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EXPONENTIAL STABILITY OF LORD SHULMAN THERMOELASTIC SYSTEM WITH POROUS DAMPING AND DISTRIBUTED DELAY TERM

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Abstract. In this paper, we consider a one-dimensional Lord-Shulman thermoelastic system [4] with porous damping and distributed delay term acting on the porous equation. Under suitable assumptions on the weight of distributed delay, we establish the well-posedness of the system by using semigroup theory and we show that the dissipations due to thermal effects with porous damping are strong enough to stabilise the system exponentially, independently of the wave speeds of the system.

1. Introduction

According to the thermoelasticity theory, a body’s deformation is connected to a change in enthalpy and, consequently, a change in temperature. In other words, thermoelasticity is the study of the relationship between a material’s thermal conductivity and pressure, as well as its elastic qualities and temperature. In the classical model of heat diffusion, heat flow is governed by Fourier’s law of thermal conductivity, which is considered as the best foundational equation for modeling heat conduction and states that heat flow is proportional to the temperature gradient. However, this model has a major drawback, as its use leads to physical inconsistency For the infinite speed of heat diffusion, in other words, any thermal disturbance at one point will instantly transfer to other parts of the body. As a result, many researchers were interested in developing new constitutive relations to resolve this paradox, such as the theories of Green and Lindsay [6], Lord and Shulman [8], and Gurtin et al [7], which take into account the acceleration of heat flow. These theories are based on Cattaneo Maxwell’s heat conduction equation [4]. Scientists have been quite interested in Lord and Shulman’s thermoelasticity during the past few years, and a lot of work has been done to explain it. This theory is based on the study of a system of four hyperbolic equations with heat dissipation; in this case, the heat equation is also hyperbolic as opposed to the parabolic one obtained from Fourier’s law. On the other hand, a number of stability results have been achieved and the theory of thermoelasticity with microtemperatures has been explored in numerous publications [5, 9, 10, 13, 14, 15, 16]. It is significant to note that the thermodynamic theory of microstructure elastic materials has lately incorporated the notion of microtemperature. The linked microelements also have microtemperatures, which reflect the temperature difference in a microvolume, in

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addition to the microstring deformations.

The Cattaneo Maxwell law’s evolution equations for the thermo-porous-elasticity with microtemperatures are:

\[
\rho_1 u_{tt} = t_x , \quad J \varphi_{tt} = h_x + g , \quad \rho_0 T_0 \eta_t = q_x , \quad \rho_1 e_t = P_x + q - Q \tag{1.1}
\]

Where \( \rho_1 \) is the mass density, \( J \) is the product of the mass density by the equilibrated inertia, \( T_0 \) is the reference temperature at the equilibrium state (which we assume to be equal to one for simplicity), \( t \) is the stress tensor, \( \eta \) is the entropy, \( q \) is the heat flux vector, \( h \) is the equilibrated stress, \( g \) is the equilibrated body force, \( P \) is the first heat flux moment, \( Q \) is the mean heat flux and \( e \) is the first moment of energy.

The variables \( u \) and \( \varphi \) are, respectively, the displacement of the solid elastic material and the volume fraction.

The constitutive equations are:

\[
\begin{align*}
t &= (\lambda + 2\mu) u_x + \mu_0 \varphi - \beta_0 (\tau \theta_t + \theta) \\
g &= -\mu_0 u_x - \xi \varphi + \beta_1 (\tau \theta_t + \theta) \\
q &= \delta \theta_x + k_1 T \\
Q &= (\delta - k_3) \theta_x + (k_1 - k_2) T \\
h &= a_0 \varphi_x - \mu_2 (\tau T_t + T) \\
\rho_1 \eta &= \beta_0 u_x + \beta_1 \varphi + a (\tau \theta_t + \theta) \\
P &= -k_2 T_x, \\
\rho_1 e &= -b (\tau T_t + T) - \mu_2 \varphi_x.
\end{align*}
\tag{1.2}
\]

The variable \( \theta \) denotes the temperature, \( T \) denotes the microtemperature, which represents the variation of the temperature inside the microelement, \( \lambda \) and \( \mu \) are the usual parameters and \( \tau \) is the relaxation parameter, which is assumed to be small but strictly positive. The coefficients \( \beta_0 \), \( \beta_1 \), \( \mu_0 \), \( \delta \), \( a_0 \) are positive constants, denotes respectively, the coupling between the displacement and the temperature, the coupling between the displacement and the volume fraction, the coupling between the displacement and the porosity, the thermal conductivity and the thermal capacity.

The rest of the constituent parameters \( k_1 \), \( k_2 \), \( k_3 \), \( \xi \) and \( \mu_2 \) define the characteristics of the material and, in particular, they define the couplings and satisfy the inequalities:

\[
\mu_0^2 < \mu_1 \xi, \tag{1.3}
\]

and

\[
k_2^2 < \delta k_1. \tag{1.4}
\]

Where \( \mu_1 = \lambda + 2\mu \), by substituting (1.2) into (1.1), we obtain the following system:

\[
\begin{align*}
\rho_1 u_{tt} &= \mu_1 u_{xx} + \mu_0 \varphi_x - \beta_0 (\tau \theta_{tx} + \theta_x), & & \text{in } (0, 1) \times (0, \infty) \\
J \varphi_{tt} &= a_0 \varphi_{xx} - \mu_2 (\tau T_{tx} + T_x) - \mu_0 u_x - \xi \varphi + \beta_1 (\tau \theta_t + \theta), & & \text{in } (0, 1) \times (0, \infty) \\
a(\tau \theta_t + \theta)_t &= -\beta_0 u_{tx} - \beta_1 \varphi_x + \delta \theta_{xx} + k_1 T_x, & & \text{in } (0, 1) \times (0, \infty) \\
b(\tau T_t + T)_t &= k_2 T_{xx} - \mu_2 \varphi_{tx} - k_2 T - k_3 \theta_x, & & \text{in } (0, 1) \times (0, \infty)
\end{align*}
\tag{1.5}
\]

Bazarr, Fernandez and Quintanilla [1] considered (1.5) and used the semigroup theory together with the method developed by Liu and Zheng [17] to establish the
exponential decay of the solution for the following boundary conditions:

\[
\begin{align*}
\rho (l, t) &= u (0, t) = \varphi (l, t) = \varphi (0, t) = \theta (l, t), \\
\gamma (0, t) &= \theta (0, t) = T (l, t) = T (0, t) = 0. 
\end{align*}
\]

(1.6)

In [2] the author established the stability of the above system in the presence of the porous \( \beta \varphi_l \) term added in the second equation, and in the case that they don’t assume the micro-temperature effect.

In the present work, we extend the results obtained in [2] by adding distributed delay term in the second equation, we consider the following system:

\[
\begin{align*}
\rho_1 u_{tt} &= \rho_2 u_{xx} + \mu_0 \varphi_x - \beta_0 (\tau \theta_x + \theta_x), \quad \text{in } (0, 1) \times (0, \infty) \\
\int_{\tau_1}^{\tau_2} \gamma_2 (s) \varphi_t (x, t - s) ds &= 0, \quad \text{in } (0, 1) \times (0, \infty) \\
\alpha (\tau \theta_t + \theta_t) &= -\beta_0 u_{tx} - \beta_1 \varphi_t + \delta \theta_{xx} = 0, \quad \text{in } (0, 1) \times (0, \infty)
\end{align*}
\]

(1.7)

With the initial data:

\[
\begin{align*}
u (x, 0) &= u_0 (x), \quad \varphi (x, 0) = \varphi_0 (x), \quad \theta (x, 0) = \theta_0 (x) \\
u_t (x, 0) &= u_1 (x), \quad \varphi_t (x, 0) = \varphi_1 (x), \quad \theta_t (x, 0) = \theta_1 (x) \\
\varphi_l (x, -t) &= f_0 (x, t), \quad x \in (0, 1), \quad t \in (0, \tau_2),
\end{align*}
\]

and Neumann-Dirichlet boundary conditions

\[
u_x (0, t) = u_x (1, t) = \varphi (0, t) = \varphi (1, t) = \theta (0, t) = \theta (1, t) = 0, \quad t \geq 0.
\]

The initial data \( u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1 \) belongs to the suitable functional space, the term \( \int_{\tau_1}^{\tau_2} \gamma_2 (s) \varphi_t (x, t - s) ds \) is a distributed delay that acts only on the porous equation and \( \gamma_2 : [\tau_1, \tau_2] \to IR \) is a bounded function, where \( \tau_1, \tau_2 \) are two real numbers satisfying \( 0 \leq \tau_1 < \tau_2 \).

We’ll show that the dissipations due to thermal effects with porous damping are strong enough to stabilize the system exponentially, independently of the wave speeds of the system.

The rest of this article is organized as follows: In section 2, we introduce some transformations and state the assumption needed in our work. In section 3, we use the semigroup method to prove the well-posedness of our problem. In section 4, we state and prove our stability results. We use \( c_0 \) throughout this paper to denote a generic positive constant.

