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Short Report

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A tension spline fitted numerical scheme for singularly perturbed reaction-diffusion problem with negative shift

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Abstract

Objective: A numerical scheme is developed and analyzed for a singularly perturbed reaction-diffusion problem with negative shift. The influence of the perturbation parameter exhibits boundary layers at the two ends of the domain, and the negative shift causes strong interior layer. The rapidly changing behavior of the solution in the layers brings significant difficulties in solving the problem analytically. The problem is treated by proposing a numerical scheme using the implicit Euler method in the temporal direction and the spline tension method in the spatial direction with uniform meshes.

Result: Error estimate is investigated for the developed numerical scheme. The scheme is demonstrated by numerical examples. The theoretical and numerical results show that the method is uniformly convergent.

Keywords: Singularly perturbed problem; Tension spline method; Boundary layers

Mathematics Subject Classification: Primary 65M06; secondary 65M12, 65M15

1 Introduction

In various areas of science and engineering, one may assume that a certain system is governed by a principal cause, which means that the current state is not dependent on the previous state and determined solely by the present one. However, under close observation, a principal cause is usually an approximation to the real situation and more existent models involve some of the past states of the system. Such systems are formulated by delay differential equations. The study of delay differential equations has recently become more active in various fields of study, such as biology, physics, engineering, robotics, and others with varying goals and expectations [1].

A singularly perturbed delay reaction-diffusion problem is a differential equation in which the diffusive term is dominated by the reaction term due to the small positive parameter $\varepsilon$ and involves one or more shifting arguments. Such problems arise frequently in the modeling of different physical phenomena. For instance, models in Bio-mathematics [2], problems in optimal control theory [3], neural dynamics and signal transmission [4] and models in the electro-optic bistable devices [5] are some applications where singularly perturbed delay differential equations are significantly applied.
Because of the presence of the perturbation parameter, the solution of a singularly perturbed delay differential equation involves two boundary layers and the term with delay causes an interior layer. The abruptly changing behaviors of the solution in the layers make it difficult to solve the problem analytically. Standard numerical methods, on the other hand, do not provide satisfactory solutions because they do not consider boundary or interior layers unless a large mesh number is used, which requires a massive computational cost. So, there is a need of developing uniformly convergent numerical methods to treat such type of problems.


Motivated by the various studies mentioned above, we treated a time dependent singularly perturbed parabolic differential equation with delay in the spatial variable. We handled the influence of the perturbation parameter and the large negative shift by developing a numerical scheme based on the implicit Euler method for the time variable and the fitted tension spline method for the spatial variable on uniform meshes. The stability estimate and the uniform convergence of the proposed numerical scheme are investigated and proved. The validity of the theoretical findings is demonstrated by carrying out numerical experiments. Based on the theoretical and numerical results, we found that the proposed scheme is uniformly convergent.

We organized the study in the following order: In Section 2, we presented the statement of the problem. Section 3 deals with the detail numerical description and methods. We presented numerical results and discussions to illustrate the theoretical results in Section 4. The result of this research work is concluded in Section 5.

**Notations.** In our study, we used $C$ as a generic constant independent of the small positive parameter and the mesh numbers. For a given function $v$ on the domain $\Omega$, the maximum norm is defined as $\|v\| = \max_{(x,t)\in \Omega} |v(x, t)|$. 
2 The continuous problem

We considered a singularly perturbed delay differential equation on \( \Omega = \Omega_x \times \Omega_t = [0, 2] \times [0, T] \) as

\[
\begin{aligned}
    &u_t - \varepsilon u_{xx} + l(x)u(x, t) + m(x)u(x - 1, t) = g(x, t), \\
    &u(x, 0) = u_0(0), \ x \in (0, 2), \\
    &u(x, t) = \alpha(x, t), \ (x, t) \in \Omega_L, \\
    &u(2, t) = \beta(t), \ (2, t) \in \Omega_R,
\end{aligned}
\]  

(1)

where \( 0 < \varepsilon \ll 1, \ \Omega_L = \{(x, t) : x \in [-1, 0]; 0 \leq t \leq T\} \) and \( \Omega_R = \{(2, t) : 0 \leq t \leq T\} \) for finite time \( T \). The functions \( l(x), m(x), g(x,t), u_0(x), \alpha(x) \) and \( \beta(x) \) are assumed to be sufficiently smooth, bounded and independent of \( \varepsilon \). Moreover, for arbitrary positive constant \( \mu \), we assumed that

\[
l(x) + m(x) \geq 2\mu > 0 \ \text{and} \ m(x) < 0, \ x \in \Omega_x.
\]  

(2)

Considering the interval boundary condition, (1) can be equivalently written as

\[
L_\varepsilon u = \begin{cases}
    u_t - \varepsilon u_{xx} + l(x)u(x, t) = g(x, t) - m(x)\alpha(x - 1, t), \ (x, t) \in (0, 1] \times (0, T], \\
    u_t - \varepsilon u_{xx} + l(x)u(x, t) + m(x)u(x - 1, t) = g(x, t), \ (x, t) \in (1, 2) \times (0, T]
\end{cases}
\]  

(3)

subjected to

\[
\begin{aligned}
    &u(x, 0) = u_0(x), \ \forall x \in \bar{\Omega}_x, \\
    &u(x, t) = \alpha(x, t), \ (x, t) \in \Omega_L, \\
    &u(2, t) = \beta(t), \ (2, t) \in \Omega_R.
\end{aligned}
\]  

(4)

If we set \( \varepsilon = 0 \) in the continuous problem, then the reduced problem is given as

\[
L_\sigma u_0 = \begin{cases}
    (u_0)_t + l(x)u_0(x, t) = g(x, t) - m(x)\alpha(x - 1, t), \ x \in (0, 1], \ t \in (0, T], \\
    (u_0)_t + l(x)u_0(x, t) + m(x)u_0(x - 1, t) = g(x, t), \ x \in (1, 2), \ t \in (0, T]
\end{cases}
\]  

(5)

with the conditions

\[
\begin{aligned}
    &u_0(x, 0) = u_0(x), \ x \in \Omega_x, \\
    &u_0(x, t) = \alpha(x, t), \ (x, t) \in \Omega_L, \\
    &u_0(2, t) = \beta(t), \ (2, t) \in \Omega_R.
\end{aligned}
\]  

(6)

From the reduced problem (5), we observed that \( u_0(x, t) \) needs not necessarily satisfy the conditions \( u_0(0, t) = \alpha(0, t), u_0(2, t) = \beta(t), u_0(1^+, t) = u_0(1^-, t) \) and \( (u_0)_x(1^+, t) = (u_0)_x(1^-, t) \) and hence, the solution \( u(x, t) \) involves two boundary layers at the ends of \([0, 2]\) and interfacing layers at \( x = 1 \) [14]. Moreover, the initial
and boundary data are assumed to satisfy Holder continuity and we impose the compatibility conditions as

\[
\begin{align*}
  u_0(0, 0) &= \alpha(0, 0), \\
  u_0(2, 0) &= \beta(2, 0), \\
  \alpha_l(0, 0) - \varepsilon(u_0)_{xx} + l(0)u_0(0) &= g(0, 0) - m(0)\alpha_l(0, 0), \\
  \beta_l(2, 0) - \varepsilon(u_0)_{xx}(2) + l(2)u_2(2) + m(2)u_0(1, 0) &= g(2, 0).
\end{align*}
\] (8)

By the above assumptions, it is possible to obtain a unique solution for the considered continuous problem. And by the approaches in [15], we can obtain that

\[|u(x, t) - u_0(x)| \leq Ct, \quad (x, t) \in \bar{\Omega}.\] (9)

The solution of (1) approaches to \(u_0(x, t)\) for small values of \(\varepsilon\). As it is described in [16], we assumed that all the considered data values in (1) are identically zero, so that the following properties hold.

