An Efficient Primal-dual Interior Point Algorithm for Linear Optimization Problems Based on a New Parameterized Kernel Function With a Logarithmic Barrier Term

El Amir DJEFFAL (l.djeffal@univ-batna2.dz)
University of BATNA 2

Fatima Boukhenchouche
University of BATNA 2

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AN EFFICIENT PRIMAL-DUAL INTERIOR POINT ALGORITHM FOR LINEAR OPTIMIZATION PROBLEMS BASED ON A NEW PARAMETERIZED KERNEL FUNCTION WITH A LOGARITHMIC BARRIER TERM

BOUKHNCHOUCHE FATIMA¹, DJEFFAL.EL.AMIR²,*

¹LEDPA Laboratory, Department of Mathematics, University of BATNA 2, Algeria,
²*,LEDPA Laboratory, Department of Mathematics, University of BATNA 2, Algeria,

Abstract. In this paper, we present a primal-dual interior point method for linear optimization problems based on a new kernel function with a new parameterized logarithmic barrier term. We prove that the proposed kernel function belongs to the eligible class. We derive the complexity bounds for large and small-update methods respectively.

Keywords. linear optimization, Kernel function, Interior point methods, large-update methods, complexity bound.

1. PRELIMINARIES

Consider (P) the standard Linear Optimization (LO) problem as below:

\[
\min \{c^T x : \quad Ax = b, \quad x \geq 0\},
\]

where: \(A \in \mathbb{R}^{m \times n}; \quad b \in \mathbb{R}^m; \quad c \in \mathbb{R}^n\) with \(m \leq n\). We assume that \(A\) is full row rank \((\text{rank}(A) = m \leq n)\). The dual problem (D) associated with (P) is:

\[
\max \{b^T y : \quad A^T y + s = c, \quad s \geq 0\}
\]

where: \(y \in \mathbb{R}^m; \quad s \in \mathbb{R}^n\)

First, we assume without loss of generality that both the primal problem (P) and its dual (D) satisfy the interior point condition (IPC), i.e., there exists \((x^0, y^0, s^0)\) such that:

\[
Ax^0 = b; \quad x^0 > 0; \quad A^T y^0 + s^0 = c; \quad s^0 > 0
\]

The IPC ensures the existence of optimal solutions \(x^*\) and \((y^*, s^*)\) solutions of (P) and (D) such that:

\[
c^T x^* = b^T y^* = x^* s^* = 0
\]

Therefore, applying the Karush-Kuhn-Tucker conditions to (P) and (D) we arrive at the system:

\[
Ax = b, \quad x \geq 0, \quad A^T y + s = c, \quad y, s \geq 0, \quad xs = 0
\]

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving (4). The basic idea of primal-dual IPMs is to replace the third equation in (4), the so-called complementarity condition for (P) and (D), by the parameterized equation \(xs = \mu e\), with \(\mu > 0\). Thus we

*Corresponding author.
E-mail addresses: f.boukhenchouche@univ-batna2.dz (F.Boukhenchouche), l.djeffal@univ-batna2.dz (E.A.Djeffal).
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consider the system:
\[
Ax = b \ , x \geq 0 \ , A^T y + s = c \ , y, s \geq 0 \ , xs = \mu e
\]  
(1.5)
This system has a unique solution denoted by \((x(\mu), y(\mu), s(\mu))\); for each \(\mu > 0\); thanks to the fact that \(A\) is full ranked. The solution is called the \(\mu\)-center of (P) and (D). Moreover, when \(\mu \to 0\); the limit of the central path exists and yields optimal solutions for (P) and (D). From a theoretical point of view, the IPC can be assumed without loss of generality. In fact we may, and will, assume that \(x^0 = s^0 = e\). In practice, this can be realized by embedding the given problems (P) and (D) into a homogeneous self-dual problem, which has two additional variables and two additional constraints. For this and the other properties mentioned above, see [14]. The IPMs follow the central path approximately. We describe briefly the usual approach. Without loss of generality, we assume that \((x(\mu), y(\mu), s(\mu))\) is known for some positive \(\mu\). For example, due to the above assumption, we may assume this for \(\mu = 1\), with \(x(1) = s(1) = e\). We then decrease \(\mu\) to \((1 - \theta)\mu\) for some fixed \(\theta \in ]0, 1[\), and we solve the following Newton system:
\[
A\Delta x = 0 \ , A^T \Delta y + \Delta s = 0 \ , s\Delta x + x\Delta s = \mu e - sx
\]  
(1.6)
This system defines uniquely a search direction \((\Delta x, \Delta y, \Delta s)\). By taking a step along the search direction, with the step size defined by some line search rules, we construct a new triple \((x, y, s)\). If necessary, we repeat the procedure until we find iterates that are “close” to \((x(\mu), y(\mu), s(\mu))\). Then \(\mu\) is again reduced by the factor \((1 - \theta)\), and we apply Newton’s method targeting the new \(\mu\)-centers, and so on. This process is repeated until \(\mu\) is small enough, say until \(\eta\mu \leq \varepsilon\); at this stage we have found an \(\varepsilon\)-solution of problems (P) and (D). The result of a Newton step with step size \(\alpha\) is denoted as:
\[
x^+ = x + \alpha \Delta x \ , \ y^+ = y + \alpha \Delta y \ , \ s^+ = s + \alpha \Delta s
\]  
(1.7)
where the step size \(\alpha\) satisfies \(0 < \alpha \leq 1\). Now, we introduce the scaled vector \(v\) and the scaled search directions \(d_x\) and \(d_s\) as follows:
\[
v = \sqrt{\frac{\mu}{\mu}} \ , \ d_x = \frac{\nabla \Delta x}{x} \ , \ d_s = \frac{\nabla \Delta s}{s}
\]  
(1.8)
Using these notations, system (6) can be rewritten as follows:
\[
\tilde{A}d_x = 0 \ , \tilde{A}^T \Delta y + d_s = 0 \ , d_x + d_s = v^{-1} - v
\]  
(1.9)
where \(\tilde{A} = \frac{1}{\mu} AV^{-1} X \ , \ V = diag(v) \ , X = diag(x)\). Note that the right-hand side of the third equation in (9) is equal to the negative gradient of the logarithmic barrier function \(\Phi_L(v)\), i.e.,
\[
d_x + d_s = -\nabla \Phi_L(v)
\]  
(1.10)
where the barrier function \(\Phi_L(v) : R^{n+}_+ \to R_+\) is defined as follows:
\[
\Phi_L(v) = \Phi_L(x, s, \mu) = \sum_{i=1}^{n} \psi_L(v_i)
\]
\[
\psi_L(v_i) = \frac{v_i^2 - 1}{2} - \log(v_i)
\]
We use \(\Phi_L(v)\) as the proximity function to measure the distance between the current iterate and the \(\mu\)-center for given \(\mu > 0\). We also define the norm-based proximity measure, \(\delta(v) : R^{n+}_+ \to R^+\), as follows:
Algorithm 1: Generic Interior Point Algorithm for LO