2. Preliminaries

As in [11] we introduce the following new dependent variable

\[
z (x, \rho, s, t) = \varphi_t (x, t - \rho s), \quad \text{in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty) \quad (2.1)
\]

Is is easy to check that \( z \) satisfies

\[
s z_t (x, \rho, s, t) = -z_p (x, \rho, s, t), \quad \text{in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty) \quad (2.2)
\]
Consequently, problem (1.7) is equivalent to
\[
\begin{align*}
\rho_1 u_{tt} &= \mu_1 u_{xx} + \mu_0 \varphi_x - \beta_0 (\tau \theta_t + \theta_x), & \text{in } (0, 1) \times (0, \infty) \\
J \varphi_{tt} &= a_0 \varphi_{xx} - \mu_0 u_x - \xi \varphi + \beta_1 (\tau \theta_t + \theta) - \gamma_1 \varphi_t - \int_{\tau_1}^{\tau_2} \gamma_2 (s) z (x, 1, s, t) ds, & \text{in } (0, 1) \times (0, \infty) \\
a (\tau \theta_t + \theta)_t &= -\beta_0 u_{tx} - \beta_1 \varphi_t + \delta \theta_{xx}, & \text{in } (0, 1) \times (0, \infty) \\
s z_t &= -z_p, & \text{in } (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty). \tag{2.3}
\end{align*}
\]

With the following initial and boundary conditions
\[
\begin{align*}
u (x, 0) &= u_0 (x), \quad \varphi (x, 0) = \varphi_0 (x), \quad \theta (x, 0) = \theta_0 (x) & \text{in } (0, 1), \\
u_t (x, 0) &= u_1 (x), \quad \varphi_t (x, 0) = \varphi_1 (x), \quad \theta_t (x, 0) = \theta_1 (x) & \text{in } (0, 1), \\
z (x, \rho, 0) &= f_0 (x, \rho) & \text{in } (0, 1) \times (0, 1) \times (0, \tau_2), \tag{2.4} \\
u_x (0, t) &= u_x (1, t) = \varphi (0, t) = \varphi_1 (1, t) = \theta (0, t) = \theta_1 (1, t) = 0, & t \geq 0 \\
z (0, 0, s, t) &= \varphi_t (x, t) & \text{in } (0, 1) \times (\tau_1, \tau_2) \times (0, \infty),
\end{align*}
\]

concerning the weight of the delay, we only assume that
\[
\int_{\tau_1}^{\tau_2} |\gamma_2 (s)| ds < \gamma_1. \tag{2.5}
\]

Meanwhile, using (2, 3), and the boundary conditions, we get
\[
\frac{d^2}{dt^2} \int_0^1 u (x, t) dx = 0. \quad \forall t \geq 0 \tag{2.6}
\]

So, by solving (2.5) and using the initial data of \( u \), we obtain
\[
\int_0^1 u (x, t) dx = t \int_0^1 u_1 (x) dx + \int_0^1 u_0 (x) dx. \quad \forall t \geq 0
\]

Consequently, if we set
\[
\pi (x, t) = u (x, t) - t \int_0^1 u_1 (x) dx - \int_0^1 u_0 (x) dx. \quad t \geq 0, \quad x \in (0, 1)
\]

We end up with
\[
\int_0^1 \pi (x, t) dx = 0, \quad t \geq 0
\]

Thus, Poincaré’s inequality can be applied on \( \pi \). In addition, simple substitution shows that \( (\pi, \varphi, \theta) \) is the solution of problem (2.3) with initial data for \( \pi \) given as
\[
\pi (x, 0) = \pi_0 (x) = u_0 (x) - \int_0^1 u_0 (x) dx,
\]

and
\[
\pi_t (x, 0) = \pi_1 (x) = u_1 (x) - \int_0^1 u_1 (x) dx.
\]
In what follows, we will work with $\pi$ but, for convenience, we write $u$ instead of $\pi$.

3. Well-Posedness of the problem

In this section, we give the existence and uniqueness result for problem (2.3)-(2.4) using the semigroup theory.

First, we introduce the vector function

$$U = (u, v, \varphi, \theta, \psi, z)^T.$$ 

and the new dependent variables

$$v = u_t, \quad \phi = \varphi_t, \quad \psi = \theta_t.$$ 

Then the system (2.3)-(2.4) can be written as follows:

$$\begin{aligned}
U_t &= AU,
U(x, 0) &= U_0(x) = (u_0, u_1, \varphi_0, \varphi_1, \theta_0, \theta_1, f_0)^T,
\end{aligned}$$

(3.1)

Where $A : D(A) \subset H \rightarrow H$ is a linear operator defined by

$$AU = \begin{pmatrix}
v \\
\frac{\mu_1}{\rho_1} u_{xx} + \frac{\mu_0}{\rho_1} \varphi_x - \frac{\beta_0}{\rho_1} (\tau \psi_x + \theta_x) \\
\phi \\
\frac{\alpha_0}{\tau_1} \varphi_{xx} - \frac{\mu_0}{\tau_1} u_x - \frac{\xi}{\tau_1} \varphi + \frac{\beta_1}{\tau_1} (\tau \psi + \theta) - \frac{\gamma_1}{\tau_1} \phi - \frac{1}{\tau_1} \int_{\tau_1}^{\tau_2} \gamma_2(s) z(x, 1, s, t) \, ds \\
\psi \\
\frac{1}{\tau} \psi - \frac{\beta_0}{\alpha \tau} v_x - \frac{\beta_1}{\alpha \tau} \phi + \frac{\delta}{\alpha \tau} \theta_{xx} \\
-\frac{1}{s} z_p(x, \rho, s, t)
\end{pmatrix}$$

and $H$ is the energy space given by

$$H = H^1_+ (0, 1) \times L^2_+ (0, 1) \times H^1_0 (0, 1) \times L^2 (0, 1) \times H^1_0 (0, 1) \times L^2 (0, 1) \times L^2 ((0, 1) \times (0, 1) \times (\tau_1, \tau_2))$$

such that

$$\begin{aligned}
H^1_+ (0, 1) &= H^1 (0, 1) \cap L^2_+ (0, 1) \\
L^2_+ (0, 1) &= \left\{ w \in L^2 (0, 1) : \int_0^1 w(x) \, dx = 0 \right\}
\end{aligned}$$
For any $U = (u, v, \varphi, \theta, \psi, z)^T \in H$ and $\bar{U} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\theta}, \bar{\psi}, \bar{z})^T \in H$ we equip $H$ with the inner product defined by

$$
\left( U, \bar{U} \right)_H = \rho_1 \int_0^1 v \bar{v} \, dx + J \int_0^1 \phi \bar{\phi} \, dx + a_0 \int_0^1 \varphi_x \bar{\varphi}_x \, dx
+ \delta \tau \int_0^1 \theta_x \bar{\theta}_x \, dx + a \int_0^1 (\tau \psi + \theta) \left( \tau \bar{\psi} + \bar{\theta} \right) \, dx
+ \mu_1 \int_0^1 \left( u_x + \frac{\mu_0}{\mu_1} \varphi \right) \left( \bar{u}_x + \frac{\mu_0}{\mu_1} \bar{\varphi} \right) \, dx
+ \left( \tau - \frac{\mu_0^2}{\mu_1} \right) \int_0^1 \varphi \bar{\varphi} \, dx + \int_0^1 \int_0^{\tau_2} s \left( \gamma_2(s) \right) z(x, \rho, s, t) \bar{z}(x, \rho, s, t) \, ds \, d\rho \, dx.
$$

The domain of $A$ is given by

$$D(A) = \{ U \in H / u \in H_x^2 (0, 1) \cap H_x^1 (0, 1) \land v \in H_x^2 (0, 1) \land \varphi, \theta \in H^2 (0, 1) \cap H_x^1 (0, 1) \land \phi, \psi \in H_x^2 (0, 1) \land z, z_\rho \in L^2 ((0, 1) \times (0, 1) \times (\tau_1, \tau_2)) \land z(x, 0, s, t) = \phi \}. $$

Clearly, $D(A)$ is dense in $H$.

Now, we can give the following existence result.

**Theorem 1.** Let $U_0 \in H$, Problem (3.1) has a unique solution $U \in C (\mathbb{R}_+, H)$. Moreover, if $U_0 \in D(A)$, then $U \in C (\mathbb{R}_+, D(A)) \cap C^1 (\mathbb{R}_+, H)$.

