Reconstruction of the shape and surface impedance from Cauchy data for the Helmholtz equation

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Research Article

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Posted Date: February 20th, 2023

DOI: https://doi.org/10.21203/rs.3.rs-2592545/v1

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Additional Declarations: No competing interests reported.
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Received: date / Accepted: date

Abstract In this paper, we are concerned with the coefficients identification for the Helmholtz equation. This problem which consists of determining an unknown inner boundary of an annular domain and surface heat transfer coefficient from boundary Cauchy data. We propose two reconstruction algorithms to simultaneously recover the shape and the surface impedance of the obstacle within a body. This problem is ill-posed, thus we apply regularization techniques in order to improve the corresponding approximation. Numerical experiments are presented for the reconstruction algorithms, which show that both the shape and the surface impedance can be reconstructed accurately.

AMS Mathematics Subject Classification 2010: 65N20, 65N21.

Keywords Helmholtz equation, Trust-Region-Reflective optimization algorithm, Levenberg-Marquardt algorithm, Ill-posed problem

1 Introduction

In this paper, we want to determine the unknown shape and the surface impedance of the obstacle within a body which are very important in inverse problems. The coefficients identification for the Helmholtz equation from measurements arises in many real life applications, such as in nondestructive obstacle detecting, medical imaging, radar or sonar applications. We know that the solution of this inverse problem is challenging due to the inherent ill-posedness and also the nonlinearity of the problem. Therefore, we consider reconstruction algorithms with regularization techniques to solve the inverse problem.

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Mathematical questions regarding existence and uniqueness of shape and impedance recovery from far-field scattering data have been considered by several investigators [2, 14]. Most of the methods are developed to reconstruct both the shape and the impedance for the inverse scattering problem [18, 24, 4, 8, 11, 9]. In [14], the author considered the inverse problem to determine the shape and the impedance of a two-dimensional scatterer from a knowledge of the far-field pattern of the scattering of time-harmonic acoustic or electromagnetic waves. In [18, 21], based on careful singularity analysis, the authors proposed the reconstruction formula to reconstruct the shape of the obstacle and surface impedance from acoustic scattering data. In [24], the author presented a hybrid method to reconstruct shape and impedance for the inverse scattering problem from the far-field pattern for one incident direction. In [4], the authors considered the inverse obstacle scattering problem of determining both the shape and the "equivalent impedance" from far-field measurements at a fixed frequency. In [8], the authors designed an algorithm to reconstruct the shape and impedance functions from far-field measurements. In [11], the authors proposed a reconstruction procedure to identify both the shape of the obstacle and the boundary impedance from the far-field pattern. In [9], based on the method of lines, the authors proposed an inversion procedure to recover both shape and impedance from scattering data for all angles and two different frequencies of the incident waves. In [12, 16], the authors applied the method of fundamental solutions to determine the unknown inner boundary of an annular domain and possibly its surface heat transfer coefficient from one or two pairs of boundary Cauchy data.

In this paper, we are interested in is to reconstruct both the shape and the surface impedance from two pairs of boundary Cauchy data [12, 16]. We consider the coefficients identification problem for the Helmholtz equation which consists of simultaneously determining an unknown shape and the surface impedance within a body. Combining with the Tikhonov regularization method, we propose Trust-Region-Reflective optimization algorithm (TRA) and Levenberg-Marquardt algorithm (LMA) to simultaneously recover the shape and the surface impedance of the obstacle from Cauchy data on the outer boundary.

The outline of this paper is given in the following. In Section 2, an inverse problem for Helmholtz equation is given. In Section 3, parametrization of boundary and discretization of integral equations are proposed. In Section 4, reconstruction algorithms for detecting the shape and the surface impedance of the obstacle are proposed to solve this inverse problem. In Section 5, numerical experiments are presented to show the efficiency of the reconstruction algorithms. In Section 6, we give some conclusions.