Input:
Aproximity the function $\Phi(v)$;
a threshold parameter $\tau > 1$;
an accuracy parameter $\varepsilon > 0$;
a fixed barrier update parameter $\theta$, $0 < \theta < 1$;

Begin:
$x = e$; $s = e$; $\mu = 1$; $v = e$.

While $n\mu \geq \varepsilon$ do

Begin (outer iteration)

$\mu = (1 - \theta)\mu$;

While $\Phi(x,s,\mu) > \tau$ do

Begin (inner iteration)

solve the system (10), $\Phi_L(v)$ replaced by $\Phi(v)$ to obtain $(\Delta x, \Delta y, \Delta s)$;
choose a suitable step size $\alpha$;
$x = x + \alpha \Delta x$;
y = $y + \alpha \Delta y$;
s = $s + \alpha \Delta s$.

End (inner iteration)

End (outer iteration)

End

FIGURE 1. Generic algorithm

$\delta(v) = \frac{1}{2} \| \nabla \Phi_L(v) \| = \frac{1}{2} \| d_x + d_s \|$ \hspace{1cm} (a)

Since $\Phi_L(v)$ is strictly convex and attains its minimum value of zero at $v = e$, we have:

$\nabla \Phi_L(v) = 0 \Leftrightarrow \delta(v) = 0 \Leftrightarrow v = e$

We call $\psi_L(t)$ the kernel function of the logarithmic barrier function $\Phi_L(v)$. In this paper, we replace $\psi_L(t)$ by a new kernel function $\psi(t)$ and $\Phi_L(v)$ by a new barrier function $\Phi(v)$, which will be defined in section 2. Note that the pair $(x,s)$ coincides with the $\mu$-center $(x(\mu), s(\mu))$ if and only if $v = e$. It is clear from the above description that the closeness of $(x,s)$ to $(x(\mu), s(\mu))$ is measured by the value of $\Phi(v)$ with $\tau > 0$ as a threshold value. If $\Phi(v) \leq \tau$, then we start a new outer iteration by performing a $\mu$-update; otherwise, we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of $\mu$ and apply (7) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), s(\mu))$. Then $\mu$ is again reduced by the factor $(1 - \theta)$ with $0 < \theta < 1$, and we apply Newton’s method targeting the new $\mu$-centers, and so on. This process is repeated until $\mu$ is small enough, say until $n\mu < \varepsilon$; at this stage we have found an $\varepsilon$-approximate solution of LO. The parameters $\tau$, $\theta$ and the step size $\alpha$ should be chosen in such a way that the algorithm is optimized in the sense that the number of iterations required by the algorithm is as small as possible. The choice of the so-called barrier update parameter $\theta$ plays an important role in both theory and practice of IPMs. Usually, if $\theta$ is a constant independent of the dimension $n$ of the problem, for instance, $\theta = \frac{1}{2}$, then we call the algorithm a large-update method. If $\theta$ depends on the dimension of the problem, such as $\theta = \frac{1}{n}$, then the
algorithm is called a small-update method. The generic form of the algorithm is shown in figure 1.

In most cases, the best complexity result obtained for small-update IPMs is \(O(\sqrt{n \log \frac{n}{\varepsilon}})\). For large-update methods the best obtained bound is \(O(\sqrt{n \log n} \log n)\), which until now has been the best known bound for such methods [2] [13]. In this paper, we define a new kernel function with logarithmic barrier term and propose primal–dual interior point methods which improve all the results of the complexity bound for large-update methods based on a logarithmic kernel function for LO. More precisely, based on the proposed kernel function, we prove that the correspondent algorithm has \(O((p+1)n^{\frac{p+2}{2p+1}})\) complexity bound for large-update method \(O(p^2 \sqrt{n \log \frac{n}{\varepsilon}})\) for small-update method. Another interesting choice is \(p\) dependent with \(n\), which minimizes the iteration complexity bound. In fact, if we take \(p = \frac{\log n}{2}\), we obtain the best known complexity bound for large-update methods namely \(O(\sqrt{n \log n} \log n)\). This bound improves the so far obtained complexity results for large-update methods based on a logarithmic kernel function given by El Ghami et al. [7].