**Proof.** The result follows from Lumer-Phillips theorem provided we prove that $A$ is a maximal dissipative operator. Firstly, we prove that $A$ is dissipative. For any $U \in D(A)$, and using the inner product, we get

$$
(AU, U)_H = - \left( \gamma_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \phi^2 \, dx - \delta \int_0^1 \theta^2_x \, dx
- \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, z^2(x, 1, s, t) \, ds \, dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} \gamma_2(s) z(x, 1, s, t) \, ds \, dx
$$

using Young’s inequality on the last term in (3.2) we obtain

$$
- \int_0^1 \phi \int_{\tau_1}^{\tau_2} \gamma_2(s) z(x, 1, s, t) \, ds \, dx \leq \left( \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \phi^2 \, dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, z^2(x, 1, s, t) \, ds \, dx
$$

(3.3)

Substituting (3.3) into (3.2) and using (2.5) we get

$$
(AU, U)_H = - \left( \gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \phi^2 \, dx - \delta \int_0^1 \theta^2_x \, dx \leq 0
$$

(3.2)
Hence, the operator $A$ is dissipative. Next, we prove that the operator $A$ is maximal. It is sufficient to show that the operator $I - A$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in H$, we prove that there exist a unique $U \in D(A)$ such that

$$(I - A) U = F$$

(3.4)

That is

$$
\begin{align*}
&u - v = f_1 \in H_1^1 (0, 1) \\
v - \frac{\mu_1}{\rho_1} u_{xx} - \frac{\mu_0}{\rho_1} \varphi_x + \frac{\beta_0}{\rho_1} (\tau \psi_x + \theta_x) = f_2 \in L_2^2 (0, 1) \\
&\varphi - \phi = f_3 \in H_0^1 (0, 1) \\
&\phi - \frac{\alpha_0}{\gamma} \varphi_{xx} + \frac{\mu_0}{\gamma} u_x + \frac{\xi}{\gamma} \phi - \frac{\beta_1}{\gamma} (\tau \psi + \theta) + \frac{\gamma_1}{\gamma} \phi + \frac{1}{\gamma} \int_{\tau_1}^{\tau_2} \gamma_2 (s) z (x, 1, s, t) \, ds = f_4 \in L_2^2 (0, 1) \\
&\theta - \psi = f_5 \in H_1^1 (0, 1) \\
&\psi + \frac{\beta_0}{\alpha} v_x + \frac{\beta_1}{\alpha} \phi - \frac{\delta}{\alpha} \theta_{xx} + \frac{1}{\alpha} \psi = f_6 \in L_2^2 (0, 1) \\
z + \frac{1}{s} z_{\rho} = f_7 \in L_2^2 ((0, 1), (0, 1), (\tau_1, \tau_2))
\end{align*}
$$

(3.5)

We note that the last equation in (3.5) with $z (x, 0, s, t) = \phi (x, t)$ has a unique solution given by

$$z (x, \rho, s, t) = \phi (x, t) \, e^{-s\rho} + s \, e^{-s\rho} \int_0^\rho e^{s\sigma} f_7 (x, \sigma, s, t) \, d\sigma$$

(3.6)

Inserting $v = u - f_1$, $\phi = \varphi - f_3$, $\psi = \theta - f_5$ and (3.6) in (3.5)\textsubscript{2}, (3.5)\textsubscript{4} and (3.5)\textsubscript{6}, we get

$$
\begin{align*}
\begin{cases}
\rho_1 u - \mu_1 u_{xx} - \mu_0 \varphi_x + \beta_0 (\tau + 1) \theta_x = h_1 \in L_2^2 (0, 1) \\
\mu_1 \varphi - \alpha_0 \varphi_{xx} + \mu_0 u_x - \beta_1 (\tau + 1) \theta = h_2 \in L_2^2 (0, 1) \\
\alpha (\tau + 1) \theta - \beta \theta_{xx} + \beta_1 \varphi = h_3 \in L_2^2 (0, 1)
\end{cases}
\end{align*}
$$

(3.7)

where

$$h_1 = \rho_1 (f_1 + f_2) + \beta_0 \tau f_{5x}$$

$$h_2 = \left( J + \gamma_1 + \int_{\tau_1}^{\tau_2} e^{-s} \gamma_2 (s) \, ds \right) f_3 + J f_4 - \beta_1 \tau f_5 + \int_{\tau_1}^{\tau_2} s \, e^{-s} \gamma_2 (s) \, \int_0^1 e^{s\sigma} f_7 (x, \sigma, s, t) \, d\sigma \, ds$$

$$h_3 = \alpha (\tau + 1) f_5 + a \tau f_6 + \beta_0 f_{1x} + \beta_1 f_3$$

$$\mu_4 = J + \xi + \gamma_1 + \int_{\tau_1}^{\tau_2} e^{-s} \gamma_2 (s) \, ds$$

To solve (3.7), we consider the following variational formulation:

$$B \left( (u, \varphi, \theta); \left( \bar{u}, \bar{\varphi}, \bar{\theta} \right) \right) = L \left( \bar{u}, \bar{\varphi}, \bar{\theta} \right)$$

(3.8)
where $B : [H_1^1 (0, 1) \times H_0^1 (0, 1) \times H_0^1 (0, 1)]^2 \to \mathbb{R}$ is the bilinear form defined by

$$B ((u, \varphi, \theta); (\tilde{u}, \tilde{\varphi}, \tilde{\theta})) = \rho_1 \int_0^1 u \tilde{u} \, dx + \mu_4 \int_0^1 \varphi \tilde{\varphi} \, dx + a (\tau + 1)^2 \int_0^1 \theta \tilde{\theta} \, dx$$

$$+ \mu_1 \int_0^1 u_x \tilde{u}_x \, dx + a_0 \int_0^1 \varphi_x \tilde{\varphi}_x \, dx + b (\tau + 1 \int_0^1 \theta_x \tilde{\theta}_x \, dx$$

$$+ \mu_0 \int_0^1 (u_x \tilde{\varphi} + \varphi \tilde{u}_x) \, dx + \beta_0 (\tau + 1 \int_0^1 (u_x \tilde{\theta} - \theta \tilde{u}_x) \, dx + \beta_1 (\tau + 1 \int_0^1 (\varphi \tilde{\theta} - \theta \tilde{\varphi}) \, dx$$

and $L : H_1^1 (0, 1) \times H_0^1 (0, 1) \times H_0^1 (0, 1) \to \mathbb{R}$ is the linear form given by

$$L (\tilde{u}, \tilde{\varphi}, \tilde{\theta}) = \int_0^1 h_1 \tilde{u} \, dx + \int_0^1 h_2 \tilde{\varphi} \, dx + (\tau + 1 \int_0^1 h_3 \tilde{\theta} \, dx$$

Now, for $V = H_1^1 (0, 1) \times H_0^1 (0, 1) \times H_0^1 (0, 1)$ equipped with the norm

$$\|(u, \varphi, \theta)\|_V^2 = \|u\|_{L^2 (0, 1)}^2 + \|u_x\|_{L^2 (0, 1)}^2 + \|\varphi\|_{L^2 (0, 1)}^2 + \|\varphi_x\|_{L^2 (0, 1)}^2 + \|\theta\|_{L^2 (0, 1)}^2 + \|\theta_x\|_{L^2 (0, 1)}^2$$

we have

$$B ((u, \varphi, \theta); (u, \varphi, \theta)) = \rho_1 \int_0^1 u^2 \, dx + \mu_4 \int_0^1 \varphi^2 \, dx + a (\tau + 1)^2 \int_0^1 \theta^2 \, dx + \mu_1 \int_0^1 u_x^2 \, dx$$

$$+ a_0 \int_0^1 \varphi_x^2 \, dx + b (\tau + 1 \int_0^1 \theta_x^2 \, dx + 2 \mu_0 \int_0^1 u_x \varphi \, dx$$

on the other hand, we can write

$$\mu_1 u_x^2 + 2 \mu_0 u_x \varphi + \mu_4 \varphi^2 = \frac{1}{2} \left[ \mu_1 \left(u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 + \mu_4 \left(\varphi + \frac{\mu_0}{\mu_4} u_x \right)^2 + \left(\mu_1 - \frac{\mu_0^2}{\mu_4} \right) u_x^2 + \left(\mu_4 - \frac{\mu_0^2}{\mu_1} \right) \varphi^2 \right]$$

using (1.3), we deduce

$$\mu_1 u_x^2 + 2 \mu_0 u_x \varphi + \mu_4 \varphi^2 > \frac{1}{2} \left[ \left(\mu_1 - \frac{\mu_0^2}{\mu_4} \right) u_x^2 + \left(\mu_4 - \frac{\mu_0^2}{\mu_1} \right) \varphi^2 \right]$$

then, for some $M_0 > 0$

$$B ((u, \varphi, \theta); (u, \varphi, \theta)) \geq M_0 \|(u, \varphi, \theta)\|_V^2$$

Thus $B$ is coercive. Similarly, we can easily prove that the bilinear and linear forms $B$ and $L$ are continue.