2 Formulation of an inverse problem for Helmholtz equation

An inverse problem of the Helmholtz equation is considered. Assume that $\Omega$ is a bounded simply-connected domain along with smooth boundary such
as $\mathcal{D} \subset \Omega$, and $\Omega \setminus \mathcal{D}$ is connected. The steady-state temperature $u$ satisfies the helmholtz equation with the boundary conditions given by

$$
\begin{cases}
\triangle u + k^2 u = 0, & \text{in } \Omega \setminus \mathcal{D}, \\
u = f, & \text{on } \partial \Omega, \\
u + \gamma u = 0, & \text{on } \partial \mathcal{D},
\end{cases}
$$

where $k > 0$ is the wave number, the boundary data $f \in H^{\frac{1}{2}}(\Gamma_0)$ is non-constant, $\gamma$ is the surface heat transfer coefficient, and $\nu$ is the outward normal to the domain $\Omega \setminus \mathcal{D}$. Assumption that $k^2$ is not a eigenvalue of $-\triangle$ in $\Omega \setminus \mathcal{D}$.

Let $\Gamma_0 = \partial \Omega$ and $\Gamma_1 = \partial \mathcal{D}$. We know that the direct problem given by (1) has a unique solution $u \in H^{1}(\Omega \setminus \mathcal{D})$, when $\mathcal{D}$ and $\gamma$ are known, which is a well-posed problem. The inverse problem is to reconstruct $\mathcal{D}$, $\gamma$ or both $\mathcal{D}$ and $\gamma$, the additional measured current in $H^{-1/2}(\Gamma_0)$ is as follows

$$
\frac{\partial u}{\partial \nu} \bigg|_{\Gamma_0} = g.
$$

In practical problems, we can only get the measurement data $g^\delta$ which is approximate function of $g$, and satisfying

$$
\| g^\delta - g \|_{L^2(\Gamma_0)} \leq \delta,
$$

where $\| \cdot \|_{L^2(\Gamma_0)}$ denotes the $L^2$-norm on the outer boundary $\Gamma_0$ and the constant $\delta > 0$ represents a noisy level.

In this paper, we are concerned with the case when both $\mathcal{D}$ and $\gamma$ are unknown. In this case, two linearly independent Dirichlet data $f$ ensure a unique solution for the pair $(\mathcal{D}, \gamma)$, i.e. we need two pairs of boundary Cauchy data $(f, g)$ to simultaneously recover the shape and the surface impedance of the obstacle, refer to the proofs of Theorem 4.5 of [22] and Theorem 3.2 of [10].

The fundamental solution for the Helmholtz equation (1) is given by

$$
\Phi(x, y) = i \frac{4}{k} H_0^{(1)}(k|x - y|), \quad x \neq y,
$$

where $H_0^{(1)}$ is a Hankel function of the first kind of order zero. In terms of the fundamental solution, we have

$$
u(x) = \int_{\Gamma_0} \Phi(x, y) \varphi_0(y) dy + \int_{\Gamma_1} \Phi(x, y) \varphi_1(y) dy, \quad x \in \Omega \setminus \mathcal{D},
$$

where $\varphi_0(y)$ and $\varphi_1(y)$ are the unknown densities on the boundary $\Gamma_0$ and $\Gamma_1$, respectively.
3 Parametrization of boundary and discretization of integral equations

In this paper, our goal is to simultaneously reconstruct the shape and the surface impedance from Cauchy data for the Helmholtz equation. In order to reconstruct the shape of the obstacle, the boundary $\Gamma_1$ is given by in the following

$$\Gamma_1 : O + r(t)(\cos t, \sin t), 0 \leq t \leq 2\pi,$$

in terms of a $2\pi$ periodic smooth function $r(t) : [0, 2\pi] \to (0, \infty)$, where $O$ is the centroid of the domain $D$. In our numerical computations, we approximate $r(t)$ by a trigonometric polynomial of degree less than or equal to $d$, refer to $[1, 15, 23]$, i.e.