2. PROPERTIES OF THE KERNEL FUNCTION AND THE BARRIER FUNCTION

2.1. Properties of the new kernel function. In this section, we first give some properties of the new kernel function which are essential to our complexity analysis. Then, we showcase its eligibility. Finally, we study the effect of updating the barrier parameter \(\mu\) on the value of the proximity function and proximity measure based on our kernel function. We call \(\psi(t) : R^+ \rightarrow R^+\) a kernel function if \(\psi\) is twice differentiable and satisfies the following conditions:

\[
\psi(1) = \psi'(1) = 0, \quad \lim_{x \to +\infty} \psi(x) = \lim_{x \to 0^+} \psi(x) = +\infty. \quad \text{Now, we define a new function } \psi(t) \text{ as follows :}
\]

\[
\psi(t) = \frac{t^2 - 1}{2} - \frac{\log(t)}{2} + \frac{t^{-p} - 1}{2p}; \quad p > 0
\]

then, for all , \(v \in R^n\), we have

\[
\phi(v) = \sum_{i=1}^{n} \psi(v_i) = \sum_{i=1}^{n} \left( \frac{v_i^2 - 1}{2} - \frac{\log(v_i)}{2} + \frac{v_i^{-p} - 1}{2p} \right); \quad p > 0
\]

In the analysis of the algorithm based on \(\psi(t)\), we need its first three derivatives. These are given by:

\[
\psi'(t) = t - \frac{1}{2t} - \frac{pt^{-p-1}}{2} = t - \frac{1}{2t} - \frac{1}{2tp+1}
\]

\[
\psi''(t) = 1 + \frac{1}{2t^2} + \frac{p+1}{2tp+2}
\]

\[
\psi'''(t) = -\frac{1}{t^3} - \frac{(p+1)(p+2)}{2tp+3} = -\left[ \frac{1}{t^3} + \frac{(p+1)(p+2)}{2tp+3} \right]
\]

From (14) we have

\[
\psi''(t) > 1 \quad t > 0; \quad p > 0
\]

It follows that \(\psi(1) = \psi'(1) = 0\), we can easily to verify that:

\[
\lim_{x \to +\infty} \psi(x) = \lim_{x \to 0^+} \psi(x) = +\infty.
\]
Due to the conditions $\psi(1) = \psi'(1) = 0$, we can completely describe $\psi(t)$ by its second derivative as follows:

$$\psi(t) = \int_1^t \int_1^x \psi''(y) dy dx$$ (2.7)

Thus the univariate function $\psi(t)$ is a kernel function.

The next lemma serves to prove that the new kernel function $\psi(t)$ is an eligible kernel function according to Bai et al. (2004).

2.2. Eligibility of the new kernel function :

Lemma 1. Let the function $\psi(t)$ be defined as in (11). Then, we have:

$$t\psi''(t) + \psi'(t) > 0, \quad t < 1.$$ (2.8)

$$t\psi''(t) - \psi'(t) > 0, \quad t > 1.$$ (2.9)

$$\psi'''(t) < 0, \quad t > 1.$$ (2.10)

$$2[\psi''(t)]^2 - \psi'(t)\psi'''(t) > 0, \quad t < 1.$$ (2.11)

$$\psi''(t)\psi'(\beta t) - \beta \psi'(t)\psi''(t) > 0, \quad t > 1, \quad \beta > 1.$$ (2.12)

The kernel function $\psi(t)$ was called eligible if it satisfies (18) to (22). It was also shown that (19) and (20) imply (22) by (Lemma 2.4 in Bai et al., 2004)[2]. So verify only the conditions from (18) to (21)

Proof.

For (18) by using (13) and (14), it follows that,

$$t\psi''(t) + \psi'(t) = t + \frac{1}{2t} + \frac{p+1}{2t^{p+1}} + t - \frac{1}{2t^{p+1}} = 2t + \frac{p}{2t^{p+1}} > 0 \quad , t < 1, \quad p > 0.$$

This proves that condition (18) is satisfied. We furthermore have:

$$t\psi''(t) - \psi'(t) = t + \frac{1}{2t} + \frac{p+1}{2t^{p+1}} - t + \frac{1}{2t} + \frac{p+1}{2t^{p+1}} = t + \frac{p+1}{2t^{p+2}} > 0 \quad , t > 1, \quad p > 0.$$

proving that (19) is satisfied as well.