**Lemma 2.1** The solution \(u(x, t)\) of the continuous problem (1) is bounded as \(|u(x, t)| \leq C, \quad (x, t) \in \bar{\Omega}\).

**Proof** From (9), it follows that \(|u(x, t)| - |u_0(x)| \leq Ct\), which implies \(|u(x, t)| \leq Ct + |u_0(x)|, \quad (x, t) \in \bar{\Omega}\). Since \(u_0(x)\) is bounded, fixing \(t \in (0, T]\), we can obtain \(|u(x, t)| \leq C, \quad (x, t) \in \bar{\Omega}\). \(\square\)

**Lemma 2.2** (Maximum principle). Let \(z(x, t)\) be a continuous function in \(\bar{\Omega}\). If \(z(x, t) \geq 0, \quad (x, t) \in \partial \Omega\) and \(L_z z(x, t) \geq 0, \quad (x, t) \in \Omega\), then \(z(x, t) \geq 0, \quad (x, t) \in \Omega\).

**Proof** Let \((\hat{x}, \hat{t}) \in \bar{\Omega}\) and \(z(\hat{x}, \hat{t}) = \min_{(x, t)} z(x, t)\). Assume that \(z(\hat{x}, \hat{t}) < 0\). By the considered hypothesis, \((\hat{x}, \hat{t}) \notin \partial \Omega\) and by the extreme value theorem, we have \(z_x(\hat{x}, \hat{t}) = 0, \quad z_{xx}(\hat{x}, \hat{t}) \geq 0\).

**Case 1**: For \(0 < \hat{x} \leq 1\), we have \(L_{z_2} z(\hat{x}, \hat{t}) = z_t - \varepsilon z_{xx} + l(\hat{x})z(\hat{x}, \hat{t}) = -\varepsilon z_{xx}(\hat{x}, \hat{t}) + l(\hat{x})z(\hat{x}, \hat{t}) < 0\).

**Case 2**: For \(1 < \hat{x} \leq 2\), we have \(L_{z_2} z(\hat{x}, \hat{t}) = z_t - \varepsilon z_{xx} + l(\hat{x})z(\hat{x}, \hat{t}) + m(\hat{x})z(\hat{x} - 1, \hat{t}) - \varepsilon z_{xx}(\hat{x}, \hat{t}) + [l(\hat{x}) + m(\hat{x})]z(\hat{x} - 1, \hat{t}) - z(\hat{x}, \hat{t}) < 0\).

The two cases contradict the hypothesis, so that our assumption fails and \(z(\hat{x}, \hat{t}) \geq 0\), which implies \(z(x, t) \geq 0, \quad (x, t) \in \bar{\Omega}\). \(\square\)

**Lemma 2.3** (Stability estimate). The solution of the continuous problem (1) is estimated as \(|u(x, t)| \leq \mu^{-1}|g| + \max \{|u_0(x)|, |\alpha(0, t)|, |\beta(2, t)|\}\).

**Proof** Let’s define barrier functions as \(\pi^\pm(x, t) = \mu^{-1}|g| + \max \{|u_0(x)|, |\alpha(0, t)|, |\beta(2, t)|\} \pm u(x, t)\). Then, we have

\[
\begin{align*}
  \pi^\pm(x, 0) &= \mu^{-1}|g| + \max \{|u_0(x)|, |\alpha(0, t)|, |\beta(2, t)|\} \pm u_0(x) \geq 0, \\
  \pi^\pm(0, t) &= \mu^{-1}|g| + \max \{|u_0(x)|, |\alpha(0, t)|, |\beta(2, t)|\} \pm \alpha(0, t) \geq 0, \\
  \pi^\pm(2, t) &= \mu^{-1}|g| + \max \{|u_0(x)|, |\alpha(0, t)|, |\beta(2, t)|\} \pm \beta(t) \geq 0.
\end{align*}
\]
For $x \in (0, 1]$, we get $L_{\varepsilon, 1} \pi^\pm = \pi_t^\pm - \varepsilon \pi_{xx}^\pm + r(x) \pi^\pm(x, t) \geq l(x) \max \{ |u_0(x)|, |\alpha(0, t)|, |\beta(2, t)| \} \geq 0$.

For $x \in (1, 2]$, we obtain $L_{\varepsilon, 2} \pi^\pm = \pi_t^\pm - \varepsilon \pi_{xx}^\pm + l(x) \pi^\pm(x, t) + m(x) \pi^\pm(x-1, t) \geq 2\mu \max \{ |u_0(x)|, |\alpha(0, t)|, |\beta(2, t)| \} \geq 0$.

Therefore, by Lemma 2.2, the stability estimate holds true.

\textbf{Lemma 2.4} The derivatives of the solution $u(x, t)$ of the continuous problem (1) is bounded as

$$\left| \frac{\partial^{k+j} u(x, t)}{\partial x^k \partial t^j} \right| \leq \begin{cases} C(1 + \varepsilon^{-k/2} \delta_1(x)), & 0 < x \leq 1, \ 0 < t \leq T], \\ C(1 + \varepsilon^{-k/2} \delta_2(x)), & 0 < x < 1, \ 0 < t \leq T \end{cases}$$

for the non-negative integers $k$ and $j$ such that $0 \leq k + 2j \leq 4$, where $\delta_1(x) = \exp(-\sqrt{\mu/\varepsilon} x) + \exp(-\sqrt{\mu/\varepsilon}(1-x)$ and $\delta_2(x) = \exp(-\sqrt{\mu/\varepsilon}(x-1)) + \exp(-\sqrt{\mu/\varepsilon}(2-x)$.

\textbf{Proof} For $(x, t) \in \bar{\Omega}$, consider the case when $k = 0$, which implies derivatives of the solution with respect to time. For $j = 0$, it becomes Lemma 2.1. For $j = 1$, rearranging (1) gives

$$\frac{\partial u}{\partial t}(x, t) = (\varepsilon \frac{\partial^2 u}{\partial x^2} - l(x))u(x, t) - m(x)u(x-1, t) + g(x, t).$$  \hspace{1cm} (10)

Assuming that $u(x, 0) = u_0 = 0$, $u(x, t) = \alpha(x, t) = 0$, $u(2, t) = \beta(t) = 0$, equation (10) becomes

$$\frac{\partial u}{\partial t}(x, 0) = \varepsilon \frac{\partial^2 u(x, 0)}{\partial x^2} - l(x)u(x, 0) - m(x)u(x-1, 0) + g(x, 0).$$  \hspace{1cm} (11)

Thus, along $x = 0$ and $x = 2$, we obtain that $u(x, t) = 0$ which gives $u_t(x, t) = 0$. And along $t = 0$, we obtain $u(x, 0) = 0$, so that $u_{xx}(x, 0) = 0$. On $[0, 1]$, we have $u(x-1, 0) = 0$ and on $[1, 2]$, we have $u(x-1, 0) = 0$. Combination of these results give $u(x-1, 0) = 0$, so that (11) becomes $u_t(x, 0) = g(x, 0)$ and since $g$ is smooth function, we have $|u(x, 0)| \leq C$ on $\partial \Omega$, which gives

$$|u_t(x, t)| \leq C \text{ on } \Omega.$$

(12)

By a similar procedure, for $j = 2$ we have $u_{tt}(x, t) - \varepsilon u_{ttx}(x, t) + l(x)u_t(x, t) + m(x)u_t(x-1, t) = g_t(x, t)$. Thus, along the side of $t = 0$, we get

$$u_{tt}(x, 0) - \varepsilon u_{ttx}(x, 0) + l(x)u_t(x, 0) + m(x)u_t(x-1, 0) = g_t(x, 0).$$  \hspace{1cm} (13)

From $u_t(x, 0) = g(x, 0)$, we have $u_{xx}(x, 0) = g_{xx}(x, 0)$ and $u_t(x-1, 0) = 0$, so that (13) becomes $u_{tt}(x, 0) = \varepsilon g_{xx}(x, 0) + l(x)g(x, 0) + g_t(x, 0)$. Hence, $|u_{tt}(x, 0)| \leq C$ on $\partial \Omega$ and since $g$ is smooth function, we obtain $|u_{tt}(x, t)| \leq C$ for $(x, t) \in \bar{\Omega}$.