$$r(t) = c_0 + \sum_{j=1}^{d} [c_j \cos(jt) + c_{j+d} \sin(jt)], \quad (5)$$

where $c_j \in \mathbb{R}, d \in \mathbb{N}$. Let $x(t) := (x_{11}, x_{12}) = O + r(t)(\cos t, \sin t)$, then

$$x_{11}^\prime = \frac{dx_{11}}{dt} = r'(t) \cos t - r(t) \sin t,$$
$$x_{12}^\prime = \frac{dx_{12}}{dt} = r'(t) \sin t + r(t) \cos t.$$ 

Thus we can get the outward unit normal vector

$$\nu(x(t)) = \frac{(x_{12}^\prime, -x_{11}^\prime)}{\sqrt{(x_{11}^\prime)^2 + (x_{12}^\prime)^2}}. \quad (6)$$

We can use $\beta = (c_0, c_1, \cdots, c_{2d})$ to describe the shape of the obstacle.

In order to reconstruct the surface impedance $\gamma(x)$, the impedance function is given by

$$\gamma(x) = \sum_j \tilde{c}_j \psi_j(x), x \in \Gamma_1, \quad (7)$$

where the basis functions $\psi_j$ are chosen as $\psi_j(x(t)) = \exp(ijt), j = 0, \pm 1, \cdots, \pm N, t \in [0, 2\pi]$. Then we can use $\tilde{\beta} = (c_{-N}, \cdots, c_{-1}, c_0, c_{1}, \cdots, c_N)$ to describe the surface impedance of the obstacle.

Assume the boundary $\Gamma_0$ is unit circle and the center is origin, the integral equation (4) can be discretized in the following

$$u(x) = \int_0^{2\pi} \Phi(x, x_0(\tau))\varphi_0(\tau)d\tau + \int_0^{2\pi} \Phi(x, x_1(\tau))\varphi_1(\tau)x_1^\prime(\tau)d\tau. \quad (8)$$

Based on the boundary conditions (1), the densities $\varphi_0(\tau)$ and $\varphi_1(\tau)$ are the solutions of the following system of integral equations.
\[ \int_{0}^{2\pi} \Phi(x_0(t), x_0(\tau))\varphi_0(\tau)d\tau + \int_{0}^{2\pi} \Phi(x_0(t), x_1(\tau))\varphi_1(\tau)|x'_1(\tau)|d\tau = f(t), \quad (9) \]

\[ \begin{align*}
\int_{0}^{2\pi} \frac{\partial \Phi(x_1(t), x_0(\tau))}{\partial \nu(x_1(t))} \varphi_0(\tau)d\tau + \int_{0}^{2\pi} \frac{\partial \Phi(x_1(t), x_1(\tau))}{\partial \nu(x_1(t))} \varphi_1(\tau)|x'_1(\tau)|d\tau + \frac{1}{2} \varphi_1(t) \\
+ \gamma \left( \int_{0}^{2\pi} \Phi(x_1(t), x_0(\tau))\varphi_0(\tau)d\tau + \int_{0}^{2\pi} \Phi(x_1(t), x_1(\tau))\varphi_1(\tau)|x'_1(\tau)|d\tau \right) = 0. \quad (10) \end{align*} \]

