolic interface models. The recently established technique is implemented to both linear and nonlinear models. Results of the method is good for both types of models as we observed in the above test models and their 3D graphs. The features of the method is very good and it is simply applicable and implemented in MATLAB. The \( R_cN \), RMSEs and MAEs are calculated for different number of grids points. The \( R_cN \) for most of the models is approaching to 2, which satisfying the theoretical work given in [32]. As whole, we conclude that performance is very good and better accurate, efficiency and simple applicability results are obtained of a newly established numerical technique.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability statement

In the current study, no data set is generated or analyzed.

Ethical approval

Not applicable

Funding

Not applicable

Authors contributions

All the authors contributed to the manuscript. The corresponding author given the idea and supervised. The second author implemented the idea and wrote the manuscript. The third author review the manuscript and made corrections where needed.

Acknowledgments

We are thankful to University Peshawar for providing us good research environment.

References