On the other hand, to obtain the bound of derivatives of the solution with respect to the spatial variable, we consider the cases for $k = 0, 1, 2, 3, 4$. For $k = 0$, it becomes
Lemma 2.1. For \( k = 1 \) and \( x \in \Omega_x \), consider a neighborhood \( I_x = (\iota, \iota + \sqrt{\varepsilon}) \), for all \( x \in I_x \). Then, for some \( t \) in \( I_x \) and \( t \in (0, T) \),

\[
|u_x(\iota, t)| = \varepsilon^{\frac{3}{2}} |u(\iota + \sqrt{\varepsilon}, t) - u(\iota, t)| \leq C\varepsilon^{\frac{3}{2}} \|u\|. 
\]  

(14)

For \( x \in \bar{I}_x \), we have \( u_x(x, t) = u_x(\iota, t) + u_x(x, t) - u_x(\iota, t) = u_x(\iota, t) + \varepsilon^{-1} \int_x^x (u_x(\eta, t) + l(\eta)u(\eta, t) + m(\eta)u(\eta - 1, t) - g(\eta, t)) \, ds |u_x(x, t)| \leq |u_x(\iota, t)| + C\varepsilon^{-1} (\|u\| + \|g\|) \, ds. \) From this, we have \( |u_x(\iota, t)| + C\varepsilon^{-1} (\|u\| + \|g\|) \|u\| \varepsilon^{\frac{1}{2}} \)

Substituting (14) in the above result gives \( |u_x(x, t)| \leq C\varepsilon^{\frac{3}{2}} \) for all \((x, t) \in \Omega\). For \( k = 2 \), rearranging the terms in (1) gives

\[
u_{xx}(x, t) = \varepsilon^{-1} (u_x + l(x)u(x, t) + m(x)u(x - 1, t) - g(x, t)) \). 
\]  

(15)

For fixed time \( t \in [0, T] \) and for \( x \in [0, 2] \), since \( |u| \leq C \), \( |u_t| \leq C \) and \( g \) is continuous function, we get \( u_{xx}(x, t) \leq C \varepsilon^{-1} \). For \( k = 2 \) and \( j = 1 \), differentiating both sides of (15) gives \( u_{xx}(x, t) = \varepsilon^{-1} (u_{xx}(x, t) + l(x)u_{xx}(x, t) + m(x)u_{xx}(x - 1, t) - g_{xx}(x, t)) \),

Since \( |u_t| \leq C \), \( |u_{xx}| \leq C \) and \( |g_{xx}(x, t)| \leq C \), we have \( u_{xxx}(x, t) \leq C \varepsilon^{-1} \). By a similar procedure for the remaining values of \( k \) and \( j \) with \( 0 \leq k + 2j \leq 4 \), the bounds on the derivatives of the solution can be obtained.

\[ \square \]

3 Numerical Method

3.1 Semi-discretization in the temporal direction

Let’s divide \((0, T)\) into equally spaced intervals and form a uniform temporal mesh as \( \Omega^M = \{ t_j = j\Delta t, \ j = 0, 1, ..., M, \ T = M\Delta t \} \). Then using implicit Euler method on time derivative, we obtain the semi-discrete scheme as

\[
L_e^M u^{j+1}(x) = \vartheta(x, t_{j+1}), \]  

(16)

where

\[
L_e^M u^{j+1}(x) = \begin{cases} 
-\varepsilon \Delta_t u_x^{j+1} + p(x)u^{j+1}(x), & x \in (0, 1], \\
-\varepsilon \Delta_x u_x^{j+1} + p(x)u^{j+1}(x) + q(x)u^{j+1}(x - 1), & x \in (1, 2)
\end{cases} \]  

(17)

and

\[
\vartheta(x, t_{j+1}) = \begin{cases} 
\Delta_t g(x, t_{j+1}) + \omega(x) - q(x)\alpha(x - 1, t_{j+1}), & x \in (0, 1], \\
\Delta_x g(x, t_{j+1}) + \omega(x), & x \in (1, 2)
\end{cases} \]  

(18)

subjected to \( u^{j+1}(x) = u_0(x), \ x \in \bar{\Omega}_x, \ u^{j+1}(x) = \alpha(x, t_{j+1}), \ (x, t_{j+1}) \in \Omega_L, \ u^{j+1}(2) = \beta(2, t_{j+1}), \ (2, t_{j+1}) \in \Omega_R, \) and for \( p(x) = 1 + \Delta t(x) \) and \( q(x) = \Delta tm(x) \).

Lemma 3.1 Let \( \psi^{j+1}(x) \) be a continuous function on \( \bar{\Omega} \). If \( \psi^{j+1}(0) \geq 0, \psi^{j+1}(2) \geq 0 \) and \( L_e \psi^{j+1}(x) \geq 0, \ x \in \Omega_x \), then \( \psi^{j+1}(x) \geq 0, \ x \in \Omega_x \).
Proof Let \( \nu \in [0, 2] \) and \( \psi^{j+1}(\nu) = \min_{\Omega_x} \psi^{j+1}(x) \) and assume that \( \psi^{j+1}(\nu) < 0 \). From the given conditions, we have \( \nu \notin \partial \Omega_x \) and \( \psi^{j+1}(\nu) = 0 \), so \( \psi^{j+1}(\nu) \geq 0 \).

**Case 1:** For \( \nu \in (0, 1] \), \( L^M_{\nu} \psi^{j+1}(\nu) = -\varepsilon \psi^{j+1}_{xx}(\nu) + p(x) \psi^{j+1}(\nu) < 0 \).

**Case 2:** For \( \nu \in (1, 2) \), \( L^M_{\nu} \psi^{j+1}(\nu) = -\varepsilon \psi^{j+1}_{xx}(\nu) + p(x) \psi^{j+1}(\nu) + q(\nu) \psi^{j+1}(\nu-1) \leq -\varepsilon \psi^{j+1}_{xx}(\nu) + (p(\nu) + q(\nu)) \psi^{j+1}(\nu) < 0 \).

By these two cases, the given condition is contradicted, which implies that our assumption is not holds and hence \( \psi^{j+1}(x) \geq 0 \), \( x \in \Omega_x \). Thus, the maximum principle is satisfied by \( L^M_{\nu} \psi^{j+1}(x) \), and we have

\[
\left\| (L^M_{\nu})^{-1} \right\| \leq (1 + \mu \Delta t)^{-1},
\]

which is used in estimating the truncation error of the semi-discrete scheme. \( \square \)

**Lemma 3.2** The solution \( u^{j+1}(x) \) of the semi-discrete problem (16) can be estimated as

\[
|u^{j+1}(x)| \leq \frac{\|\psi\|_{1+\mu \Delta t}}{1+\mu \Delta t} + \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\}, \forall x \in \Omega_x.
\]

Proof Define barrier functions as \( \pi^{j+1}_\pm(x) = \frac{\|\psi\|}{1+\mu \Delta t} + \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\} \pm u^{j+1}(x) \).