We know that the kernels of the first term of (9), the second term and the fourth term of (10) are smooth that the trapezoidal rule can be applied for numerical experiments. However, the kernels of the second term of (9), the first term and the third term of (10) have singularities \[13\]. Let

\[ Q(t, \tau) = \frac{i}{4} H_0^{(1)}(k|x(t) - x(\tau)|), \]

for \( t \neq \tau \), then

\[ Q(t, \tau) = Q_1(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + Q_2(t, \tau), \]

where

\[ Q_1(t, \tau) := -\frac{1}{4\pi} J_0(k|x(t) - x(\tau)|) \]

and the diagonal term for \( Q_2 \) is as follows

\[ Q_2(t, t) = \frac{i}{4} - \frac{E}{2\pi} \frac{1}{4\pi} \ln \left( \frac{k^2}{4} |x'(t)|^2 \right), \]

where \( E \) denotes the Euler’s constant. The kernels of (10) have singularities, let

\[ P(t, \tau) = -\frac{ik}{4} H_1^{(1)}(k|x(t) - x(\tau)|) \frac{|x'(t)|^2 \cdot |x(t) - x(\tau)|}{|x(t) - x(\tau)|}, \]

which can be decomposed in the form

\[ P(t, \tau) = P_1(t, \tau) \ln \left( \frac{4 \sin^2 \frac{t - \tau}{2}}{2} \right) + P_2(t, \tau), \]

where

\[ P_1(t, \tau) = \frac{k}{4\pi} J_1(k|x(t) - x(\tau)|) \frac{|x'(t)|^2 \cdot |x(t) - x(\tau)|}{|x(t) - x(\tau)|} \]
and note that the diagonal term $P_2(t, t)$ is given by
\[
P_2(t, t) = \frac{x'(t)}{4\pi |x'(t)|^2}.
\]

The interval $[0, 2\pi]$ is partitioned as $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 2\pi$ and $0 = t_0 < t_1 < \cdots < t_n = 2\pi$ where $\tau_j = j h_\tau$ ($j = 0, 1, \cdots, m$), $t_j = j h_t$ ($j = 0, 1, \cdots, n$) and $h_\tau = \frac{2\pi}{m}$, $h_t = \frac{2\pi}{n}$ are the step sizes. The discrete vector of $\varphi(\tau)$ is given by
\[
\Psi = [\varphi_0(\tau_0), \varphi_0(\tau_1), \cdots, \varphi_0(\tau_{m-1}), \varphi_1(\tau_0), \varphi_1(\tau_1), \cdots, \varphi_1(\tau_{m-1})]^T, \tag{11}
\]
and the discrete vectors of $f(t)$ is given by
\[
F = [f(t_0), f(t_1), \cdots, f(t_{n-1})]. \tag{12}
\]
Then, we can apply the well-estimated quadrature rules and the trapezoidal rule to obtain the system of algebraic equations from the system of integral equations (9)-(10)
\[
A\Psi = b, \tag{13}
\]
where $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{20} + \gamma A_{10} & A_{21} + \gamma A_{11} \end{bmatrix}$, $b = [F; 0]^T$. The element $A_{ji,j2}(j_1, j_2 = 0, 1, 2)$ is an $m \times n$ matrix corresponding to an integral in the system of integral equations (9)-(10). In our numerical computations, $(A_{ji,j2})_{n \times n}$ is a square matrix with $m = n$. such as, $(A_{01})_{n \times n} = h_\tau [\Phi_0(x_0(t_j), x_1(\tau_j)) \varphi_1(\tau_j)|x_1(\tau_j)|]_{n \times n}$ in the second term on the left hand side of (9).

From (8), we can get the Neumann data $g$ on the boundary $I_0$
\[
g(t) = \int_0^{2\pi} \frac{\Phi_0(x_0(t), x_0(\tau))}{\partial \nu(x_0(t))} \varphi_0(\tau) d\tau + \frac{1}{2} \varphi_0(t) \\
+ \int_0^{2\pi} \frac{\partial \Phi(x_0(t), x_1(\tau))}{\partial \nu(x_0(t))} \varphi_1(\tau)|x_1(\tau)| d\tau. \tag{14}
\]
The discrete vector of $g(t)$ is as follows
\[
G = [g(t_0), g(t_1), \cdots, g(t_{n-1})]^T. \tag{15}
\]
Based on the solution $\Psi$ from (13), then we can get the flux
\[
G = B\Psi, \tag{16}
\]
where $B = [B_{00}, B_{01}]$, the element $B_{0j}(j_1 = 0, 1)$ in the matrix $B$ is an $n \times n$ matrix corresponding to an integral in integral equation (14).