Consequently, by the Lax-Miligram Lemma the variational problem (3.8) has unique solution

$$(u, \varphi, \theta) \in H_1^1 (0, 1) \times H_0^1 (0, 1) \times H_0^1 (0, 1)$$

satisfying

$$B ((u, \varphi, \theta); (\tilde{u}, \tilde{\varphi}, \tilde{\theta})) = L (\tilde{u}, \tilde{\varphi}, \tilde{\theta}) \quad \forall (\tilde{u}, \tilde{\varphi}, \tilde{\theta}) \in V$$

The substitution of $u$, $\varphi$ and $\theta$ into (3.5)$_1$, (3.5)$_3$ and (3.5)$_3$, respectively, we obtain

$$(v, \phi, \psi) \in H_1^1 (0, 1) \times H_0^1 (0, 1) \times H_0^1 (0, 1)$$
Similarly, inserting $\phi$ in (3.6) and bearing in mind (3.5), we find

$$z, z_\rho \in L^2((0, 1), (0, 1), (\tau_1, \tau_2))$$

Now, if $(\varphi, \theta) \equiv (0, 0) \in H^1_0(0, 1) \times H^1_0(0, 1)$, then (3.8) reduces to

$$\rho_1 \int_0^1 u \tilde{\varphi} \, dx + \mu_1 \int_0^1 u \varphi \, dx + \mu_0 \int_0^1 \varphi \tilde{\varphi} \, dx - \beta_0 (\tau + 1) \int_0^1 \theta \tilde{\varphi} \, dx = \int_0^1 h_1 h \, dx \quad \forall \tilde{u} \in H^1_0(0, 1)$$

(3.9)

which implies

$$-\mu_1 u_{xx} = -\rho_1 u + \mu_0 \varphi_x - \beta_0 (\tau + 1) \theta_x + h_1 \in L^2(0, 1)$$

(3.10)

Consequently, by the regularity theory for the linear elliptic equation, it follows that

$$u \in H^2(0, 1) \cap H^1_0(0, 1)$$

Moreover, (3.9) is also true for any $\vartheta \in C^1([0, 1]) \subset H^1_0(0, 1)$. Hence, we have

$$\rho_1 \int_0^1 u \varphi \, dx + \mu_1 \int_0^1 u \varphi \, dx + \mu_0 \int_0^1 \varphi \tilde{\varphi} \, dx - \beta_0 (\tau + 1) \int_0^1 \theta \tilde{\varphi} \, dx = 0$$

for all $\vartheta \in C^1([0, 1])$. Thus, using integration by parts and bearing in mind (3.10), we obtain

$$u_x(1) \vartheta(1) - u_x(0) \vartheta(0) = 0 \quad \forall \vartheta \in C^1([0, 1])$$

Therefore, $u_x(0) = u_x(1) = 0$. Consequently, we obtain

$$u \in H^2_0(0, 1) \cap H^1_0(0, 1)$$

Similarly, we obtain

$$-a_0 \varphi_{xx} = -\mu_2 \varphi - \mu_0 \varphi_x - \beta_1 \varphi + h_2 \in L^2(0, 1)$$

$$-\delta \theta_{xx} = -a (\tau + 1) \theta - \beta_0 \varphi_x - \beta_1 \varphi + h_3 \in L^2(0, 1)$$

thus, we have

$$\varphi, \theta \in H^2(0, 1) \cap H^1_0(0, 1)$$

Finally, the application of the regularity theory for the linear elliptic equations guarantees the existence of unique $U \in D(A)$ such that (3.4) is satisfied. Consequently, $A$ is a maximal operator. Hence, the result of Theorem 1 follows from Lumer-Phillips theorem (see [12, 17]) \hfill \Box

4. Exponential stability

In this section, we use the energy method to prove that system (2.3)-(2.4) is exponentially stable. To achieve our goal, we need the following lemmas:

**Lemma 1.** The energy functional, $E$, defined by

$$E(t) = \frac{1}{2} \int_0^1 \left( \rho_1 u_x^2 + J \varphi_x^2 + a_0 \varphi^2 + \delta \theta_x^2 + \alpha (\tau + 1) \theta^2 + \mu_1 \left( u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 + \left( \xi - \frac{\mu_0^2}{\mu_1} \right) \varphi^2 \right) \, dx$$

$$+ \frac{1}{2} \int_0^1 \int_0^{\tau_2} \int_0^{s} \gamma_2(s) \left| z^2(x, \rho, s, t) \right| \, ds \, d\rho \, dx \quad t \geq 0.$$

(4.1)
satisfies
\[ E'(t) \leq - \left( \gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)| ds \right) \int_0^1 \varphi_2^2 dx - \delta \int_0^1 \theta_x^2 dx. \] (4.2)

Proof. Multiplying (2.3)_1 by \( u_t \), (2.3)_2 by \( \varphi_t \), (2.3)_3 by \((\tau \theta_t + \theta)\) and integrating over \((0,1)\) and summing them up we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \left( \rho_1 u_t^2 + \mu_1 u_x^2 + 2 \mu_0 \varphi u_x + J \varphi_t^2 + a_0 \varphi_x^2 + \xi \varphi^2 + a(\tau \theta_t + \theta)^2 + \delta \theta_x^2 \right) dx
\]
\[+ \gamma_1 \int_0^1 \varphi_t^2 dx + \delta \int_0^1 \theta_x^2 dx + \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s) z(x,1,s,t) ds dx = 0 \] (4.3)

Multiplying (2.3)_4 by \( |\gamma_2(s)| \) and integrating over \((0,1) \times (0,1) \times (\tau_1,\tau_2)\) we get
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 \int_0^{\tau_2} s |\gamma_2(s)| z^2(x,\rho,s,t) ds d\rho dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| z^2(x,1,s,t) ds dx
\]
\[- \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| z^2(x,0,s,t) ds dx = 0 \] (4.4)

by summing (4.3),(4.4) and using the fact that \( z(x,0,s,t) = \varphi_t(x,t) \), we have
\[
d dt \left\{ \frac{1}{2} \int_0^1 \int_0^{\tau_2} \left( \rho_1 u_t^2 + \mu_1 u_x^2 + 2 \mu_0 \varphi u_x + J \varphi_t^2 + a_0 \varphi_x^2 + \xi \varphi^2 + a(\tau \theta_t + \theta)^2 + \delta \theta_x^2 \right) dx 
\right\}
\[+ \frac{1}{2} \int_0^1 \int_0^{\tau_2} s |\gamma_2(s)| z^2(x,\rho,s,t) ds d\rho dx \]
\[= - \gamma_1 \int_0^1 \varphi_t^2 dx - \delta \int_0^1 \theta_x^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s) z(x,1,s,t) ds dx
\]
\[- \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| z^2(x,1,s,t) ds dx + \frac{1}{2} \int_0^{\tau_2} |\gamma_2(s)| \int_0^1 \varphi_t^2 dx \]
using the fact that
\[
\mu_1 u_x^2 + 2 \mu_0 \varphi u_x + \xi \varphi^2 = \mu_1 \left( u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 + (\xi - \frac{\mu_0^2}{\mu_1}) \varphi^2
\]
we find
\[
E(t) = \frac{1}{2} \int_0^1 \left( \rho_1 u_t^2 + J \varphi_t^2 + a_0 \varphi_x^2 + \delta \theta_x^2 + a(\tau \theta_t + \theta)^2 + \mu_1 \left( u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 + (\xi - \frac{\mu_0^2}{\mu_1}) \varphi^2 \right) dx
\]
\[+ \frac{1}{2} \int_0^1 \int_0^{\tau_2} s |\gamma_2(s)| z^2(x,\rho,s,t) ds d\rho dx \]
and

\[ E'(t) = - \left( \gamma_1 - \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \varphi_t^2 \, dx - \delta \int_0^1 \theta_x^2 \, dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s) \, z(x,1,s,t) \, ds \, dx \]

\[ - \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, z^2(x,1,s,t) \, ds \, dx \]

(4.5)

using Young’s inequality, we get

\[ - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \gamma_2(s) \, z(x,1,s,t) \, ds \, dx \leq \left( \frac{1}{2} \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \varphi_t^2 \, dx + \frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, z^2(x,1,s,t) \, ds \, dx \]

(4.6)

inserting (4.6) in (4.5), and using (2.5), we find

\[ E'(t) \leq - \left( \gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \int_0^1 \varphi_t^2 \, dx - \delta \int_0^1 \theta_x^2 \, dx \leq 0 \]

Thus, the energy is decreasing and bounded above by \( E(0) \), which concludes the proof. \( \square \)

**Lemma 2.** Let \((u, \varphi, \theta, z)\) be the solution of (2.3)-(2.4) then, the functional \( I_1(t) \) defined by

\[ I_1(t) = \rho_1 \int_0^1 u_t \, u \, dx - \beta_0 \tau \int_0^1 \theta_x \, dx \quad t \geq 0, \]

(4.7)

satisfies

\[ I_1'(t) \leq - \frac{\mu_1}{2} \int_0^1 u_x^2 \, dx + c_0 \int_0^1 (\theta_x^2 + u_t^2 + \varphi^2) \, dx \quad t \geq 0. \]

(4.8)