At \( x = 0 \), we have \( \pi^{j+1}_\pm(0) = \frac{\|\psi\|}{1+\mu \Delta t} + \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\} \pm u^{j+1}(0) \geq \frac{\|\psi\|}{1+\mu \Delta t} \geq 0 \).

At \( x = 2 \), we have \( \pi^{j+1}_\pm(2) = \frac{\|\psi\|}{1+\mu \Delta t} + \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\} \pm u^{j+1}(2) \geq \frac{\|\psi\|}{1+\mu \Delta t} \geq 0 \).

For \( x \in (0, 1] \), we have

\[
L^M_{\nu} \pi^{j+1}_\pm(x) = -\varepsilon (\pi^{j+1}_\pm)_{xx} + p(x) \pi^{j+1}_\pm(x)
= \pm \psi^{j+1}(x) + p(x) \frac{\|\psi\|}{1+\mu \Delta t} + p(x) \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\}
\geq \mu \left( \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\} \right) \geq 0.
\]

For \( x \in (1, 2) \), we have

\[
L^M_{\nu} \pi^{j+1}_\pm(x) = -\varepsilon (\pi^{j+1}_\pm)_{xx} + p(x) \pi^{j+1}_\pm(x) + q(x) \pi^{j+1}_\pm(x-1)
= \pm \psi^{j+1}(x) + (p(x) + q(x)) \frac{\|\psi\|}{1+\mu \Delta t} + (p(x) + q(x)) \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\}
\geq \mu \left( \max \{|u^{j+1}(0)|, |u^{j+1}(2)|\} \right) \geq 0
\]

Thus, we obtained that \( L^M_{\nu} \pi^{j+1}_\pm(x) \geq 0 \) for all \( x \in [0, 2] \). Hence, by the semi-discrete maximum principle, the required estimation of \( u^{j+1}(x) \) is attained. \( \square \)

At the \((j+1)^{th}\) level, we can define the local truncation error \( e^{j+1} \) as the difference between the exact solution \( u(x, t_{j+1}) \) and the approximate solution \( u^{j+1}(x) \) of (16) and the global error estimate \( E^{j+1} \) as the contribution of local truncation error up to the \((j+1)^{th}\) time level.
Lemma 3.3 (Local truncation error estimate). Suppose that \(|u^{(k)}(x, t)| \leq C, (x, t) \in \Omega, k = 0, 1, 2\). Then at the \((j + 1)^{th}\) time level, local truncation error is given as \(\|e^{j+1}\| \leq C(\Delta t)^2\).

Proof For the proof, we refer Lemma 6 of [17].

Lemma 3.4 (Estimation of the global error). Suppose that Lemma 3.3 holds. Then the global truncation error is estimated as \(\|E^{j+1}\| \leq C(\Delta t), j=0(1)M\).

Proof Considering the local truncation error in Lemma 3.3 up to the \((j + 1)^{th}\) time level, we have

\[
\|E^{j+1}\| = \left\| \sum_{i=1}^{j} e^i \right\|, \quad j \leq T/\Delta t \\
= \|e^1 + e^2 + \ldots + e^j\| \leq \|e^1\| + \|e^2\| + \ldots + \|e^j\| \leq C(\Delta t), \quad j = 0(1)M.
\]

Thus, the semi-discrete scheme is convergent of order one in time.

Lemma 3.5 The derivatives of the solution \(u^{j+1}(x), j + 1 = 1(1)M\) of (16) can be bounded as

\[
\left| \frac{d^k u^{j+1}(x)}{dx^k} \right| \leq \begin{cases} 
C \left[ 1 + \varepsilon^{k/2} \left( \exp(-\mu/\varepsilon x) + \exp(-\mu/\varepsilon(1 - x)) \right) \right], \\
\quad x \in \Omega_x, \quad k = 0(1)4, \\
C \left[ 1 + \varepsilon^{k/2} \left( \exp(-\mu/\varepsilon(x - 1)) + \exp(-\mu/\varepsilon(2 - x)) \right) \right],
\end{cases} \\
\quad x \in \Omega_x, \quad k = 0(1)4.
\]

Proof For the proof, we refer [18].

3.2 Spatial discretization

Suppose the domain \([0, 2]\) be subdivided into \(N\) equal intervals of step size \(h\) and form a uniform mesh as \(\Omega^N_x = \{0 = x_0, x_1, \ldots, x_{N/2} = 1, x_{N/2+1}, \ldots, x_N = 2, \quad x_i = ih, \quad i = 0(1)N, \quad h = 2/N\}\).

3.2.1 Description and derivation of the tension spline method

On a uniform mesh \(\Omega^N_x\), a function \(\$(x, \tau)\) of class \(C^2[0, 2]\) that interpolates \(u(x)\) at \(x_i\) depends on the compression parameter \(\tau\) and reduced to a cubic spline on the interval \([0, 2]\) for \(\tau\) approaching to zero is known as parametric cubic spline function [19]. In any interval \([x_i, x_{i+1}]\subset [0, 2]\), the spline function \(\$(x, \tau) = \$(x)\), which satisfies the linear second order differential equation

\[
\$(x, t_{j+1}) - \tau\$(x, t_{j+1}) = \left[\$(x_i, t_{j+1}) - \tau\$(x_i, t_{j+1})\right] \left( \frac{x_{i+1} - x}{h} \right) \\
+ \left[\$(x_{i+1}, t_{j+1}) - \tau\$(x_{i+1}, t_{j+1})\right] \left( \frac{x - x_i}{h} \right),
\]

(20)
where \( u_{i}^{\tau+1} \) for \( \tau > 0 \) is called cubic spline in compression. Solving the homogeneous part (20) and setting \( \sqrt{\tau} = \frac{\lambda}{h} \) gives

\[
\$1(x, t_{j+1}) = A \exp\left(\frac{\lambda}{h}(x - x_{i})\right) + B \exp\left(\frac{\lambda}{h}(x_{i+1} - x)\right),
\]

where \( A \) and \( B \) are arbitrary constants. For the non-homogeneous part, let

\[
\$2(x, t_{j+1}) = k\left[\$xx(x_{i}, t_{j+1}) - \tau\$x(x_{i}, t_{j+1})\right]\left(\frac{x_{i+1} - x}{h}\right) + \left[\$xx(x_{i+1}, t_{j+1}) - \tau\$x(x_{i+1}, t_{j+1})\right]\left(\frac{x - x_{i}}{h}\right).
\]

Substituting in (20) and simplifying gives \( k = -1/\tau \), so that

\[
\$2(x, t_{j+1}) = -\left(\frac{h}{\lambda}\right)^{2} \left[ M_{i} - \left(\frac{\lambda}{h}\right)^{2} u_{i}^{\tau+1} \right] \left(\frac{x_{i+1} - x}{h}\right) - \left(\frac{h}{\lambda}\right)^{2} \left[ M_{i+1} - \left(\frac{\lambda}{h}\right)^{2} u_{i+1}^{\tau+1} \right] \left(\frac{x - x_{i}}{h}\right),
\]

where \( M_{i} = \$xx(x_{i}, t_{j+1}) \) and \( M_{i+1} = \$xx(x_{i+1}, t_{j+1}) \). From (21) and (22) we get

\[
\$(x, t_{j+1}) = A \exp\left(\frac{\lambda}{h}(x - x_{i})\right) + B \exp\left(\frac{\lambda}{h}(x_{i+1} - x)\right) - \left(\frac{h}{\lambda}\right)^{2} \left[ M_{i} - \left(\frac{\lambda}{h}\right)^{2} u_{i}^{\tau+1} \right] \left(\frac{x_{i+1} - x}{h}\right)
\]

\[
- \left(\frac{h}{\lambda}\right)^{2} \left[ M_{i+1} - \left(\frac{\lambda}{h}\right)^{2} u_{i+1}^{\tau+1} \right] \left(\frac{x - x_{i}}{h}\right).
\]