According to the Cauchy data $(f, g)$, we can get the system of algebraic equations from the system of integral equations (9) and (14)
\[
\tilde{A}\Psi = \tilde{b}, \tag{17}
\]
where \( \tilde{A} = \begin{bmatrix} A_{00} & A_{01} \\ B_{00} & B_{01} \end{bmatrix} \), \( \tilde{b} = [F, G]^T \). We apply the Tikhonov regularization method [3, 7, 25] to solve (17) to obtain a regularized solution \( \tilde{\Psi} \) along with noise data \( g^\delta \), which consists in replacing the least squares problem by the following functional minimization problem

\[
\min \{ \| \tilde{A} \tilde{\Psi} - \tilde{b} \|_2^2 + \sigma \| \tilde{\Psi} \|_2^2 \},
\]

(18)

where \( \sigma \) is a regularization parameter. Then the surface impedance is given by, refer to (10)

\[
\mathcal{T} * ([A_{10}, A_{11}] * \tilde{\Psi}) = -[A_{20}, A_{21}] * \tilde{\Psi},
\]

(19)

\[
\mathcal{T} = \tilde{\beta} * C,
\]

(20)

where \( C = \begin{bmatrix} \psi_{-N}(x(t_0)) & \cdots & \psi_{-N}(x(t_m)) \\ \vdots & \ddots & \vdots \\ \psi_0(x(t_0)) & \cdots & \psi_0(x(t_m)) \\ \vdots & \ddots & \vdots \\ \psi_N(x(t_0)) & \cdots & \psi_N(x(t_m)) \end{bmatrix} \). Combing (19) and (20), we can get

\[
(C * \tilde{\beta}) * ([A_{10}, A_{11}] * \tilde{\Psi}) = -[A_{20}, A_{21}] * \tilde{\Psi},
\]

(21)

where \(*\) representation group operation. From (21), we compute \( \tilde{\beta} \) which can describe the surface impedance of the obstacle.

4 Reconstruction algorithms for reconstructing the shape and the surface impedance of the obstacle

In this section, we propose two reconstruction algorithms to recover the shape and the surface impedance of the obstacle within a body. In practical applications, we can only get measured data with errors on the outer boundary. The inverse problem for reconstructing the shape and the surface impedance is ill-posed in the sense that the solution does not depend continuously on the input measurement data. Therefore, we should apply regularization techniques to solve this inverse problem.

We consider the cost functional

\[
J(\beta, \tilde{\beta}) = \frac{1}{2} \| g^\delta - \frac{\partial u(\cdot, \beta, \tilde{\beta})}{\partial \nu} \|_{L^2(\Gamma_0)}^2,
\]

(22)

where \( g^\delta \) are the measured data on the outer boundary \( \Gamma_0 \), and

\[
\beta = (c_0, c_1, \cdots, c_{2d}) \in R^{2d+1}, \tilde{\beta} = (c_{-N}, \cdots, c_{-1}, c_0, c_1, \cdots, c_N) \in R^{2N+1}.
\]
To our knowledge, this inverse problem is a nonlinear least squares optimization problem in order to find the minimum of the objective function in (22). Our proposed reconstruction algorithms update the parameters values is to reduce the error of $J(\beta, \tilde{\beta})$ for every iteration. Starting with an initial guess $\beta(0)$, the proposed algorithms proceed by the iterations

$$
\beta^{(s+1)} = \beta^{(s)} + \Delta,
$$

where $\Delta$ is the increment vector.

4.1 Trust-Region-Reflective optimization algorithm (TRA)

Based on the interior-reflective Newton method, TRA is a subspace trust-region method refer to [5, 6]. TRA exhibits global convergence properties, which is used to minimize a nonlinear function subject to simple bound. For each iteration, the method of preconditioned conjugate gradient is employed to solve the approximate solution of a large linear system.