**Proof.** Multiplying (2,3) by \( u \) and integrating over \((0,1)\) we obtain

\[ \rho_1 \int_0^1 u_{tt} \, dx = \mu_1 \int_0^1 u_{xx} \, dx + \mu_0 \int_0^1 \varphi_x \, dx - \beta_0 \int_0^1 (\tau \theta_{tx} + \theta_x) \, u \, dx \]
and then integrating by parts together with the boundary conditions, we get
\[
\rho_1 \int_0^1 u_{tt} \, dx + \rho_1 \int_0^1 u_x^2 \, dx - \rho_1 \int_0^1 u_t^2 \, dx = -\mu_1 \int_0^1 u_x^2 \, dx - \mu_0 \int_0^1 \varphi u_x \, dx + \beta_0 \int_0^1 (\tau \theta_x + \theta) u_x \, dx
\]
\[
= -\mu_1 \int_0^1 u_x^2 \, dx - \mu_0 \int_0^1 \varphi u_x \, dx + \beta_0 \tau \int_0^1 \theta u_x \, dx - \beta_0 \int_0^1 \theta u_{tx} \, dx
\]
\[
= -\mu_1 \int_0^1 u_x^2 \, dx - \mu_0 \int_0^1 \varphi u_x \, dx + \beta_0 \tau \int_0^1 \theta u_x \, dx + \beta_0 \int_0^1 \theta u_x \, dx
\]
\[
+ \beta_0 \tau \int_0^1 \theta u_x \, dx - \beta_0 \int_0^1 \theta u_{tx} \, dx
\]
which is equivalent to
\[
\frac{d}{dt} \left( \rho_1 \int_0^1 u_t \, dx - \beta_0 \tau \int_0^1 \theta u_x \, dx \right) = \rho_1 \int_0^1 u_t^2 \, dx - \mu_1 \int_0^1 u_x^2 \, dx - \mu_0 \int_0^1 \varphi u_x \, dx
\]
\[
+ \beta_0 \tau \int_0^1 \theta u_x \, dx + \beta_0 \int_0^1 \theta u_x \, dx
\]
then, we get
\[
I_1 (t) = \rho_1 \int_0^1 u_t \, dx - \beta_0 \tau \int_0^1 \theta u_x \, dx \quad t \geq 0
\] (4.9)
and
\[
I_1' (t) = \rho_1 \int_0^1 u_t^2 \, dx - \mu_1 \int_0^1 u_x^2 \, dx - \mu_0 \int_0^1 \varphi u_x \, dx + \beta_0 \tau \int_0^1 \theta u_t \, dx + \beta_0 \int_0^1 \theta u_x \, dx
\] (4.10)
Using Young’s and Poincaré’s inequalities and (1.3) we find
\[
-\mu_0 \int_0^1 \varphi u_x \, dx \leq \xi^2 \int_0^1 \varphi^2 \, dx + \mu_1 \int_0^1 u_x^2 \, dx
\] (4.11)
\[
\beta_0 \tau \int_0^1 \theta u_t \, dx \leq \frac{\beta_0 \tau}{2} \int_0^1 \theta^2 \, dx + \beta_0 \tau \int_0^1 u_t^2 \, dx
\] (4.12)
\[
\beta_0 \int_0^1 \theta u_x \, dx \leq \frac{\beta_0}{2} \int_0^1 \theta^2 \, dx + \frac{\mu_1}{4} \int_0^1 u_x^2 \, dx
\] (4.13)
inserting (4.11), (4.12), (4.13) in (4.10), we get

\[
I_1'(t) \leq -\frac{\mu_1}{2} \int_0^1 u_x^2 \, dx + \left(\rho_1 + \frac{\beta_0 \tau}{2}\right) \int_0^1 u_t^2 \, dx + \left(\frac{\beta_2^2}{\mu_1} + \frac{\beta_0 \tau}{2}\right) \int_0^1 \theta_x^2 \, dx + \xi \int_0^1 \varphi^2 \, dx
\]

\[
\leq -\frac{\mu_1}{2} \int_0^1 u_x^2 \, dx + c_0 \int_0^1 \left(u_t^2 + \theta_x^2 + \varphi^2\right) \, dx
\]

Lemma 3. Let \((u, \varphi, \theta, z)\) be the solution of (2.3)-(2.4) then, the functional \(I_2(t)\) defined by

\[
I_2(t) = J \int_0^1 \varphi_t \varphi + \frac{\gamma_1}{2} \int_0^1 \varphi^2 \, dx - \frac{\mu_0 \rho_1}{\mu_1} \int_0^1 u_t \left(\int_0^x \varphi(y) \, dy\right) \, dx \quad t \geq 0,
\]

satisfies for any \(\varepsilon_1 > 0\) the estimate

\[
I_2'(t) \leq -a_0 \int_0^1 \varphi_x^2 \, dx - \frac{\mu_3}{2} \int_0^1 \varphi^2 \, dx + \varepsilon_1 \int_0^1 u_t^2 \, dx + c_0 \int_0^1 (\tau \theta_t + \theta^2) \, dx
\]

\[
+ c_0 \left(1 + \frac{1}{\varepsilon_1}\right) \int_0^1 \varphi_t^2 \, dx + \frac{1}{\mu_3} \left(\int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds\right) \int_0^{\tau_2} |\gamma_2(s)| z^2(x,1,s,t) \, ds \, dx \quad t \geq 0,
\]

where \(\mu_3 = \xi - \frac{\beta_2}{\mu_1}\).

Proof. Multiplying (2.3) by \(\varphi\), integrating over \((0,1)\) we obtain

\[
J \int_0^1 \varphi_t \varphi \, dx = a_0 \int_0^1 \varphi_{xx} \varphi \, dx - \mu_0 \int_0^1 u_x \varphi \, dx - \xi \int_0^1 \varphi^2 \, dx + \beta_1 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx
\]

\[
- \gamma_1 \int_0^1 \varphi_t \varphi \, dx - \int_0^{\tau_2} \varphi_\gamma_2(s) z(x,1,s,t) \, ds \, dx
\]

using integration by parts together with the boundary conditions, we get

\[
J \int_0^1 \varphi_t \varphi \, dx + J \int_0^1 \varphi_x^2 \, dx - J \int_0^1 \varphi^2 \, dx = -a_0 \int_0^1 \varphi_x^2 \, dx - \mu_0 \int_0^1 u_x \varphi \, dx - \xi \int_0^1 \varphi^2 \, dx + \beta_1 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx
\]

\[
- \gamma_1 \int_0^1 \varphi_t \varphi \, dx - \int_0^{\tau_2} \varphi \gamma_2(s) z(x,1,s,t) \, ds \, dx
\]
which is equivalent to
\[
\frac{d}{dt} \left( J \int_0^1 \varphi_1 \varphi \, dx + \frac{\gamma_1}{2} \int_0^1 \varphi^2 \, dx \right) = J \int_0^1 \varphi_1^2 \, dx - a_0 \int_0^1 \varphi_x^2 \, dx - \mu_0 \int u_x \varphi \, dx - \xi \int \varphi^2 \, dx
\]
\[
+ \beta_1 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx - \int_0^1 \varphi \int_{\tau_1}^2 \gamma_2 (s) \zeta (x, 1, s, t) \, ds \, dx
\]
which is equivalent to
\[
\frac{d}{dt} \left( J \int_0^1 \varphi_1 \varphi \, dx + \frac{\gamma_1}{2} \int_0^1 \varphi^2 \, dx \right) = J \int_0^1 \varphi_1^2 \, dx - a_0 \int_0^1 \varphi_x^2 \, dx - \mu_0 \int u_x \varphi \, dx - \xi \int \varphi^2 \, dx
\]
\[
+ \beta_1 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx - \int_0^1 \varphi \int_{\tau_1}^2 \gamma_2 (s) \zeta (x, 1, s, t) \, ds \, dx
\]
\[\text{(4.16)}\]

Multiplying (2, 3) by \(-\frac{\mu_0}{\mu_1} \int_0^x \varphi (y) \, dy\), integrating over (0,1) and then integrating by parts together with the boundary conditions, we get
\[
-\frac{\mu_0}{\mu_1} \int_0^1 u_{tt} \left( \int_0^x \varphi (y) \, dy \right) \, dx = \mu_0 \int_0^1 u_x \varphi \, dx + \frac{\mu_0^2}{\mu_1} \int u_{tt}^2 \, dx - \frac{\beta_0}{\mu_1} \int_0^1 (\tau \theta_t + \theta) \varphi \, dx
\]
which is equivalent to
\[
\frac{d}{dt} \left( -\frac{\mu_0}{\mu_1} \int_0^1 u_{tt} \left( \int_0^x \varphi (y) \, dy \right) \, dx \right) = \mu_0 \int_0^1 u_x \varphi \, dx + \frac{\mu_0^2}{\mu_1} \int u_{tt}^2 \, dx - \frac{\beta_0}{\mu_1} \int_0^1 (\tau \theta_t + \theta) \varphi \, dx
\]
\[
- \frac{\mu_0}{\mu_1} \int_0^1 u_t \left( \int_0^x \varphi (y) \, dy \right) \, dx
\]
\[\text{(4.17)}\]
we sum (4.16) and (4.17) we find
\[
\frac{d}{dt} \left( J \int_0^1 \varphi_1 \varphi \, dx + \frac{\gamma_1}{2} \int_0^1 \varphi^2 \, dx - \frac{\mu_0}{\mu_1} \int u_{tt} \left( \int_0^x \varphi (y) \, dy \right) \, dx \right) = J \int_0^1 \varphi_1^2 \, dx - a_0 \int_0^1 \varphi_x^2 \, dx - \mu_3 \int \varphi^2 \, dx
\]
\[
+ \left( \beta_1 - \frac{\beta_0}{\mu_1} \right) \int_0^1 (\tau \theta_t + \theta) \varphi \, dx
\]
\[
- \frac{\mu_0}{\mu_1} \int_0^1 u_t \left( \int_0^x \varphi (y) \, dy \right) \, dx
\]
\[
- \int_0^1 \varphi \int_{\tau_1}^2 \gamma_2 (s) \zeta (x, 1, s, t) \, ds \, dx
\]
where \(\mu_3 = \xi - \frac{\mu_0^2}{\mu_1}\) then, we get
\[
I_2 (t) = J \int_0^1 \varphi_1 \varphi \, dx + \frac{\gamma_1}{2} \int_0^1 \varphi^2 \, dx - \frac{\mu_0}{\mu_1} \int u_t \left( \int_0^x \varphi (y) \, dy \right) \, dx \quad t \geq 0 \quad \text{(4.18)}
\]
and