For (20) it is clear from (15), we have:

$$\psi'''(t) = -\left[\frac{1}{2t^3} + \frac{(p+1)(p+2)}{2t^{p+3}}\right] < 0 \quad , t > 1, \quad p > 0$$

Finally for (21), we have:

$$2\left[\psi''(t)\right]^2 = 2 + \frac{1}{2t^4} + \frac{2}{t^2} + \frac{(p+1)^2}{2t^{2p+4}} + \frac{(2t^2+1)(p+1)}{t^{p+4}}$$

and

$$-\psi''(t)\psi'(t) = \frac{1}{t^2} - \frac{1}{2t^4} + \frac{(p+1)(p+2)}{2t^{p+2}} - \frac{(p+1)(p+2)}{4t^{p+4}} - \frac{(p+1)(p+2)}{4t^{2p+4}}$$
Then
\[2 \left[ \psi''(t) \right]^2 - \psi''(t) \psi'(t) = 2 + \frac{3}{t^2} + \frac{2(p+1)^2 - (p+1)(p+2)}{4t^2p+4} + \frac{4(2t^2+1)(p+1) - 2 - (p+1) + (p+1)(p+2)}{2t^2p+2} + \frac{4(2pt^2 + 2t^2 + p + 1) - 2 - p^2 - 3p - 2}{4t^2p+4} = 2 + \frac{3}{t^2} + \frac{(p+1)(p+2)}{2tp+2} + \frac{p(p+1)}{4t^2p+4} + \frac{2t^2}{tp+4} + \frac{p(8t^2 - p+1)}{2tp+4}\]
condition (21) is satisfied for \(p > 0\). Then \(2[\psi''(t)]^2 - \psi'(t) \psi'''(t) > 0\) if
\[\frac{p(8t^2 - p+1)}{2tp+4} > 0 \iff p(8t^2 - p+1) > 0\]
This certainly holds if:
\[8t^2 - p+1 > 0 \iff t^2 > \frac{p-1}{8} \iff t > \sqrt{\frac{p-1}{8}} \]
\[\square\]
This is a valid inequality, since \(0 < t \leq 1\) as one may easily verify. So \(\psi(t)\) also satisfies the condition (21).

**Remark 1.** The property (18) in Lemma 1 is equivalent to convexity of composed functions
\[t \rightarrow \psi(\exp(t))\]
and this holds if only if
\[\psi(\sqrt{t_1t_2}) \leq \frac{\psi(t_1) + \psi(t_2)}{2}, t_1, t_2 > 0\]
This property is known in the literature, and it was demonstrated by several researchers see [6].

Next, we present the some technical results of the new kernel function.

**Lemma 2.** For , \(\psi(t)\) we have the following.

1. \(\psi(t)\) is exponentially convex for all \(t > 0\); that is,
\[\psi(\sqrt{t_1t_2}) \leq \frac{\psi(t_1) + \psi(t_2)}{2}, t_1, t_2 > 0\]
2. \(\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2, t > 0\)
3. \(\psi(t) \leq \frac{4+p}{4} (t-1)^2\)

**Proof.**

For 1. its clear in remark 1.

For 2. using (16) and (17), we have
\[\psi(t) = \int_{t}^{t} \int_{1}^{x} \psi''(y)dydx \geq \int_{t}^{t} \int_{1}^{x} 1dydx = \int_{t}^{t} (x-1)dx = [\frac{t^2}{2} - t - \frac{1}{2} + 1] = \frac{1}{2}(t-1)^2.\]
then we have
\[
\psi(t) = \int_1^t \int_1^x \psi''(y) dy dx \leq \int_1^t \int_1^x \psi''(x) \psi''(y) dy dx.
\]
\[
\leq \int_1^t \psi''(x) \psi'(x) dx \leq \int_1^t \psi'(x) d(\psi'(x)) dx \leq \frac{1}{2} [\psi'(x)]^2.
\]
Finally
\[
\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2, \quad t > 0
\]
For 3. by using Taylor’s development and the fact
\[
\psi(1) = \psi'(1) = 0, \quad \psi''(1) = 1 + \frac{1}{2} + \frac{p+1}{2} = \frac{4+p}{2}, \quad \psi'''(t) < 0, \quad t > 0, \quad p > 0
\]
\[
\psi(t) = \psi(1) + (t-1) \psi'(1) + \frac{1}{2} (t-1)^2 \psi''(1) + \frac{1}{6} (t-1)^3 \psi'''(\xi).
\]
\[
= \frac{1}{2} (t-1)^2 \left( \frac{4+p}{2} \right) + \frac{1}{6} (t-1)^3 \psi'''(\xi) \leq \left( \frac{4+p}{4} \right) (t-1)^2.
\]
for some \( \xi \), \( 1 \leq \xi \leq t \). This completes the proof. \( \square \)

Let \( \sigma : [0, +\infty[ \rightarrow [1, +\infty[ \) be the inverse function of \( (\psi(t)) \) for \( t \geq 1 \) and
\( \rho : [0, +\infty[ \rightarrow ]0, 1] \) be the inverse function of \(-\frac{1}{2} \psi'(t)\) for all \( t \in ]0, 1] \). Then we have the following lemma.