In terms of (22), the shape derivative with respect to the parameters $\beta$ is given by

$$
\tilde{J}'(\beta) = \frac{\partial \nabla u(\cdot, \beta, \tilde{\beta})}{\partial \nu}.
$$

TRA is to solve the quadratic subproblem with a bound step

$$
\min_{\Delta} \psi(\Delta) = \tilde{J}'(\beta)^T \Delta + \frac{1}{2} \Delta^T M \Delta : |\Theta \Delta| \leq \Lambda,
$$

where $\Theta$ is a positive diagonal scaling matrix refer to [5, 6], $\Lambda$ is the trust region size with a positive scalar and

$$
M(\beta) = \tilde{J}'(\beta)^T \tilde{J}'(\beta) + \Theta \text{diag}(\tilde{J}'(\beta)) \text{diag}(|\text{sign}(\tilde{J}'(\beta))|) \Theta.
$$

Taking $\Delta$ as the initial descent direction, we determine the piecewise linear reflective path $p(\alpha)$. Perform an approximate piecewise line minimization $J(\beta^{(s)} + p(\alpha), \tilde{\beta})$ with respect to $\alpha$ to determine an acceptable stepsize $\alpha$ refer to [5]. Then, we can obtain the final iteration relationship

$$
\beta^{(s+1)} = \beta^{(s)} + p(\alpha).
$$

4.2 Levenberg-Marquardt algorithm (LMA)

LMA interpolates between the Gauss-Newton algorithm and the method of gradient descent which is applied to solve non-linear least squares problems, refer to [17, 19, 20]. LMA is more robust than the Gauss-Newton algorithm, which can get a solution even if it starts very far off the final minimum.
According to (24), the increment vector $\Delta$ is as follows

$$
\Delta = -(\tilde{J}'(\beta)^T \tilde{J}'(\beta) + \lambda \text{diag}(\tilde{J}'(\beta)^T \tilde{J}'(\beta)))^{-1} \tilde{J}(\beta)^T \eta(\beta),
$$

(27)

where

$$
\eta(\beta, \tilde{\beta}) = g^\delta - \frac{\partial u(\cdot, \beta, \tilde{\beta})}{\partial \nu}
$$

and $\lambda$ is the Marquardt parameter.

Marquardt recommended starting with a value $\lambda_0$ and a factor $v > 1$. Initially setting $\lambda = \lambda_0$ and computing the residual sum of squares $\eta(\beta, \tilde{\beta})$ after one step from the starting point with $\lambda = \lambda_0$ and secondly with $\lambda/v$. If use of the Marquardt parameter $\lambda/v$ results in a reduction in squared residual then this is taken as the new value of $\lambda$ and the process continues; if using $\lambda/v$ resulted in a worse residual, but using $\lambda$ resulted in a better residual, then $\lambda$ is left unchanged and the new optimum is taken as the value obtained with $\lambda$ as Marquardt parameter.

So we can get the final iteration relationship

$$
\beta^{(s+1)} = \beta^{(s)} + \alpha \Delta.
$$

(28)

where $\alpha$ is the step size of iteration.

5 Numerical experiments

We present some results of numerical experiments using the proposed reconstruction algorithms in the previous section. The noisy measured data are as follows

$$
g^\delta = g(1 + \delta \cdot \text{rand}(\text{size}(g)),
$$

where $g$ is the exact data and the magnitude $\delta$ indicates a relative noise level.

When both the shape and the impedance are unknown, our algorithms to reconstruct the shape and the impedance is as follows:

Step 1. The 0th iteration: taking $\Gamma_1^{(0)}$ and $\gamma^{(0)}$ as the starting guess.