\[ I_2'(t) = J \left[ \int_0^1 \varphi_1^2 \, dx - a_0 \int_0^1 \varphi_1^2 \, dx - \mu_3 \int_0^1 \varphi^2 \, dx - \int_0^{\tau_2} \varphi \int_{\tau_1}^{\tau_2} \gamma_2(s) \, z(x,1,s,t) \, ds \, dx \right] \]

\[ + \left( \beta_1 - \frac{\beta_0 \mu_0}{\mu_1} \right) \left[ \int_0^1 (\tau \theta_1 + \theta) \varphi \, dx - \frac{\mu_0 \rho_1}{\mu_1} \int_0^1 u_0 \left( \int_0^1 \varphi_1(y) \, dy \right) \, dx \right] \quad t \geq 0 \]

Using Young’s and Cauchy Schwarz’s inequalities we obtain

\[ - \frac{\mu_0 \rho_1}{\mu_1} \int_0^1 u_1 \left( \int_0^1 \varphi_1(y) \, dy \right) \, dx \leq \varepsilon_1 \int_0^1 u_0^2 \, dx + \frac{1}{4\varepsilon_1} \left( \frac{\mu_0 \rho_1}{\mu_1} \right)^2 \int_0^1 \varphi_1^2 \, dx \quad (4.20) \]

\[ - \int_0^{\tau_2} \varphi \int_{\tau_1}^{\tau_2} \gamma_2(s) \, z(x,1,s,t) \, ds \, dx \leq \frac{\mu_3}{4} \int_0^1 \varphi^2 \, dx + \frac{1}{\mu_3} \left( \int_0^{\tau_2} \gamma_2(s) \, ds \right) \int_0^{\tau_2} \left| \gamma_2(s) \right| \, z^2(x,1,s,t) \, ds \, dx \quad (4.21) \]

\[ \left( \beta_1 - \frac{\beta_0 \mu_0}{\mu_1} \right) \int_0^1 (\tau \theta_1 + \theta) \varphi \, dx \leq \frac{\mu_3}{4} \int_0^1 \varphi^2 \, dx + \frac{1}{\mu_3} \left( \beta_1 - \frac{\beta_0 \mu_0}{\mu_1} \right)^2 \int_0^1 (\tau \theta_1 + \theta)^2 \, dx \quad (4.22) \]

inserting (4.20), (4.21) and (4.22) in (4.19) we get

\[ I_2'(t) \leq -a_0 \int_0^1 \varphi_1^2 \, dx - \frac{\mu_3}{2} \int_0^1 \varphi^2 \, dx + \varepsilon_1 \int_0^1 u_0^2 \, dx + c_0 \int_0^1 (\tau \theta_1 + \theta)^2 \, dx + c_0 \left( 1 + \frac{1}{\varepsilon_1} \right) \int_0^1 \varphi_1^2 \, dx \]

\[ + \frac{1}{\mu_3} \left( \int_0^{\tau_2} \left| \gamma_2(s) \right| \, ds \right) \int_0^{\tau_2} \left| \gamma_2(s) \right| \, z^2(x,1,s,t) \, ds \, dx \quad t \geq 0 \]

\[ \square \]

**Lemma 4.** Let \((u, \varphi, \theta, z)\) be the solution of (2.3)-(2.4) then, the functional \(I_3(t)\) defined by

\[ I_3(t) = -a \int_0^1 \theta^2 \theta_1 \, dx - \frac{a \tau^2}{2} \int_0^1 \theta^2 \, dx \quad t \geq 0, \quad (4.23) \]

satisfies for any \(\varepsilon_2 > 0\) the estimate

\[ I_3'(t) \leq -a \int_0^1 (\tau \theta_1 + \theta)^2 \, dx + \varepsilon_2 \int_0^1 u_0^2 \, dx + c_0 \int_0^1 \varphi_1^2 \, dx + c_0 \left( 1 + \frac{1}{\varepsilon_2} \right) \int_0^1 \theta_1^2 \, dx \quad t \geq 0. \quad (4.24) \]

**Proof.** Multiplying (2, 3) by \(-\tau\), integrating over \((0,1)\) we obtain

\[ -a \tau^2 \int_0^1 \theta_1 \, dx + a \int_0^1 \theta_1 \, dx = \beta_0 \tau \int_0^1 u_1 \theta_1 \, dx + \beta_1 \tau \int_0^1 \varphi_1 \theta_1 \, dx - \delta \int_0^1 \theta_{xx} \theta_1 \, dx \]
and then integrating by parts together with the boundary conditions, we get

\[-a \frac{\partial^2}{\partial t^2} \int_0^1 \theta_t \, dx - a \frac{\partial}{\partial t} \int_0^1 \theta \, dx = -\beta_0 \frac{\partial}{\partial t} \int_0^1 u_t \theta_t \, dx + \beta_1 \int_0^1 \varphi_t \theta \, dx + \delta \int_0^1 \theta_x^2 \, dx\]

which is equivalent to

\[\frac{d}{dt} \left( -a \int_0^1 \tau^2 \theta_t \, dx - \frac{a \tau}{2} \int_0^1 \theta^2 \, dx \right) = -a \tau^2 \int_0^1 \theta_t^2 \, dx - \beta_0 \tau \int_0^1 u_t \theta_t \, dx + \beta_1 \tau \int_0^1 \varphi_t \theta \, dx + \delta \tau \int_0^1 \theta_x^2 \, dx\]

then, we obtain

\[I_3(t) = -a \int_0^1 \tau^2 \theta_t \, dx - \frac{a \tau}{2} \int_0^1 \theta^2 \, dx \quad t \geq 0\]

and

\[I_3'(t) = -a \tau^2 \int_0^1 \theta_t^2 \, dx - \beta_0 \tau \int_0^1 u_t \theta_t \, dx + \beta_1 \tau \int_0^1 \varphi_t \theta \, dx + \delta \tau \int_0^1 \theta_x^2 \, dx\] (4.25)

using Young’s and Poincaré’s inequalities, we find

\[-\beta_0 \tau \int_0^1 u_t \theta_t \, dx \leq \varepsilon_2 \int_0^1 u_t^2 \, dx + \frac{1}{4 \varepsilon_2} (\beta_0 \tau)^2 \int_0^1 \theta_x^2 \, dx\] (4.26)

\[\beta_1 \tau \int_0^1 \varphi_t \theta \, dx \leq \frac{1}{2} (\beta_1 \tau)^2 \int_0^1 \varphi_t^2 \, dx + \frac{1}{2} \int_0^1 \theta_x^2 \, dx\] (4.27)

making use of the fact

\[-\int_0^1 (\tau \theta_t)^2 \, dx \leq -\frac{1}{2} \int_0^1 (\tau \theta_t + \theta)^2 \, dx + \int_0^1 \theta_x^2 \, dx\]

we get

\[-a \int_0^1 (\tau \theta_t)^2 \, dx \leq -a \frac{1}{2} \int_0^1 (\tau \theta_t + \theta)^2 \, dx + a \int_0^1 \theta_x^2 \, dx\] (4.28)

inserting (4.26), (4.27) and (4.28) in (4.25) we get

\[I_3'(t) \leq -\frac{a}{2} \int_0^1 (\tau \theta_t + \theta)^2 \, dx + \varepsilon_2 \int_0^1 u_t^2 \, dx + c_0 \int_0^1 \varphi_t^2 \, dx + c_0 \int_0^1 \left( 1 + \frac{1}{\varepsilon_2} \right) \int_0^1 \theta_x^2 \, dx \quad t \geq 0\]

\[\square\]

**Lemma 5.** Let \((u, \varphi, \theta, z)\) be the solution of (2.3)-(2.4) then, the functional \(I_4(t)\) defined by

\[I_4(t) = -\rho_1 a \int_0^1 \left( \int_0^x u_t(y) \, dy \right) (\tau \theta_t + \theta) \, dx \quad t \geq 0,\] (4.29)
satisfies for any $\varepsilon_3 > 0$ the estimate

$$\quad I_4'(t) \leq -\frac{\beta_0 \rho_1}{2} \int_0^1 u_t^2 \, dx + \varepsilon_3 \int_0^1 u_x^2 \, dx + c_0 \int_0^1 \varphi^2 \, dx + c_0 \int_0^1 \varphi_t^2 \, dx$$

$$+ c_0 \int_0^1 \theta_x^2 \, dx + c_0 \left( 1 + \frac{1}{\varepsilon_3} \right) \int_0^1 (\tau \theta_t + \theta)^2 \, dx \quad t \geq 0. \quad (4.30)$$

