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Ishfaq Nazir (ishfaqnazir02@gmail.com)
University of Kashmir

Mohammad Ibrahim Mir
University of Kashmir

Irfan Ahmad Wani
University of Kashmir

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A note on Erdős-Lax type inequalities for polynomials

Ishfaq Nazir*, Mohammad Ibrahim Mir*, Irfan Ahmad Wani*

*Department of Mathematics, University of Kashmir, South Campus, Anantnag-192101, Jammu and Kashmir, India.
E-mail:ishfaqnazir02@gmail.com; ibrahimmath80@gmail.com; irfanmushtaq62@gmail.com

Abstract. The goal of this paper is to establish some results for the polar derivative of a polynomial in the plane that are inspired by a classical result of Erdős-Lax that relates the sup-norm of the derivative on the unit circle to that of the polynomial itself (on the unit circle) by using extreme coefficients of the given polynomial. The obtained results may be useful in various applications that required the Erdős-Lax type inequalities. Moreover, a numerical example is presented, showing that in some situations, the bounds obtained by our results can be considerably sharper than the previous ones known in very rich literature on this subject.

Keywords: Polynomials, Inequalities, Zeros, Polar derivative.

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1. Introduction and Main Results

Let \( P(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu} \) be a polynomial of degree \( n \) and \( P'(z) \) be its derivative. The study of comparison inequalities that relate the norm between polynomials on a disk in the plane is a fertile area in analysis. This is significant in particular for its applications in the geometric function theory and in application domains like physical systems. Various inequalities in both directions relating the norm of the derivative and the underlying polynomial play a key role in the literature for proving the inverse theorems in approximation theory and, of course have their own intrinsic value. The Bernstein and Erdős-Lax type inequalities and their various generalizations are very well-known for various norms and for many classes of functions such as polynomials with various constraints, and on various regions of the complex plane. A classical inequality that relates an estimate to the size of the derivative of a polynomial to that of the polynomial itself in the uniform-norm on the unit disk in the plane is the famous Bernstein-inequality [4]. It states that, if \( P(z) \) is a polynomial of degree \( n \), then

\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}
\]

Equality holds in (1.1) if and only if \( P(z) \) has all its zeros at the origin. It might easily be observed that the restriction on the zeros of \( P(z) \) imply an improvement in (1.1). It turns out that to have any hope of a lower bound or an improved upper bound, one must have some control over the location of the zeros of polynomial \( P(z) \). It was conjectured by P. Erdős and later proved by Lax [6] that if \( P(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then
The inequality (1.2) is best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$.

Definitely, the bound $\frac{n}{2}$ given in inequality (1.2) does not depend on how far the zeros lie outside the unit circle. Aziz and Dawood [2] made an attempt to address this issue to some extent and improved the inequality (1.2) by proving that if $P(z)$ is a polynomial of degree $n$ having no zero in $|z| < 1$, then

$$
\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left( \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right) \tag{1.3}
$$

**Definition 1.** Let $P(z)$ be a polynomial of degree $n$ with complex coefficients and $\alpha \in \mathbb{C}$ be a complex number, then the polynomial

$$
D_{\alpha} P(z) = nP(z) + (\alpha - z)P'(z)
$$

is called polar derivative of $P(z)$ with pole $\alpha$. Note that $D_{\alpha} P(z)$ is a polynomial of degree $n - 1$ and it is a generalisation of the ordinary derivative in the sense that

$$
\lim_{\alpha \to \infty} \frac{D_{\alpha} P(z)}{\alpha} = P'(z)
$$

uniformly with respect to $z$ for $|z| \leq R$, $R > 0$.

For more information on polar derivatives of polynomials one can refer to monographs by Rahman and Schmeisser [10] or Milovanovic et al. [9].

Erdős-Lax type inequalities on complex polynomials have been extended extensively from ordinary derivative to polar derivative of complex polynomials. For the latest publications in this direction one can refer to [8] and [11]. In this context, Aziz [1] obtained the following result which extends inequality (1.2), to the polar derivative of a polynomial.

**Theorem 1.1.** If $P(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n$ having no zero in the disk $|z| < 1$, then for any complex number $\alpha$, with $|\alpha| \geq 1$, we have

$$
\max_{|z|=1} |D_\alpha P(z)| \leq \frac{n(|\alpha| + 1)}{2} \max_{|z|=1} |P(z)|, \tag{1.4}
$$

Aziz and Shah [3] proved that if $P(z)$ is a polynomial of degree $n$ that does not vanish in $|z| < 1$, then for every complex number $\alpha$ with $|\alpha| \geq 1$
\[
\max_{|z|=1}|D_\alpha P(z)| \leq \frac{n}{2} \left( (|\alpha| + 1) \max_{|z|=1}|P(z)| - (|\alpha| - 1) \min_{|z|=1}|P(z)| \right).
\] (1.5)

Although inequality (1.5) sharpens inequality (1.4), it has the drawback that if there is even one zero on \(|z| = 1\), then \(\min_{|z|=1}|P(z)| = 0\) and so the inequality (1.5) fails to give any improvement over (1.4). Now a question naturally arises: Is there a way to capture the information on the moduli of zeros in order to refine inequality (1.4) for the class of polynomials satisfying the Erdős-Lax inequality hypothesis? Can we obtain a bound that describes the extreme coefficients of \(P(z)\) and whose ratio tells us something about how far the zeros are from the origin? In this paper, we approach the Erdős-Lax inequality from this side and obtain a bound that sharpens the inequality (1.4) considerably. In fact, we prove the following result.