Step 2. The s-th iteration ($s > 0$): solving (9) and (10) to get the densities $\varphi^{(s)}_0$ and $\varphi^{(s)}_1$ along with $f_1$, computing the Neumann data $g_1$ from (14), and then apply TRA or LMA to compute the cost functional $J(\beta, \tilde{\beta})$ along with $g_1$.

Step 3. If a stopping criterion is satisfied, we stop here, let $\beta = \beta^{(s)}$ and $\tilde{\beta} = \tilde{\beta}^{(s)}$; Otherwise let $\beta^{(s+1)} = \beta^{(s)} + \Delta$ and $\tilde{\beta}^{(s+1)} = \tilde{\beta}^{(s)}$, compute the surface impedance from (17) and (21) along with $(f_2, g_2)$. Let $\beta^{(s+1)} = \beta^{(s)}$, then we move on to the (s+1)th iteration by going back to the second step.

Example 5.1: In this case, we suppose the shape of inner boundary $\Gamma_1$ is a peanut which is located in origin $(0, 0)$. Polar radius of a peanut is parameterized by

$$
r(t) = 0.32 \sqrt{\cos^2 t + 0.25 \sin^2 t}, 0 \leq t \leq 2\pi.
$$

(29)
And the surface heat transfer coefficient is given by
\[ \gamma(t) = \cos t, \quad 0 \leq t \leq 2\pi. \]

We take the wave number \( k = 0.3 \) and the Dirichlet data on the outer boundary \( \Gamma_0 \) in the following
\[ f_1(t) = \sin t, f_2(t) = \cos^2 t. \] (30)

We apply TRA to simultaneously recover the shape and the surface impedance of the obstacle within a body along with random noise in the data. We assume the initial guess for polar radius of a peanut is \( r(0) = 0.01 \) and the initial guess for the surface heat transfer coefficient is \( \gamma(0) = 0.01 \). We take 40 measured points on the outer boundary \( \Gamma_0 \). From Figures 1 and 2, we can see that the recovered solution match the exact solution very well along with noise data 0.01 and 0.001, respectively. For TRA, we also know that the smaller the error, the better the result.

**Example 5.2:** In this case, we suppose the shape of inner boundary \( \Gamma_1 \) is a peanut which is located in origin \((0, 0)\). Polar radius of a peanut is parameterized by (29). And the surface heat transfer coefficient is given by
\[ \gamma(t) = \cos^3 t, \quad 0 \leq t \leq 2\pi. \]

We take the wave number \( k = 2 \) and the Dirichlet data on the outer boundary \( \Gamma_0 \) is given by (30).

We apply TRA and LMA to simultaneously recover the shape and the surface impedance of the obstacle within a body along with random noise 0.01 in the data. We assume the initial guess for polar radius of a peanut is \( r(0) = 0.1 \) and the initial guess for the surface heat transfer coefficient is \( \gamma(0) = 0.1 \). We take 40 measured points on the outer boundary \( \Gamma_0 \). From Figures 3 and 4, we can see that the recovered solution match the exact solution very well for TRA and LMA, respectively.
6 Conclusions

In this paper, we consider an inverse problem to simultaneously reconstruct the shape and surface impedance from Cauchy data for the Helmholtz equation. This problem is nonlinear and ill-posed. We propose two stable and efficient reconstruction algorithms to solve this inverse problem. Numerical results show that our proposed reconstruction algorithms are efficient, feasible and stable to simultaneously reconstruct the shape and surface impedance.
Fig. 4 (a) the recovery of the shape $\Gamma_0$; (b) the recovery of the surface impedance $\gamma$. Using LMA to estimate the shape and the surface impedance of the obstacle with 0.01 noise data along with exact solution (Red, Solid) and recovered solution (Blue, Solid, o), respectively for Example 5.2.

Acknowledgments

The research of J.C. Liu was supported by the Fundamental Research Funds for the Central Universities (No.2014QNA57) and the NSF of China (No.11601512, No.11326236).

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