Proof. Multiplying $(2,3)_3$ by $-\rho_1 \int_0^x u_t (y) \, dy$, integrating over $(0,1)$ we obtain

$$\quad -\rho_1 a \int_0^1 \left( \int_0^x u_t (y) \, dy \right) (\tau \theta_t + \theta) \, dx = \beta_0 \rho_1 \int_0^1 u_{xx} \left( \int_0^x u_t (y) \, dy \right) \, dx + \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t (y) \, dy \right) \, dx$$

$$- \delta \rho_1 \int_0^1 \theta_{xx} \left( \int_0^x u_t (y) \, dy \right) \, dx$$

using integration by parts together with the boundary conditions, we get

$$\quad -\rho_1 a \int_0^1 \left( \int_0^x u_t (y) \, dy \right) (\tau \theta_t + \theta) \, dx = -\beta_0 \rho_1 \int_0^1 u_t^2 \, dx + \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t (y) \, dy \right) \, dx + \delta \rho_1 \int_0^1 \theta_x \, u_t \, dx$$

which is equivalent to

$$\quad \frac{d}{dt} \left( -a \rho_1 \int_0^1 (\tau \theta_t + \theta) \left( \int_0^x u_t (y) \, dy \right) \, dx \right) = -a \rho_1 \int_0^1 (\tau \theta_t + \theta) \left( \int_0^x u_t (y) \, dy \right) \, dx - \beta_0 \rho_1 \int_0^1 u_t^2 \, dx$$

$$+ \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t (y) \, dy \right) \, dx + \delta \rho_1 \int_0^1 \theta_x \, u_t \, dx$$

using $(2,3)_1$ we obtain

$$\quad \frac{d}{dt} \left( -a \rho_1 \int_0^1 (\tau \theta_t + \theta) \left( \int_0^x u_t (y) \, dy \right) \, dx \right) = -a \rho_1 \int_0^1 (\tau \theta_t + \theta) u_x \, dx - a \rho_0 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx$$

$$+ a \beta_0 \int_0^1 (\tau \theta_t + \theta)^2 \, dx - \beta_0 \rho_1 \int_0^1 u_t^2 \, dx$$

$$+ \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t (y) \, dy \right) \, dx + \delta \rho_1 \int_0^1 \theta_x \, u_t \, dx$$

then, we have

$$\quad I_4(t) = -a \rho_1 \int_0^1 (\tau \theta_t + \theta) \left( \int_0^x u_t (y) \, dy \right) \, dx \quad t \geq 0$$
and
\[ I_4'(t) = -a\mu_1 \int_0^1 (\tau \theta_t + \theta) u_x \, dx - a\mu_0 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx + a\beta_0 \int_0^1 (\tau \theta_t + \theta)^2 \, dx \]
\[ -\beta_0 \rho_1 \int_0^1 u_t^2 \, dx + \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t(y) \, dy \right) \, dx + \delta \rho_1 \int_0^1 \theta_x u_t \, dx \]  \hspace{1cm} (4.31)

Using Young’s and Cauchy Schwarz’s inequalities we find
\[ \delta \rho_1 \int_0^1 \theta_x u_t \, dx \leq \frac{\beta_0 \rho_1}{4} \int_0^1 u_t^2 \, dx + \frac{\delta \rho_1}{\beta_0} \int_0^1 \varphi_t^2 \, dx \]  \hspace{1cm} (4.32)
\[ \beta_1 \rho_1 \int_0^1 \varphi_t \left( \int_0^x u_t(y) \, dy \right) \, dx \leq \frac{\beta_0 \rho_1}{4} \int_0^1 u_t^2 \, dx + \frac{\beta_1 \rho_1}{\beta_0} \int_0^1 \varphi_t^2 \, dx \]  \hspace{1cm} (4.33)
\[ -a\mu_0 \int_0^1 (\tau \theta_t + \theta) \varphi \, dx \leq \frac{a\mu_0}{2 \varepsilon_2} \int_0^1 (\tau \theta_t + \theta)^2 \, dx + \frac{a \varepsilon_3 \mu_0}{2} \int_0^1 \varphi_t^2 \, dx \]  \hspace{1cm} (4.34)
\[ -a\mu_1 \int_0^1 (\tau \theta_t + \theta) u_x \, dx \leq \varepsilon_3 \int_0^1 u_t^2 \, dx + \frac{1}{4 \varepsilon_3} (a\mu_1)^2 \int_0^1 (\tau \theta_t + \theta)^2 \, dx \]  \hspace{1cm} (4.35)

inserting (4.32), (4.33), (4.34) and (4.35) in (4.31) we get
\[ I_4'(t) \leq -\frac{\beta_0 \rho_1}{2} \int_0^1 u_t^2 \, dx + \varepsilon_3 \int_0^1 u_t^2 \, dx + c_0 \int_0^1 \varphi_t^2 \, dx + c_0 \int_0^1 \varphi_t^2 \, dx \]
\[ + c_0 \int_0^1 \theta_x^2 \, dx + c_0 \left( 1 + \frac{1}{\varepsilon_3} \right) \int_0^1 (\tau \theta_t + \theta)^2 \, dx \quad t \geq 0 \]

\[ \square \]

**Lemma 6.** Let \((u, \varphi, \theta, z)\) be the solution of (2.3)-(2.4) then, the functional \(I_5(t)\) defined by
\[ I_5(t) = \int_0^1 \int_{\tau_2}^{\tau_3} s e^{-s \rho} |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx \quad t \geq 0, \]  \hspace{1cm} (4.36)
satisfies the estimate
\[ I_5'(t) \leq -\eta_1 \int_0^{\tau_2} |\gamma_2(s)| z^2(x, 1, s, t) \, ds \, dx - \eta_1 \int_0^1 \int_{\tau_2}^{\tau_3} s |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx \]
\[ + \gamma_1 \int_0^1 \varphi_t^2 \, dx \quad t \geq 0. \]  \hspace{1cm} (4.37)
Proof. Multiplying (2.3) by \( e^{-s \rho} |\gamma_2(s)| \) \( z \), integrating over \((0, 1) \times (0, 1) \times (\tau_1, \tau_2)\) we obtain

\[
\int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| \, z \, ds \, d\rho \, dx = - \int_0^1 \int_0^{\tau_2} e^{-s \rho} |\gamma_2(s)| \, z_{\rho} \, z \, ds \, d\rho \, dx
\]

which is equivalent to

\[
\frac{d}{dt} \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx = - \int_0^1 \int_0^{\tau_2} e^{-s \rho} |\gamma_2(s)| \frac{\partial}{\partial \rho} z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

Integration by part and using the fact that \( z(x, 0, s, t) = \varphi_t(x, t) \) gives

\[
\frac{d}{dt} \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx = - \int_0^1 \int_0^{\tau_2} e^{-\tau} |\gamma_2(s)| z^2(x, 1, s, t) \, ds \, dx
\]

\[
- \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx + \left( \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \frac{d}{dt} \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

then, we have

\[
I_5(t) = \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx \quad t \geq 0
\]

and

\[
I_5'(t) = - \int_0^1 \int_0^{\tau_2} e^{-s} |\gamma_2(s)| z^2(x, 1, s, t) \, ds \, dx + \left( \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \frac{d}{dt} \int_0^1 \varphi_t^2 \, dx
\]

\[
- \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

using the fact that \( e^{-s} \leq e^{-s \rho} \leq 1 \) we get for all \( \rho \in [0, 1] \)

\[
I_5'(t) \leq - \int_0^1 \int_0^{\tau_2} e^{-s} |\gamma_2(s)| z^2(x, 1, s, t) \, ds \, dx + \left( \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) \frac{d}{dt} \int_0^1 \varphi_t^2 \, dx
\]

\[
- \int_0^1 \int_0^{\tau_2} s e^{-s} |\gamma_2(s)| z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

Since \(-e^{-s}\) is an increasing function, we have \(-e^{-s} \leq -e^{-\tau_2}\) for all \( s \in [\tau_1, \tau_2] \).

Finally, setting \( \eta_1 = e^{-\tau_2} \) and recalling (2.5) we get

\[
I_5'(t) \leq -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, z^2(x, 1, s, t) \, ds \, dx - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\gamma_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

\[
+ \eta_1 \int_0^1 \varphi_t^2 \, dx \quad t \geq 0
\]
Now, we define the Lyapunov functional $L(t)$ by

$$L(t) = NE(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t) + N_4 I_4(t) + N_5 I_5(t) \quad \forall t \geq 0,$$

where $N, N_1, N_2, N_3, N_4$ and $N_5$ are positive real numbers to be chosen appropriately later.