The values of the constants \( A \) and \( B \) can be determined by the interpolation conditions. That is, in \([x_{i}, x_{i+1}]\) from (23), we can get

\[
\$(x_{i}, t_{j+1}) = A + B \exp(\lambda) - \left(\frac{h}{\lambda}\right)^{2} \left( M_{i} - \left(\frac{\lambda}{h}\right)^{2} u_{i}^{\tau+1} \right)
\]

and

\[
\$(x_{i+1}, t_{j+1}) = A \exp(\lambda) + B - \left(\frac{h}{\lambda}\right)^{2} \left( M_{i+1} - \left(\frac{\lambda}{h}\right)^{2} u_{i+1}^{\tau+1} \right).
\]

From (24) and (25), we can obtain that \( A = \frac{h^{2}}{2\lambda^{2}\sinh(\lambda)} [M_{i+1} - e^{\lambda}M_{i}] \) and \( B = \frac{h^{2}}{2\lambda^{2}\sinh(\lambda)} [M_{i} - e^{\lambda}M_{i+1}] \). Thus, (23) becomes

\[
\$(x, t_{j+1}) = \frac{h^{2}}{2\lambda^{2}\sinh(\lambda)} \left[ M_{i+1} \sinh\left(\frac{\lambda(x - x_{i})}{h}\right) + M_{i} \sinh\left(\frac{\lambda(x_{i+1} - x)}{h}\right)\right]
\]

\[
- \left[ \frac{h}{\lambda^{2}} M_{i} - \frac{1}{h} u_{i}^{\tau+1} \right] (x_{i+1} - x) - \left[ \frac{h}{\lambda^{2}} M_{i+1} - \frac{1}{h} u_{i+1}^{\tau+1} \right] (x - x_{i}),
\]

which is the cubic spline in compression on \([x_{i}, x_{i+1}]\), where \( M_{i} = \$xx(x_{i}, t_{j+1}) \). The derivative of (26) at \((x_{i}^{+}, t_{j+1})\) is

\[
\$x(x_{i}^{+}, t_{j+1}) = \frac{u_{i+1}^{\tau+1} - u_{i}^{\tau+1}}{h} + \frac{hM_{i+1}}{\lambda^{2}} \left(1 - \frac{\lambda}{\sinh(\lambda)}\right) - \frac{hM_{i}}{\lambda^{2}} (\lambda \coth(\lambda) - 1).
\]
On the other hand, for \( x \in [x_{i-1}, x_i] \), by a similar procedure we can obtain

\[
\$x(x_i^-, t_{j+1}) = \frac{u_i^{j+1} - u_i^{j+1}}{h} + \frac{hM_1(\lambda \coth(\lambda) - 1)}{\lambda^2} + \frac{hM_{i-1}}{\lambda^2} \left( 1 - \frac{\lambda}{\sinh(\lambda)} \right). \tag{28}
\]

From (27) and (28) at the mesh point \( x_i \), we obtain

\[
h^2(\lambda_1 M_{i-1} + 2\lambda_2 M_i + \lambda_1 M_{i+1}) = u_i^{j+1} - 2u_i^{j+1} + u_i^{j+1}, \tag{29}
\]

where \( \lambda_1 = \frac{1}{h^2} \left( 1 - \frac{\lambda}{\sinh(\lambda)} \right) \) and \( \lambda_2 = \frac{1}{h^2}(\lambda \coth(\lambda) - 1) \). The consistency condition in (29) is a guarantee for the continuity of the first derivative of the spline function at the interior points. Now, from the time semi-discrete problem (16), we have

\[
\begin{align*}
\varepsilon \Delta t M_i &= p_i u_i^{j+1} + q_i u_i^{j+1} - N/2 - \Delta tg_i^{j+1} - u_i^j, \tag{30a} \\
\varepsilon \Delta t M_{i \pm 1} &= p_{i \pm 1} u_{i \pm 1}^{j+1} + q_{i \pm 1} u_{i \pm 1}^{j+1} - N/2 - \Delta tg_i^{j+1} - u_{i \pm 1}^j, \tag{30b}
\end{align*}
\]

where \( p_i = 1 + \Delta tl(x_i) \) and \( q_i = \Delta tm(x_i) \). Inserting (30a) and (30b) into (29) yields

\[
\begin{align*}
(-\varepsilon \Delta t + \lambda_1 h^2 p_{i-1}) u_i^{j+1} + (2\varepsilon \Delta t + 2\lambda_2 h^2 p_i) u_i^{j+1} + (-\varepsilon \Delta t + \lambda_1 h^2 p_{i+1}) u_i^{j+1} \\
= \lambda_1 h^2 u_{i-1}^{j+1} + 2\lambda_2 h^2 u_i^{j+1} + \lambda_1 h^2 u_{i+1}^{j+1} - \lambda_1 h^2 q_{i-1} u_i^{j+1}(x_{i-1} - 1) - 2\lambda_2 h^2 q_i u_i^{j+1}(x_i - 1) \\
- \lambda_1 h^2 q_{i+1} u_i^{j+1}(x_{i+1} - 1) + \lambda_1 h^2 \Delta tg_{i-1}^{j+1} + 2\lambda_2 h^2 \Delta tg_i^{j+1} + \lambda_1 h^2 \Delta tg_{i+1}^{j+1}, \\
i = 1(1)N - 1, j = 0(1)M - 1. \tag{31}
\end{align*}
\]

### 3.2.2 Exponential fitting factor

To control the influence of \( \varepsilon \) in the layers, we introduce an exponential fitting factor. By analogous procedures in [20], the analytical solution of (16) is written as

\[
u^{j+1}(x) = \eta_1 \exp(\sqrt{\rho_i/\varepsilon} \Delta t(x - x_i)) + \eta_2 \exp(-\sqrt{\rho_i/\varepsilon} \Delta t(x - x_i)) - \frac{1}{p_i}(q_i u^{j+1}(x_i - 1) - \Delta mg_{x_i, t_{j+1}} - u^j(x_i)), x \in (x_{i-1}, x_{i+1}), \tag{32}
\]

where the arbitrary constants \( \eta_1 \) and \( \eta_2 \) are determined using the conditions \( u^{j+1}(x_{i \pm 1}) = u_{i \pm 1}^{j+1} \) and \( u^{j+1}(x_i) = u_i^{j+1} \) as

\[
\begin{align*}
\eta_1 = \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{2(\exp(\rho \sqrt{p_i/\Delta t}) - 2 + \exp(-\rho \sqrt{p_i/\Delta t}))} + \frac{u_{i-1}^{j+1} - u_{i+1}^{j+1}}{2(\exp(\rho \sqrt{p_i/\Delta t}) + \exp(-\rho \sqrt{p_i/\Delta t}))}, \tag{33} \\
\eta_2 = \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{2(\exp(\rho \sqrt{p_i/\Delta t}) - 2 + \exp(-\rho \sqrt{p_i/\Delta t}))} - \frac{u_{i-1}^{j+1} - u_{i+1}^{j+1}}{2(\exp(\rho \sqrt{p_i/\Delta t}) + \exp(-\rho \sqrt{p_i/\Delta t}))}. \tag{34}
\end{align*}
\]
Then, introducing a fitting factor $\sigma$ on $(0, 1]$, we obtain

$$
\frac{\varepsilon \Delta t \sigma}{h^2} (u_i^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}) - p_i[\eta_1 \exp(\sqrt{p_i/\varepsilon \Delta t}(x - x_i)) + \eta_2 \exp(-\sqrt{p_i/\varepsilon \Delta t}(x - x_i)) - \frac{1}{p_i} (q_i u_i^{j+1}(x_i - 1) - \Delta t g_i^{j+1}(x_i) - u^j_i(x_i)) - q_i u_i^{j+1}(x_i - 1) = \Delta t g_i^{j+1}(x_i) - u^j_i
$$