**Lemma 3.** For \( \psi(t) \), we have:
\[
\sqrt{s+1} \leq \sigma(s) \leq 1 + \sqrt{2s}, \quad s \geq 0.
\]
\[
\rho(a) \geq \left( \frac{1}{4a+1} \right)^{\frac{p+1}{p+2}}, \quad a \geq 0, \quad p > 0.
\]

**Proof.**

for (23) \( s = \psi(t) \), \( t \geq 1 \), i.e. \( \sigma(s) = t \), \( t \geq 1 \)
for \( s = \psi(t) = \frac{t^2-1}{2} - \log(t) + \frac{t^{-p-1}}{2p} \), \( p > 0 \) then
\[
\frac{t^2-1}{2} + f(t).
\]
where \( f(t) = \frac{t^{-p-1}}{2p} - \frac{\log(t)}{2} \) the barrier function of \( \psi(t) \), then
\[
f'(t) = -\frac{1}{2}(t^{-p-1} + \frac{1}{t}) < 0
\]
\( f(t) \) is montically decreasing with respect to \( t \geq 1 \). and \( f(1) = 0 \), we have \( f(t) \in [-\infty, 0] \). This implies that:
\[
s = t^2 - 1 + f(t) \leq t^2 - 1 \Rightarrow s + 1 \leq t^2 \Rightarrow \sigma(s) = t \geq \sqrt{s+1}
\]
to proves right inequality, using 2. in lemma 2, we have for all \( t \geq 1 \),
\[
s = \psi(t) \geq \frac{1}{2} (t-1)^2 \Rightarrow 2s \geq (t-1)^2 \Rightarrow \sqrt{2s} + 1 \geq t = \sigma(s)
\]
for (24). Let $a = -\frac{1}{2} \psi'(t), t \in [0,1]$ Due to the definition of $\rho, \rho(a) = t$, and $a \geq 0$

$$2a = \frac{1}{2} + \frac{t^{-p^{-1}}}{2} - t \Rightarrow \frac{1}{t^{p^{-1}}} = 4a + 2t - t^{-1}$$

because $g : t \to 2t - t^{-1}$, monotone increasing with respect to $t \in [0,1], g'(t) = 2 + \frac{1}{t^2}, g(1) = 1$

and hence

$$\frac{1}{t^{p^{-1}}} \leq 4a + 1 \iff t^{p+1} \geq \frac{1}{(4a+1)^{p^{-1}}} \iff \rho(a) = t \geq (4a+1)^{\frac{1}{p^{-1}}}, a \geq 0, p > 0$$

This completes this proof. \hfill \Box

**Lemma 4.** Let $\sigma : [0, +\infty[ \to [1, +\infty[ be the inverse function of $\psi(t)$ for $t \geq 1$ then we have

$$\phi(\beta v) \leq n^2 \psi \left( \beta \sigma \left( \frac{\phi(v)}{n} \right) \right), \quad v \in R_{++}, \quad \beta \geq 1$$

**Proof.**

Using (22), and Theorem 3.2 in [2], we can get the result. This completes the proof. \hfill \Box

**Lemma 5.** Let $0 \leq \theta \leq 1, \ v^+ = \frac{v}{\sqrt{1-\theta}},$ if $\phi(v) \leq \tau$, then we have

$$\phi(v^+) \leq \frac{p+4}{4(1-\theta)} \left( \sqrt{2\tau} + \theta \sqrt{n} \right)^2 \quad (2.15)$$

$$\phi(v^+) \leq \frac{\tau + \theta (\tau + n + 2\sqrt{2n\tau})}{1-\theta} \quad (2.16)$$

**Proof.**

For (25) we have

0 \leq \theta \leq 1 then 0 \leq 1 - \theta \leq 1 since $\frac{1}{\sqrt{1-\theta}},$ and $\sigma \left( \frac{\phi(v)}{n} \right) \geq 1$, then $\frac{\sigma(\phi(v))}{\sqrt{1-\theta}} \geq 1$,

using lemma 4 with $\beta = \frac{1}{\sqrt{1-\theta}}$

$$\phi(\beta v) = \phi \left( \frac{1}{\sqrt{1-\theta}} \cdot v = v^+ \right) \leq n^2 \psi \left( \frac{\sigma \left( \frac{\phi(v)}{n} \right)}{\sqrt{1-\theta}} \right)$$

using 3. in lemma 2

$$\phi(v^+) \leq n \left( \frac{p+4}{4} \right) \left( \frac{\sigma \left( \frac{\phi(v)}{n} \right)}{\sqrt{1-\theta}} - 1 \right)^2 \leq \frac{n}{1-\theta} \left( \frac{p+4}{4} \right) \left( \sigma \left( \frac{\phi(v)}{n} \right) - \sqrt{1-\theta} \right)^2$$

using (23) we have

$$\phi(v^+) \leq \frac{n}{1-\theta} \left( \frac{p+4}{4} \right) \left( \sqrt{\frac{2\phi(v)}{n} + 1} - \sqrt{1-\theta} \right)^2 = \frac{n}{1-\theta} \left( \frac{p+4}{4} \right) \left( \sqrt{\frac{2\phi(v)}{n} + \frac{\theta}{1+\sqrt{1-\theta}}} \right)^2$$

with $\phi(v) \leq \tau$ and $\frac{\theta}{1+\sqrt{1-\theta}} < \theta$

$$\phi(v^+) \leq \frac{n}{1-\theta} \left( \frac{p+4}{4} \right) \left( \sqrt{\frac{2\tau}{n} + \frac{\theta}{1+\sqrt{1-\theta}}} \right)^2 \leq \frac{p+4}{4(1-\theta)} \left( \sqrt{2\tau + \theta \sqrt{n} \right)^2$$

For (26) we have this Lemma

**Lemma*.** Let $\beta \geq 1$ then $\psi(\beta t) \leq \psi(t) + (\beta^2 - 1) t^2$
Proof. using f(t) define in proof of lemma 3 we have \( f(\beta t) - f(t) \leq 0 \) for \( \beta \geq 1 \) hence

\[
\psi(\beta t) = \frac{\beta^2 t^2}{2} + f(\beta t) + \frac{t^2}{2} - f(t) - \frac{t^2}{2} = \frac{\beta^2 - 1}{2} t^2 + f(\beta t) + \psi(t) - f(t)
\]

This completes this proof.