**Lemma 7.** Let $(u, \varphi, \theta, z)$ be the solution of (2.3)-(2.4) then, there exists two positive constants $b_1$ and $b_2$ such that the Lyapunov functional $L(t)$ satisfies

$$b_1 E(t) \leq L(t) \leq b_2 E(t) \quad \forall t \geq 0,$$

and

$$L'(t) \leq -\varsigma E(t) \quad , \quad \varsigma > 0.$$

**Proof.** From (4.38), we can write

$$|L(t) - NE(t)| \leq N_1 |I_1(t)| + N_2 |I_2(t)| + N_3 |I_3(t)| + N_4 |I_4(t)| + N_5 |I_5(t)|$$

$$\leq N_1 \rho_1 \int_0^1 |u_t| |u| dx + N_1 \beta_0 \tau \int_0^1 |u_x| |\theta| dx + N_2 J_1 \int_0^1 |\varphi_t| |\varphi| dx + \frac{N_2 \gamma_1}{2} \int_0^1 \varphi^2 dx$$

$$+ \frac{N_3 \mu_0 \rho_1}{\mu_1} \int_0^1 |u_t| \left( \int_0^x |\varphi(y)| dy \right) dx + N_3 \alpha \tau^2 \int_0^1 |\theta_t| |\theta| dx + \frac{N_3 \alpha \tau}{2} \int_0^1 \theta^2 dx$$

$$+ N_4 \rho_1 \int_0^1 |\tau| \theta_t + \theta| \left( \int_0^x |u_t(x)| dy \right) dx + N_5 \int_0^1 \int_0^{\tau_2} s e^{-s \rho} |\gamma_2(s)| z^2 ds d\rho dx$$

Exploiting Young's, Poincaré's and Cauchy-Schwarz inequalities, (4.1) and the fact that $e^{-s \rho} \leq 1$ for all $\rho \in [0, 1]$ we obtain

$$|L(t) - NE(t)| \leq c \int_0^1 \left( u_t^2 + \varphi_t^2 + (\tau + \theta)^2 + \varphi_x^2 + \varphi^2 + \theta_x^2 + \left( u_x + \frac{\mu_0}{\mu_1} \varphi \right)^2 \right) dx$$

$$+ c \int_0^1 \int_0^{\tau_2} s |\gamma_2(s)| z^2(x, \rho, s, t) ds d\rho dx$$

$$\leq c E(t)$$

Consequently, we have

$$(N - c) E(t) \leq L(t) \leq (N + c) E(t)$$

Choosing $N$ is sufficiently large and depends on $N_i$, $i = 1, \ldots, 5$, we obtain (4.40).
Now, By differentiating (4.38) and recalling (4.2), (4.8), (4.15), (4.24), (4.30), (4.37) we get

\[
L'(t) \leq - \left( \frac{N_4 \beta_0 \rho_1}{8} - N_1 \varepsilon_2 - N_2 \varepsilon_1 - N_1 c_0 \right) \int_0^1 u_t^2 dx - N_2 a_0 \int_0^1 \varphi_t^2 dx
\]

\[
- \left( N \left( \gamma_1 \int_{\tau_1}^{\tau_2} |\varphi_2(s)| \, ds \right) - N_2 c_0 \left( 1 + \frac{1}{\varepsilon_1} \right) - N_3 c_0 - N_4 c_0 - N_5 \gamma_1 \right) \int_0^1 \varphi_t^2 dx
\]

\[
- \left( \delta N - N_1 c_0 - N_3 c_0 \left( 1 + \frac{1}{\varepsilon_2} \right) - N_4 c_0 \right) \int_0^1 \theta_t^2 dx - \left( \frac{N_1 \mu_1}{2} - N_4 \varepsilon_3 \right) \int_0^1 u_x^2 dx
\]

\[
- \left( \frac{N_2 \mu_3}{2} - N_1 c_0 - N_4 c_0 \right) \int_0^1 \varphi_t^2 dx - \left( \frac{N_3 a}{2} - N_2 c_0 - N_4 c_0 \left( 1 + \frac{1}{\varepsilon_3} \right) \right) \int_0^1 (\tau \theta_t + \theta)^2 dx
\]

\[
- \left( N_5 \eta_1 - \frac{N_2 \gamma_1}{\mu_3} \right) \int_0^1 \int_{\tau_1}^{\tau_2} |\varphi_2(s)| \, z^2(x, s) \, ds \, dx - N_5 \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s \, |\varphi_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

At this point, we set \( \varepsilon_1 = \frac{\beta_0 \rho_1 N_4}{8N_2} \), \( \varepsilon_2 = \frac{\beta_0 \rho_1 N_4}{4N_2} \) and \( \varepsilon_3 = \frac{\mu_1 N_4}{4N_4} \), we end up with

\[
L'(t) \leq - \left( \frac{N_4 \beta_0 \rho_1}{8} - N_1 c_0 \right) \int_0^1 u_t^2 dx - N_2 a_0 \int_0^1 \varphi_t^2 dx
\]

\[
- \left( N \left( \gamma_1 \int_{\tau_1}^{\tau_2} |\varphi_2(s)| \, ds \right) - N_2 c_0 \left( 1 + \frac{8N_2}{N_4 \beta_0 \rho_1} \right) - N_3 c_0 - N_4 c_0 - N_5 \gamma_1 \right) \int_0^1 \varphi_t^2 dx
\]

\[
- \left( \delta N - N_1 c_0 - N_3 c_0 \left( 1 + \frac{4N_4}{N_4 \beta_0 \rho_1} \right) - N_4 c_0 \right) \int_0^1 \theta_t^2 dx - \left( \frac{N_1 \mu_1}{4} - \frac{N_4 \mu_1}{4} \right) \int_0^1 u_x^2 dx
\]

\[
- \left( \frac{N_2 \mu_3}{2} - N_1 c_0 - N_4 c_0 \right) \int_0^1 \varphi_t^2 dx - \left( \frac{N_3 a}{2} - N_2 c_0 - N_4 c_0 \left( 1 + \frac{4N_4}{N_4 \mu_1} \right) \right) \int_0^1 (\tau \theta_t + \theta)^2 dx
\]

\[
- \left( N_5 \eta_1 - \frac{N_2 \gamma_1}{\mu_3} \right) \int_0^1 \int_{\tau_1}^{\tau_2} |\varphi_2(s)| \, z^2(x, s) \, ds \, dx - N_5 \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} s \, |\varphi_2(s)| \, z^2(x, \rho, s, t) \, ds \, d\rho \, dx
\]

Then, we fixed \( N_1 \) and we choose \( N_4 \) large enough such that

\( \kappa_1 = \frac{N_4 \beta_0 \rho_1}{8} - N_1 c_0 > 0 \)

once \( N_4 \) is fixed, we take \( N_2 \) large enough such that

\( \kappa_2 = \frac{N_2 \mu_3}{2} - N_1 c_0 - N_4 c_0 > 0 \)

after we fixed \( N_2 \), we choose \( N_3 \) and \( N_3 \) large enough such that

\( \kappa_3 = \frac{N_3 a}{2} - N_2 c_0 - N_4 c_0 \left( 1 + \frac{4N_4}{N_1 \mu_1} \right) > 0 \), \( \kappa_4 = N_5 \eta_1 - \frac{N_2 \gamma_1}{\mu_3} > 0 \)
Finally, select $N$ that is large enough (even larger such that (4.39) is still valid) to
\[
\kappa_5 = N \left( \gamma_1 - \int_{\tau_1}^{\tau_2} |\gamma_2(s)| \, ds \right) - N_2 c_0 \left( 1 + \frac{8N_2}{N_4 \beta_0 \rho_1} \right) - N_3 c_0 - N_4 c_0 - N_5 \gamma_1 > 0
\]
and
\[
\kappa_6 = \delta N - N_1 c_0 - N_3 c_0 \left( 1 + \frac{4N_3}{N_4 \beta_0 \rho_1} \right) - N_4 c_0 > 0
\]
Moreover, we set
\[
\kappa_7 = \delta \frac{N_4 \mu_1}{4}, \quad \kappa_8 = N_2 a_0, \quad \kappa_9 = N_5 \eta_1
\]
we obtain (4.40)

In what follows, we'll use the equivalence relationship (4.39) to estimate the system's energy (2.3)-(2.4) using the estimated (4.40) based on the assumption (2.5). The stability result can now be stated as follows.

**Theorem 2.** Let $(u, \varphi, \theta, z)$ be the solution of (2.3)-(2.4) and we assume that (2.5) hold, Then, the solution $(u, \varphi, \theta, z)$ decays exponentially, i.e. there exist two positive constants $\lambda_1$ and $\lambda_2$ such that
\[
E(t) \leq \lambda_2 e^{-\lambda_1 t} \quad \forall t \geq 0.
\] (4.41)

**Proof.** using the equivalence of $E(t)$ and $L(t)$ we deduce that
\[
L'(t) \leq -\lambda_1 L(t) \quad \forall t \geq 0
\] (4.42)
where $\lambda_1 = \frac{\kappa_7}{\kappa_8} > 0$, A simple integration of (4.42) gives
\[
L(t) \leq L(0) e^{-\lambda_1 t} \quad \forall t \geq 0
\]
which yields the serial result (4.41) and by using the other side of (4.39) again. The proof is complete

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**Conflict of interest**

The authors declare there is no conflicts of interest.
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