**Theorem 1.2.** If \(P(z) = \sum_{\nu=0}^n a_\nu z^n\) is a polynomial of degree \(n\) having no zero in the disk \(|z| < 1\), then for any complex number \(\alpha\), with \(|\alpha| \geq 1\), we have

\[
\max_{|z|=1}|D_\alpha P(z)| \leq \frac{n}{2} \left( (|\alpha| + 1) - (|\alpha| - 1) \left( \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \right) \right) \max_{|z|=1}|P(z)|.
\] (1.6)

The result is the best possible and equality holds in (1.6) for the polynomial \(P(z) = z^n + a z^{n-1} + z + a\), where \(a \geq 1\).

It is a straightforward fact that the term \(\frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \geq 0\) for any polynomial satisfying the hypothesis of Theorem 1.2, and hence (1.6) clearly sharpens (1.4). One can observe that \(\frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\) is a function of the modulus of the product of the zeros of \(P(z)\), which is \(\frac{|a_0|}{|a_n|}\).

There is a reason why we deem Theorem 1.2 interesting. The inequality (1.6) sharpens the inequality (1.4) strictly for the class of polynomials having no zeros in the open unit disc with at least one zero lying outside the closed unit disc, or more precisely whenever \(|a_0| \neq |a_n|\). One can observe that, as the zeros go farther and farther from the circle \(|z| < 1\) the value of the term \(\frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)}\) increases considerably and hence the bound given in (1.6) will be much closer to the value of \(\max_{|z|=1}|D_\alpha P(z)|\) than the one given in (1.4), which does not take into account, the distance of zeros from the unit circle.

**Remark 1.** Theorem 1.2 in general gives the bound sharper than the bound obtained from Theorem 1.1, in some cases the improvement can be considerably significant, and this we show by means of the following example.

**Example 1.1.** Consider \(P(z) = z^4 - 2z^3 + 4z - 4\), which is polynomial of degree 4. Clearly, \(P(z)\) has all its zeros \(\{-\sqrt{2}, \sqrt{2}, 1-i, 1+i\}\) on the circle \(|z| = \sqrt{2}\). So, \(P(z)\) has no zero in \(|z| < 1\). For this polynomial we find that \(\max_{|z|=1}|P(z)| = 9.614\). Taking \(\alpha = 1 + \sqrt{3}i\), so that \(|\alpha| = 2\), we obtain by Theorem 1.1 that
\[
\max_{|z|=1} |D_\alpha P(z)| \leq 57.68
\]
while as Theorem 1.2 yields
\[
\max_{|z|=1} |D_\alpha P(z)| \leq 54.79
\]
showing that Theorem 1.2 gives a considerable improvement over the bound obtained from Theorem 1.1.

Next, we prove the result which generalizes Theorem 1.2. In fact, we prove the following.

**Theorem 1.3.** If \( P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \) is a polynomial of degree \( n \) having no zero in the disk \(|z| < 1\), then for any complex number \( \beta \), with \(|\beta| \leq 1\), we have
\[
\max_{|z|=1} |D_\alpha (P(z) + \beta m z^n)| \leq \frac{n}{2} \left( (|\alpha| + 1) - (|\alpha| - 1) \left( \frac{|a_0| - |a_n + \beta m|}{n(|a_0| + |a_n + \beta m|)} \right) \right) \max_{|z|=1} |P(z) + \beta m z^n|.
\]
(1.7)
The result is the best possible and equality holds in (1.7) for the polynomial \( P(z) = z^n + a z^{n-1} + z + a \), where \( a \geq 1 \).

**Remark 2.** For \( \beta = 0 \), Theorem 1.3 reduces to Theorem 1.2, and thus Theorem 1.3 contains Theorem 1.2.

**Remark 3.** In the proof of Theorem 1.3 (given in next section), the polynomial \( Q(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + (a_n + \beta m) z^n \) with \(|\beta| \leq 1\) does not vanish in the open unit disc if \( P(z) = a_0 + a_1 z + \ldots + a_{n-1} z^{n-1} + a_n z^n \neq 0 \) in the open unit disc, and hence in the bound given in (1.7) \(|a_0| - |a_n + \beta m| \geq 0\), which implies
\[
0 < 1 - \frac{|a_0| - |a_n + \beta m|}{n(|a_0| + |a_n + \beta m|)} \leq 1.
\]

By ignoring the coefficients in the bound in Theorem 1.3, we obtain the following generalisation of Theorem 1.1.