Now, for (26) and by using lemma 4 and lemma*, and 3.in lemma 2, with \( \phi(v) \leq \tau \), and \( \beta = \frac{1}{\sqrt{1-\theta}} \), we obtain the other upper bound of \( \psi(v) \) as follows:

\[
\phi(v^+) \leq n \psi\left(\sigma\left(\frac{\phi(v)}{n}\right)\right) \leq n \psi\left(\frac{1}{\sqrt{1-\theta}} \sigma\left(\frac{\phi(v)}{n}\right)\right) \leq n \psi\left(\sigma\left(\frac{\phi(v)}{n}\right) + \left(\frac{1}{\sqrt{1-\theta}} - 1\right) \sigma^2\left(\frac{\phi(v)}{n}\right)\right)
\]

Denote

\[
\tilde{\phi}_0 = \frac{p + 4}{4(1-\theta)}(\sqrt{2\tau} + \theta\sqrt{n})^2 \\
(\phi)_0 = \frac{\tau + \theta(\tau + n + 2\sqrt{2n\tau})}{1-\theta} = \frac{\tau(1 + \theta) + \theta(n + 2\sqrt{2n\tau})}{1-\theta}
\]

then \((\phi)_0\) is an upper bound for \(\psi(v^+)\) during the process of the algorithm.

2.3. Determining a stepsize. In this section, we compute a default step size \(\alpha\) and the resulting decrease in the barrier function. After a damped step, we have

\[
x^+ = x + \alpha \Delta x, \quad y^+ = y + \alpha \Delta y, \quad s^+ = s + \alpha \Delta s
\]

using (8), we have

\[
x^+ = x(e + \alpha \frac{\Delta x}{x}) = x(e + \alpha \frac{dx}{v}), \quad s^+ = s(e + \alpha \frac{\Delta s}{s}) = s(e + \alpha \frac{ds}{v})
\]

so we have \(v^+ = \sqrt{\frac{x^+ s^+}{\mu}} = \sqrt{(v + \alpha dx)(v + \alpha ds)}\)

Define for \(\alpha > 0\), \(f(\alpha) = \phi(v^+) - \phi(v)\)
Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed $\mu$. By 1.in lemma 2 we have:

$$\phi(v^+) = \phi(\sqrt{(v + \alpha dx)(v + \alpha ds)}) \leq \frac{\phi(v + \alpha dx) + \phi(v + \alpha ds)}{2}$$

Therefore, we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) = \frac{\phi(v + \alpha dx) + \phi(v + \alpha ds)}{2} - \phi(v) \quad (2.18)$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1(\alpha)$ with respect to $\alpha$, we have

$$f'_1(\alpha) = \frac{1}{2} \sum_{i=1}^{n} \left( \psi'(v_i + \alpha d_{x_i})d_{x_i} + \psi'(v_i + \alpha d_{s_i})d_{s_i} \right)$$

$$f''_1(\alpha) = \frac{1}{2} \sum_{i=1}^{n} \left( \psi''(v_i + \alpha d_{x_i})d_{x_i}^2 + \psi''(v_i + \alpha d_{s_i})d_{s_i}^2 \right)$$

where $d_{x_i}$ and $d_{s_i}$ denote the components of the vectors $dx$ and $ds$. Using (10) and $\delta(v)$ be as defined in (a), 1, we have

$$f'(0) = \frac{1}{2} \langle \nabla \phi(v), (d_{x} + d_{s}) \rangle = -\frac{1}{2} \langle \nabla \phi(v), \nabla \phi(v) \rangle = -2\delta(v)^2$$

For convenience, we denote $v_1 = \min(v), \quad \delta = \delta(v), \quad \phi = \phi(v)$.

The following results introduce the conditions on $\alpha$ in which we have $f_1(\alpha) \leq 0$ which implies that the function $f(\alpha)$ decreases during an inner iteration using the fact that $f(0) = f_1(0) = 0$ and $f(\alpha) \leq f_1(\alpha)$.

**Lemma 6.** Let $\delta(v)$ be as defined in (a) 1, then we have

$$\delta(v) \geq \sqrt{\frac{1}{2} \phi(v)} \quad (2.19)$$

**Proof.**

Using 2. in Lemma 2

$$\phi(v) = \sum_{i=1}^{n} \psi(v_i) \leq \sum_{i=1}^{n} \frac{1}{2} (\psi'(v_i))^2 = \frac{1}{2} \| \nabla \phi(v) \|^2 = 2\delta(v)^2$$

$\square$

**Remark 2.** Throughout the paper, we assume that $\tau \geq 1$, Using Lemma 6 and $\phi(v) \geq \tau$ we have:

$$\delta(v) \geq \frac{1}{\sqrt{2}}, \quad v \in \mathbb{R}_+^n$$

From Lemmas 4.1–4.4 in [2], we have the following Lemmas 7–3.10, because $\psi(t)$ is kernel function and $\psi'(t)$ is monotonically decreasing.