(35)

On simplification of (35) for $i = 1, 2, \ldots, N/2$, we obtain the fitting factor

$$
\sigma_1 = \left( \frac{\rho/2}{\sqrt{p(0)/\Delta t}} \right)^2 \left( \frac{\sinh(\rho/2)}{\sinh(p/2)} \right),
$$

(36)

where $p(0) = 1 + \Delta t h(0)$ and $p = h/\sqrt{\varepsilon}$. And for $i = N/2 + 1, N/2 + 2, \ldots, N$, we obtain the fitting factor as

$$
\sigma_2 = \left( \frac{\rho/2}{\sqrt{p(0)/\Delta t}} \right)^2 \left( \frac{\sinh(\rho/2)}{\sinh(p/2)} \right),
$$

(37)

Thus, with the fitting factor $\sigma_1$ and $\sigma_2$ in (31), we obtain a fully-discrete numerical scheme as

$$
L_{\varepsilon}^{N,M} u_i^{j+1} = \vartheta(x_i, t_j),
$$

(38)

where

$$
L_{\varepsilon}^{N,M} u_i^{j+1} = \begin{cases} 
(-\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{i-1}) u_{i-1}^{j+1} + (2\varepsilon \sigma_1 \Delta t + 2\lambda_2 h^2 p_i) u_{i}^{j+1} + (-\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{i+1}) u_{i+1}^{j+1}, 
& i = 1(1)N/2, \\
(-\varepsilon \sigma_2 \Delta t + \lambda_1 h^2 p_{i-1}) u_{i-1}^{j+1} + (2\varepsilon \sigma_2 \Delta t + 2\lambda_2 h^2 p_i) u_{i}^{j+1} + (-\varepsilon \sigma_2 \Delta t + \lambda_1 h^2 p_{i+1}) u_{i+1}^{j+1} \\
& + \lambda_1 h^2 q_{i-1} u_{i-1-N/2}^{j+1} + 2\lambda_2 h^2 q_i u_{i-N/2}^{j+1} + \lambda_1 h^2 q_{i+1} u_{i+1-N/2}^{j+1}, 
& i = N/2 + 1(1)N, 
\end{cases}
$$

and

$$
\vartheta(x_i, t_j) = \begin{cases} 
\lambda_1 h^2 u_{i-1}^{j+1} + 2\lambda_2 h^2 u_{i}^{j+1} + \lambda_1 h^2 u_{i+1}^{j+1} - \lambda_1 h^2 q_{i-1} \alpha_{i-1-N/2}^{j+1} - 2\lambda_2 h^2 q_i \alpha_{i-N/2}^{j+1} \\
-\lambda_1 h^2 q_{i+1} \alpha_{i+1-N/2}^{j+1} + \lambda_1 h^2 \Delta t g_{i-1}^{j+1} + 2\lambda_2 h^2 \Delta t g_{i}^{j+1} + \lambda_1 h^2 \Delta t g_{i+1}^{j+1}, 
& i = 1(1)N/2, \\
\lambda_1 h^2 u_{i-1}^{j+1} + 2\lambda_2 h^2 u_{i}^{j+1} + \lambda_1 h^2 u_{i+1}^{j+1} + \lambda_1 h^2 \Delta t g_{i-1}^{j+1} + 2\lambda_2 h^2 \Delta t g_{i}^{j+1} + \lambda_1 h^2 \Delta t g_{i+1}^{j+1}, 
& i = N/2 + 1(1)N. 
\end{cases}
$$

From (38), we get a system of equation as

$$
\gamma_1^{-1} u_{i-1}^{j+1} + \gamma_1^0 u_i^{j+1} + \gamma_1^+ u_{i+1}^{j+1} = G_{i,j}
$$

(39)
with \( u_{i}^{j+1} = u_{i}^{j+1}(0) \) and \( u_{i}^{j+1} = u_{i}^{j+1}(x_N) \), where

\[
\begin{align*}
\gamma_1^- &= -\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{i-1}, \\
\gamma_1^0 &= 2\varepsilon \sigma_1 \Delta t + 2\lambda_2 h^2 p_i, \\
\gamma_1^+ &= -\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{i+1}, \\
G_{i,j} &= \begin{cases} \\
\lambda_1 h^2 u_{i-1}^{j} + 2\lambda_2 h^2 u_{i}^{j} + \lambda_1 h^2 u_{i+1}^{j} - \lambda_1 h^2 q_i - \lambda_2 h^2 q_{i-1} - \lambda_2 h^2 q_{i+1} & \text{for } i = 1, N/2, \text{ and assume that } \lambda_1 h^2 u_{i-1}^{j} + 2\lambda_2 h^2 u_{i}^{j} + \lambda_1 h^2 u_{i+1}^{j} - \lambda_1 h^2 q_i - \lambda_2 h^2 q_{i-1} - \lambda_2 h^2 q_{i+1} \\
\lambda_1 h^2 q_{i+1} + \lambda_1 h^2 u_{i-1}^{j} + 2\lambda_2 h^2 u_{i}^{j} + \lambda_1 h^2 u_{i+1}^{j} - \lambda_1 h^2 q_i - \lambda_2 h^2 q_{i-1} - \lambda_2 h^2 q_{i+1} & \text{for } i = N/2, N.
\end{cases}
\end{align*}
\]

The systems in (39) is solved easily using suitable solver of system of equations.

3.3 Discrete stability and uniform convergence

**Lemma 3.6** Let \( \nu \in [0, 2] \) and \( \psi^{j+1}(\nu) = \min_{0\leq \lambda \leq \nu} \psi^{j+1}(x) \) and assume that \( \psi^{j+1}(\nu) < 0 \). For a mesh function \( \psi^{j+1} \) if \( \psi^{j+1}_{\nu} \geq 0 \) and \( L^{N, M}_{\xi} \psi^{j+1}(x) \geq 0 \), then \( \psi^{j+1}_{\nu} \geq 0 \), \( i = 0(1)N \).

**Proof** Let \( \xi = 0(1)N \) and \( \psi^{j+1}_{\xi} = \min_{0\leq \lambda \leq \xi} \psi^{j+1}(x) \) and suppose that \( \psi^{j+1}_{\xi} < 0 \). From the given condition, it is clear that \( k \neq 0, N \). So, we consider the following two cases.