**Corollary 1.** If \( P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \) is a polynomial of degree \( n \) having no zero in the disk \(|z| \leq 1\), then for any complex number \( \beta \), with \(|\beta| \leq 1\), we have
\[
\max_{|z|=1} |D_\alpha (P(z) + \beta m z^n)| \leq \frac{n(|\alpha| + 1)}{2} \max_{|z|=1} |P(z) + \beta m z^n|.
\]
(1.8)
The result is the best possible and equality holds in (1.8) for the polynomial \( P(z) = (z + 1)^n \).
Remark 4. If we divide inequality (1.8) by $|\alpha|$ and take $|\alpha| \to \infty$ and $\beta = 0$, we get Theorem 1.1 and thus Corollary 1 contains Theorem 1.1.

2. Lemmas

For the proof our results, we need the following lemmas.

**Lemma 1.** If $P(z)$ is a polynomial of degree $n$ then on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n P(1/z)$.

The above lemma is due to Govil and Rahman [5].

**Lemma 2.** If $P(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ is a polynomial of degree $n$ having no zero in the disk $|z| < 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left[ 1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \right] \max_{|z|=1} |P(z)|.$$ 

The above lemma is due to Kummar [6].

3. Proofs of Theorems

**Proof of Theorem 1.2.** If $Q(z) = z^n P(1/z)$, then it easily follows that for $|z| = 1$

$$|Q'(z)| = |nP(z) - zP'(z)|$$

Now for any complex number $\alpha$ with $|\alpha| \geq 1$, we have on $|z| = 1$

$$|D_\alpha P(z)| = |nP(z) + (\alpha - z)P'(z)|$$

$$= |nP(z) - zP'(z) + \alpha P'(z)|$$

$$\leq |nP(z) - zP'(z)| + |\alpha||P'(z)|$$

$$= |Q'(z)| + |\alpha||P'(z)|$$

$$= n \max_{|z|=1} |P(z)| - |P'(z)| + |\alpha||P'(z)| \quad \text{(by Lemma 1)}$$

$$= n \max_{|z|=1} |P(z)| + (|\alpha| - 1)|P'(z)|$$

$$\leq n \max_{|z|=1} |P(z)| + (|\alpha| - 1) \max_{|z|=1} |P'(z)|.$$ 

Therefore using Lemma 2, we have

$$\max_{|z|=1} |D_\alpha P(z)| \leq n \max_{|z|=1} |P(z)| + n \frac{(|\alpha| - 1)}{2} \left[ 1 - \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \right] \max_{|z|=1} |P(z)|.$$ 

That is
max|D_αP(z)| \leq \frac{n}{2} \left[ (|α| + 1) - (|α| - 1) \left( \frac{|a_0| - |a_n|}{n(|a_0| + |a_n|)} \right) \right] \max_{|z|=1} |P(z)|

This completes the proof of Theorem 1.2.

□

**Proof of Theorem 1.3.** Since $P(z) \neq 0$ in $|z| < 1$, we must have

\[ m = \min_{|z|=1} |P(z)| \leq |P(z)| \tag{3.1} \]

for $|z| < 1$. Let $Q(z) = P(z) + βmz^n$. If $m = 0$, then $Q(z) = P(z)$ and therefore $Q(z) \neq 0$ for $|z| < 1$ in this case. Let us assume that $m > 0$. Our claim is $Q(z) \neq 0$ for $|z| < 1$ in this case also. Let us assume there exists a zero $z_0$ of $Q(z)$ in $|z| < 1$, then we will have

\[ Q(z_0) = P(z_0) + βmz_0^n = 0 \]

which implies

\[ |P(z_0)| = |βz_0|m < m. \tag{3.2} \]

Since $|z_0| < 1$ and $|β| \leq 1$. But the inequality (3.2) contradicts the fact given in (3.1) therefore our claim is true. Hence $Q(z) \neq 0$ for $|z| < 1$. By applying above Theorem 1.2 to $Q(z)$ we get for every complex number $α$ with $|α| \geq 1$

\[ \max_{|z|=1} |D_αQ(z)| \leq \frac{n}{2} \left[ (|α| + 1) - (|α| - 1) \left( \frac{|a_0| - |a_n + βm|}{n(|a_0| + |a_n + βm|)} \right) \right] \max_{|z|=1} |Q(z)|. \]

That is

\[ \max_{|z|=1} |D_α(P(z) + βmz^n)| \leq \frac{n}{2} \left[ (|α| + 1) - (|α| - 1) \left( \frac{|a_0| - |a_n + βm|}{n(|a_0| + |a_n + βm|)} \right) \right] \max_{|z|=1} |P(z) + βmz^n|. \]

This completes the proof of Theorem 1.3.

□

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