**Lemma 7.** [2], Let $f_1(\alpha)$ be as defined in(29) and $\delta(v)$ be as defined in(a). Then we have $f_1(\alpha) \leq 2\delta^2(v_{\min} - 2\alpha \delta)$. Since $f_1(\alpha)$ is convex, we will have $f_1(\alpha) \leq 0$ for all $\alpha$ less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa.

The previous Lemma leads to the following three Lemmas:
Lemma 8. [2], Let $f'(\alpha) \leq 0$
\[
\psi'(v_{\min}) - \psi'(v_{\min} - 2\alpha \delta) \leq 2\delta
\tag{2.20}
\]

Lemma 9. [2], Let $\rho : [0, +\infty] \rightarrow [0, 1]$ be the inverse function of $-1/2 \psi'(t)$ The largest step size $\bar{\alpha}$ holding (31) is given by
\[
\bar{\alpha} = \frac{1}{2\delta} (\rho(\delta) - \rho(2\delta))
\]

Lemma 10. [2], Let $\bar{\alpha}$ be as defined in Lemma (9) Then
\[
\bar{\alpha} \geq \frac{1}{\psi'(\rho(2\delta))}
\]

Lemma 11. Let $\bar{\alpha}$ and $\rho$ be as defined in Lemma (10). if $\phi(v) \geq \tau \geq 1$ then we have
\[
\bar{\alpha} \geq \frac{2}{2 + (p+2)(8\delta + 1)^{p+1}}
\]

Proof.
Applying Lemma 10 , 2.4 and (24) we have
\[
\bar{\alpha} \geq \frac{1}{2\psi''(\rho 2\delta)} = \frac{1}{1 + \frac{1}{2(\rho(2\delta))^2} + \frac{p+1}{2(\rho(2\delta))^{p+2}}} \geq \frac{1}{1 + \frac{(8\delta + 1)^{p+1}}{2} + \frac{(p+1)(8\delta + 1)^{p+2}}{2}} = \frac{2}{2 + (p+2)(8\delta + 1)^{p+1}}
\]
with $(8\delta + 1)^{p+1} < (8\delta + 1)^{p+2}$, $p \in \mathbb{R}^+$

Denote
\[
\tilde{\alpha} = \frac{2}{2 + (p+2)(8\delta + 1)^{p+1}}
\tag{2.21}
\]
we have that $\tilde{\alpha}$ is the default step size and that $\tilde{\alpha} \leq \bar{\alpha}$.

Next lemma shows that our proximity function $\phi$ with the default step size $\tilde{\alpha}$ is decreasing.

Lemma 12. [13] Let $h$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$, which attains its minimum at $t^* > 0$. If $h$ is increasing for $t \in [0, t^*]$ , then
\[
h(t) \leq \frac{th(0)}{2}, \quad 0 \leq t \leq t^*
\]

Lemma 13. [6]If the step size $\alpha$ satisfies $\alpha \leq \bar{\alpha}$, then $f(\alpha) \leq -\alpha \delta^2$

Lemma 14. Let $\tilde{\alpha}$ be as defined in (32), and $\phi(v) \geq 1$ let $\phi \geq 1$, then
\[
f(\tilde{\alpha}) \leq \frac{-\sqrt{2}}{104(p+2)^{p+1}} \phi(v)
\tag{2.22}
Proof. Since $\phi(v) \geq 1$, then from lemma 6

$$\delta \geq \sqrt{\frac{1}{2} \phi} \geq \sqrt{\frac{1}{2}}$$

using lemma 13(4.5in[2]), and (30), (32), with $\alpha = \bar{\alpha}$ we have

$$f(\bar{\alpha}) \leq -\alpha \delta^2 = -\frac{2\delta^2}{2 + (p+2)(8\delta + 1)\frac{p+2}{p+1}} \leq -\frac{2\delta^2}{2(2\delta) + (p+2)(4(2\delta) + 2\delta)\frac{p+2}{p+1}}$$

$$\leq -\frac{2\delta^2}{2(2\delta) + (p+2)(10)\frac{p+2}{p+1}} \leq -\frac{2\delta^2}{2(2\delta) + (p+2)(10)\frac{p+2}{p+1}}$$

$$\leq -\frac{2\delta^2}{(p+2)104} \leq -\frac{\sqrt{2}\phi(\nu)}{(p+2)104}$$

This completes the proof. □

3. Complexity of the algorithm

3.1. Inner iteration bound. After the update of $\mu$ to $(1 - \theta)\mu$, we have

$$\phi(v^+) \leq (\phi)_0 = \frac{\tau(1 + \theta)}{1 - \theta} + \theta(n + 2\sqrt{2n\tau})$$

We need to count how many inner iterations are required to return to the situation where $\phi(v) \leq \tau$. We denote the value of $\phi(v)$ after the $\mu$ update as $(\phi)_0$, the subsequent values in the same outer iteration are denoted by $(\phi)_k$, $k = 1, 2, ..., K$, where $K$ denotes the total number of inner iterations in the outer iteration. The decrease in each inner iteration is given by (33). In [2], we can find the appropriate values of $\gamma$ and $\gamma \in [0, 1]$:

$$\gamma = 1 - \frac{p}{2(P+1)} = \frac{p+2}{2(p+1)}$$

**Lemma 15.** Let $K$ be the total number of inner iterations in the outer iteration. Then we have

$$K \leq 104\sqrt{2}(p+1)[(\phi)_0]^{\frac{p+2}{(p+1)}}$$

*Proof.* By Lemma 1.3.2 in[13], we have

$$K \leq \frac{[(\phi)_0]^{\gamma}}{\gamma^{\gamma}} \leq 104\sqrt{2}(p+1)[(\phi)_0]^{\frac{p+2}{2(p+1)}}$$