**Case 1:** When \( \xi = 1(1)N/2 \), we have

\[
L^{N, M}_{\epsilon_{\nu}} \psi^{j+1}_{\xi} = (\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{-1}) \psi^{j+1}_{\xi-1} + (\varepsilon \sigma_1 \Delta t + 2\lambda_2 h^2 p_i) \psi^{j+1}_{\xi} + (\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_{i+1}) \psi^{j+1}_{\xi+1} < 0.
\]

**Case 2:** When \( \xi = N/2 + 1(1)N - 1 \), we have

\[
L^{N, M}_{\epsilon_{\nu}} \psi^{j+1}_{\xi} = (\varepsilon \sigma_2 \Delta t + \lambda_1 h^2 p_{-1}) \psi^{j+1}_{\xi-1} + (\varepsilon \sigma_2 \Delta t + 2\lambda_2 h^2 p_i) \psi^{j+1}_{\xi} + (\varepsilon \sigma_2 \Delta t + \lambda_1 h^2 p_{i+1}) \psi^{j+1}_{\xi+1} + \lambda_1 h^2 q_{i-1} \psi^{j+1}_{\xi-1} + 2\lambda_2 h^2 q_i \psi^{j+1}_{\xi} + \lambda_1 h^2 q_{i+1} \psi^{j+1}_{\xi+1} < 0.
\]

From the two cases, we see that \( L^{N, M}_{\epsilon_{\nu}} \psi^{j+1}_{\xi} \leq 0 \), which contradicts the given hypothesis. Thus, our assumption is not hold, and hence \( \psi^{j+1}_{\nu} \geq 0 \), \( i = 0(1)N \).

**Lemma 3.7** The solution \( u^{j+1} \) of the difference scheme (38) is estimated as \( |u^{j+1}| \leq (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \left\{ \| u^{j+1}_0 \|, |u^{j+1}_N| \right\}, \forall i = 0, 1, ..., N. \)

**Proof** Let \( \pi^{j+1}_{i, \pm} \) be barrier functions defined by \( \pi^{j+1}_{i, \pm} = (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \left\{ \| u^{j+1}_0 \|, |u^{j+1}_N| \right\} \pm u^{j+1}_i \). Then, by these functions

- When \( i = 0 \), we have \( \pi^{j+1}_{i, \pm} = (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \left\{ \| u^{j+1}_0 \|, |u^{j+1}_N| \right\} \pm u^{j+1}_i \geq (1 + \mu \Delta t)^{-1} \| \vartheta \| \geq 0 \).
- When \( i = N \), we have \( \pi^{j+1}_{N, \pm} = (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \left\{ \| u^{j+1}_0 \|, |u^{j+1}_N| \right\} \pm u^{j+1}_i \geq (1 + \mu \Delta t)^{-1} \| \vartheta \| \geq 0 \).
Now, let \( \omega = (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \). Then, when \( i = 1, 2, \ldots, N/2 \), we have
\[
L_{\varepsilon, 1}^{N, M} \pi_{i, \pm}^{j+1} = (-\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_i - 1)(\omega \pm u_{i, \pm}^{j+1}) + (2 \varepsilon \sigma_1 \Delta t + 2 \lambda_2 h^2 p_i)(\omega \pm u_{i, \pm}^{j+1}) \\
+ (-\varepsilon \sigma_1 \Delta t + \lambda_1 h^2 p_i - 1)(\omega \pm u_{i, \pm}^{j+1}) \\
= h^2(\lambda_1 (p_{i-1} + p_{i+1}) + 2 \lambda_2 p_i) \left[ (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \right] \pm \vartheta(x_i, t_j) \\
\geq h^2(\lambda_1 p_{i-1} + 2 \lambda_2 p_i + \lambda_1 p_{i+1} + 2 \lambda_1 q_i) \left[ \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \right] \geq 0.
\]

And for \( i = N/2 + 1, N/2 + 2, \ldots, N - 1 \), we have
\[
L_{\varepsilon, 2}^{N, M} \pi_{i, \pm}^{j+1} = (-\varepsilon \sigma_2 \Delta t + \lambda_2 h^2 p_i - 1)(\omega \pm u_{i, \pm}^{j+1}) + (2 \varepsilon \sigma_2 \Delta t + 2 \lambda_2 h^2 p_i)(\omega \pm u_{i, \pm}^{j+1}) \\
+ (-\varepsilon \sigma_2 \Delta t + \lambda_2 h^2 p_i - 1)(\omega \pm u_{i, \pm}^{j+1}) + \lambda_2 h^2 q_i(\omega \pm u_{i, \pm}^{j+1}) \\
= h^2(\lambda_1 p_{i-1} + 2 \lambda_2 p_i + \lambda_1 p_{i+1} + \lambda_1 q_i + 1 + 2 \lambda_1 q_i) \left[ (1 + \mu \Delta t)^{-1} \| \vartheta \| + \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \right] \pm \vartheta(x_i, t_j) \\
\geq h^2[\lambda_1 p_{i-1} + p_{i+1} + q_i - 1 + q_i + 1] \left[ \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \right] \geq 0.
\]

Therefore, we have \( L_{\varepsilon}^{N, M} \pi_{i, \pm}^{j+1} \geq 0, \ i = 0, 1, 2, \ldots, N \), and applying Lemma 3.6, the required stability estimate of \( w_i^{j+1} \) is implied.

**Theorem 3.1** Let \( w^{j+1}(x) \) and \( u_i^{j+1} \) be the solutions of the schemes (16) and (38), respectively. Then, the error estimate in the spatial discretization is given by
\[
|w^{j+1}(x) - u_i^{j+1}| \leq CN^{-2}, \ i = 0, 1, 2, \ldots, N.
\]

**Proof** For \( i = 1, 2, \ldots, N/2 \), the truncation error is
\[
|L_{\varepsilon}^{N, M} w^{j+1}(x_i) - L_{\varepsilon, 1}^{N, M} w^{j+1}(x_i)| = | - \varepsilon \Delta t w_{xx}^{j+1} + p(x_i) w^{j+1}(x_i) + \varepsilon \sigma_1 \Delta t \vartheta^{j+1} u_i^{j+1} - \lambda_1 p_{i+1} u_i^{j+1} - 2 \lambda_2 p_i u_i^{j+1} - \lambda_1 p_{i-1} u_i^{j+1}|
\]
\[
= h^2(\lambda_1 p_{i-1} + p_{i+1} + q_i + 1 + q_i + 1) \left[ \max \{ |w_{0, i}^{j+1}|, |w_{N, i}^{j+1}| \} \right] \geq 0.
\]

Using Taylor’s series expansion for \( u_i^{j+1} \), we obtain
\[
|L_{\varepsilon}^{N, M} w^{j+1}(x_i) - L_{\varepsilon, 1}^{N, M} w^{j+1}(x_i)| = | - \varepsilon \Delta t w_{xx}^{j+1} + p(x_i) w^{j+1}(x_i) + \varepsilon \sigma_1 \Delta t (w^{j+1} + \frac{h^2}{12} w_{xxxx}^{j+1})
+ \frac{h^4}{360} w_{xxxxx}^{j+1} + O(h^6) - \lambda_1 p_{i+1} u_i^{j+1} + h w_i^{j+1} + \frac{h^2}{2} w_{x}^{j+1} + \frac{h^3}{6} w_{xx}^{j+1} + \frac{h^5}{120} w_{xxxx}^{j+1} + O(h^6) - 2 \lambda_2 p_i u_i^{j+1}
- \lambda_1 p_{i-1} (u_i^{j+1} - h w_i^{j+1} + \frac{h^2}{2} w_{x}^{j+1} - \frac{h^3}{6} w_{xx}^{j+1} + \frac{h^5}{24} w_{xxxx}^{j+1}
+ \frac{h^5}{120} w_{xxxxx}^{j+1} + O(h^6))|.
\]
For a $\lambda_1$ and $\lambda_2$ satisfying $2\lambda_2 = 1 - 2\lambda_1$, the use of Taylor’s series expansion on $p_{i \pm 1}$ and on the fitting factor $\sigma_1$, we obtain a reduced form of (40) as