This completes the proof. □
3.2. **Total iteration bound.** The number of outer iterations is bounded above by \( \frac{\log \frac{n}{\varepsilon}}{\theta} \) (see [14] Lemma II.17, page 116). By multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely,

\[
104\sqrt{2}(p + 1)[(\phi)_{0}]^{\frac{p+2}{2(p+1)}} \log \frac{n}{\varepsilon}
\]

for large-update methods. For large-update methods with \( \tau = \mathcal{O}(n) \) and \( \theta = \Theta(1) \), we have

\[
\mathcal{O}((p + 1)n^{\frac{p+2}{2(p+1)}} \log \frac{n}{\varepsilon}) \text{ iterations complexity.}
\]

In case of a small-update methods, we have \( \tau = \mathcal{O}(1) \) and \( \theta = \Theta(\sqrt{n}) \). Substitution of these values into (33) does not give the best possible bound. A better bound is obtained in (27), using this upper bound for \((\bar{\phi})_{0}\), we get the following iteration bound:

\[
\frac{1}{\theta}104\sqrt{2}(p + 1)[(\bar{\phi})_{0}]^{\frac{p+2}{2(p+1)}} \log \frac{n}{\varepsilon}
\]

For small-update methods note now \((\bar{\phi})_{0} = \mathcal{O}(p)\), and the iteration bound becomes

\[
\mathcal{O}(p^2 \sqrt{n} \log \frac{n}{\varepsilon}) \text{ iteration complexity.}
\]

4. **Comparison of algorithms**

In this section we offer compared of their algorithm given by EL Ghami et al. [7] and the result with ours. To prove the effectiveness of our new kernel function and evaluate its effect on the behavior of the algorithm, we offer a comparison between the results of algorithms of IPMs, the following two kernel functions.

(1) The first kernel function, given by El Ghami et al. [7], is noted kernel function \( E \), defined by

\[
\psi_{E}(t) = \frac{t^{p+1} - 1}{1 + p} - \log t, \quad p \in [0, 1], \quad t > 0
\]

(2) Our new kernel function is noted kernel function \( F \), defined in (11) by

\[
\psi_{F}(t) = \frac{t^2 - 1}{2} - \frac{\log(t)}{2} + \frac{t^p - 1}{2p}, \quad t > 0, \quad p > 0
\]

We summarize these results in the Tables

**Table 1** The conditions(18),(19)and(20)

<table>
<thead>
<tr>
<th>( \psi(t) )</th>
<th>( E )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi'(t) )</td>
<td>( t^{p} - \frac{1}{t} )</td>
<td>( t - \frac{1}{2t} - \frac{1}{2p+1} )</td>
</tr>
<tr>
<td>( \psi''(t) )</td>
<td>( pt^{p-1} + \frac{1}{t^2} &gt; 1 )</td>
<td>( 1 - \frac{1}{2t^2} + \frac{p+1}{2p+3} &gt; 1 )</td>
</tr>
<tr>
<td>( \psi'''(t) )</td>
<td>( -p(1-p)t^{p-2} - \frac{2}{t^3} &lt; 0 )</td>
<td>( -\frac{1}{t^3} + \frac{(p+1)(p+2)}{2p+3} &lt; 0 )</td>
</tr>
<tr>
<td>( t\psi''(t) + \psi'(t) &gt; 0 )</td>
<td>( (1+p)t^p &gt; 0 )</td>
<td>( 2t + \frac{p}{2p+1} &gt; 0 )</td>
</tr>
<tr>
<td>( t\psi'''(t) - \psi'(t) )</td>
<td>( -(1-p)t^p + \frac{2}{t} ) is not always positive</td>
<td>( t + \frac{p+1}{2p+2} &gt; 0 )</td>
</tr>
</tbody>
</table>

**Table 2:** The default stepsize and the complexity results for large- and small-update methods
Kernel function | E | F
---|---|---
The default stepsize | $\frac{1}{2+(4\delta+1)^2}$ | $\frac{2}{2+(p+2)(8\delta+1)^{\frac{p+2}{p}}}$
Large-update methods | $O\left(n\log\frac{n}{\epsilon}\right)$ | $O\left((p+1)n^{2(p+1)}\log\frac{n}{\epsilon}\right)$
Small-update methods | $O\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ | $O\left(p^2\sqrt{n}\log\frac{n}{\epsilon}\right)$

5. Conclusion

In this paper, our objective is to propose a new efficient parameterized logarithmic kernel function. We study the properties of the kernel functions and investigate the effects of parameters. Moreover, we improve the algorithmic complexity of the interior points method. We analyze large-update and small-update versions of the primal–dual interior algorithm described in Figure 1, which is based on the new parameterized kernel functions defined by (11). The results obtained in this paper represent important contributions to improve the convergence and the complexity analysis of primal-dual IPMs for LO. So far, and to our knowledge, these results are the best known complexity bound for large-update with a logarithmic barrier term.

References

[1] Anane Nassima, méthodes de points intérieurs pour la programmation linéaire basées sur les fonctions noyaux, at the massachusetts institute of technology.