$$
|L^M_{c} u_i^{j+1}(x_i) - L^N_{c} u_i^{j+1}| = |(\frac{\varepsilon \Delta t}{12} u_{xxxx}^{j+1} - \lambda_1 p_i u_{xx}^{j+1})h^2 + (\frac{\varepsilon \Delta t}{360} u_{xxxxx}^{j+1} - \frac{\Delta t p_i}{144} u_{xxxx}^{j+1} - \frac{\lambda_1 p_i}{24} u_{xx}^{j+1})h^4 + O(h^6)|
$$

$$
\leq |(\frac{\varepsilon \Delta t}{12} u_{xxxx}^{j+1} - \lambda_1 p_i u_{xx}^{j+1})h^2 + (\frac{\varepsilon \Delta t}{360} u_{xxxxx}^{j+1} - \frac{\Delta t p_i}{144} u_{xxxx}^{j+1} - \frac{\lambda_1 p_i}{24} u_{xx}^{j+1})h^4 + O(h^6)| \leq C h^2.
$$

Since $h^2 \leq N^{-2}$, invoking Lemma 3.6 yields $|u_i^{j+1}(x) - u_i^{j+1}| \leq CN^{-2}$ for $i = 0, 1, 2, ..., N$. □

**Theorem 3.2** Let $u(x)$ be the solution of (1) and $w_i^{j+1}$ be the solution of (38). Then, the uniform error is estimated as

$$
\sup_{i=0(1)N, j=0(1)M} |u(x_i, t_{j+1}) - w_i^{j+1}| \leq C(\Delta t + N^{-2})
$$

**Proof** The proof can be obtained easily by combining the proofs of Lemma 3.4 and Theorem 3.1. □

**4 Numerical experiments, results and discussions**

To illustrate the implementation of the present numerical scheme, we solved model problems. Since the exact solutions of both problems are not known, we apply the double mesh principle [21] to determine the maximum nodal error as $E_{e}^{N,M} = \max_{1 \leq i \leq N} (u_i^{N,M} - u_i^{2N,2M})$, where $u_i^{2N,2M}(x_i, t_j)$ is obtained by doubling the mesh numbers for a fixed transition parameter. The parameter-uniform maximum error is determined as $E_{e}^{N,M} = \max_{e} E_{e}^{N,M}$. The maximum convergence rate of the method is computed as $R_{e}^{N,M} = \frac{\log(E_{e}^{N,M}/E_{e}^{2N,2M})}{\log(2)}$ and its uniform convergence rate is determined by $R_{e}^{N,M} = \max_{e} R_{e}^{N,M}$.

**Example 4.1** [12]. Consider $\frac{\partial u}{\partial t} + \varepsilon \frac{\partial^2 u}{\partial x^2} - 5u(x, t) + 2u(x-1, t) = -2$, $u(x, 0) = \sin(\pi x)$ for $(x, 0) \in [0, 2]$, $u(x, t) = 0$ for $(x, t) \in \{(x, t) : x \in [-1, 0] \text{ and } t \in [0, 2]\}$ and $u(2, t) = 0$ for $(2, t) \in \{(2, t) : 0 \leq t \leq 2\}$.

**Example 4.2** [13]. Consider $\frac{\partial u}{\partial t} + \varepsilon \frac{\partial^2 u}{\partial x^2} - (x+6)u(x, t) + (x^2+1)u(x-1, t) = -3$, $u(x, 0) = 0$, $(x, 0) \in [0, 2]$, $u(x, t) = 0$, $(x, t) \in \{(x, t) : x \in [-1, 0] \text{ and } t \in [0, 2]\}$ and $u(2, t) = 0$ for $(2, t) \in \{(2, t) : t \in [0, 2]\}$.

The numerical solutions and error analysis of both examples are computed by using MATLAB R2019a package and taking $\lambda_1 = 1/24$ and $\lambda_1 = 11/24$. The maximum nodal error and convergence rate of both examples are computed as shown in Table 1 and Table 2, from which we see that by increasing the number if mesh, the maximum error decreases and decreasing $\varepsilon$ resulted in stabled maximum.
error. This confirms the convergence of the developed scheme for all values of \( \varepsilon \).

Table 3 shows the accuracy of our scheme compared to other works in literature. Graphical simulations of the solutions for the two examples are shown in Figures 1-4. From the line plots in Figure 1 and Figure 3, we observe the solution behaviors at different time levels. Also, physical behavior of the solutions and changes in the layers' width for different values of the perturbation parameter can be observed from Figures 2 and 4. Figure 5 shows log-log plots of the maximum error and the number of spatial meshes for both examples, which indicates that the developed numerical method is convergent independent of the perturbation parameter.

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Table 1: \( E_{\text{e}}^{N,M}, E^{N,M}, R_{\text{e}}^{N,M} \) and \( R^{N,M} \) of Example 4.1.

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| \( E^{N,M} \) | 5.2038e-02 | 1.8492e-02 | 6.1191e-03 | 2.0704e-03 | 2.0704e-03 | 1.0428e-03 |
| \( R^{N,M} \) | 1.4927 | 1.5766 | 1.1966 | 1.3751 |

Table 2: \( E_{\text{e}}^{N,M}, E^{N,M}, R_{\text{e}}^{N,M} \) and \( R^{N,M} \) of Example 4.2.
Table 3 Comparison of the proposed scheme and other results in literature.

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$E^{N,M}$ and $R^{N,M}$ of Example 4.2 for $T = 1$

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5 Conclusion

In this study, we considered a time dependent singularly perturbed reaction-diffusion problem involving spatial delay. The influence of the perturbation parameter forms strong boundary layers in the solution and the delay term brings interior layer. The problem is treated by developing a scheme applying the implicit Euler method in the temporal variable and the spline tension method in the spatial variable. The stability estimate and the uniform error bound are investigated and proved. To validate the theoretical findings, we solved two numerical examples. Based on the theoretical and experimental results, we concluded that the proposed numerical scheme is uniformly convergent.

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Availability of data and materials
There is no additional data used for this study.

Ethics approval and consent to participate
Not applicable.

Competing interests
There is no competing interest in this research work.

Consent for publication
Not Applicable.

Authors’ contributions
G. F. D. started and prepared the plan for this research work. A. H. E. formulated the numerical scheme and investigated the numerical analysis of the study. M. M. W. and T. G. D. revised the procedures, analysis, and results of the study. All authors have equal contributions to the paper and agreed on the submitted version.

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References
Figure 1

Line plots of the solution for Example 4.1 for $N = 128$ at four time levels (a) $\varepsilon = 2^0$ and (b) $\varepsilon = 2^{16}$. 
Figure 2

Surface plots of the solution for Example 4.1 for $N = 128$ and $M = 64$ (a) $\varepsilon = 2^0$ and (b) $\varepsilon = 2^{16}$.

Figure 3
Line plots of the solution for Example 4.2 for N = 144 at four time levels (a) $\varepsilon = 2^0$ and (b) $\varepsilon = 2^{14}$.

Figure 4

Surface plots of the solution for Example 4.2 for N = 144 and M = 144 (a) $\varepsilon = 2^0$ and (b) $\varepsilon = 2^{14}$. 
Figure 5

The log-log plots of the Maximum absolute errors with the mesh numbers for Example 4.1 in (a) and for Example 4.2 in (b).